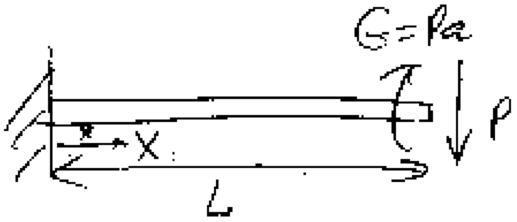
4ME06 Stress Analysis

(1)



Total energy: $W = \underbrace{\frac{1}{2}EI\int_0^L (y'')^2 dx}_{\text{REMEMBER!}} - Py(L) - Gy'(L)$

Analytical treatment to obtain exact soln:

$\delta W = 0$

$$\delta W = EI\int_0^L y''\,\delta y''\,dx - P\delta y(L) - G\delta y'(L) =$$

$$= -EI\int_0^L y'''\,\delta y'\,dx + \left[EI\,y''\delta y'\right]_0^L - P\delta y(L) - G\delta y'(L) =$$

$$= EI\int_0^L y''''\,\delta y\,dx - \left[EI\,y'''\delta y\right]_0^L + \left[EI\,y''\delta y'\right]_0^L - P\delta y(L) - G\delta y'(L)$$

B.c.: for $x=0$ $\delta y = \delta y' = 0$, since $y = y' = 0$ @ $x=0$

Hence for $\delta W = 0$ require

(i) $EI\,y'''' = 0$, $0 < x < L$ - this is EULER.

(ii) $P = -EI\,y'''(L)$, since $\delta y(L)$ is arbitrary
(iii) $G = EI\,y''(L)$, since $\delta y'(L)$ is arbitrary

} Nat'l b.c.

Hence $y = A + Bx + Cx^2 + Dx^3$

$x=0$, $y = y' = 0 \iff A = B = 0$

$x=L$, $y'' = 2C + 6DL = G/EI = Pa/EI$

$x=L$, $y''' = 6D = -P/EI$

$\boxed{\text{EXACT}}$

$C = \dfrac{PL + Pa}{2EI}$
$D = -\dfrac{P}{6EI}$

(a) Rayleigh - Ritz :

$$y = A + Bx + Cx^2 + Dx^3$$

B.c's $y(0) = y'(0) = 0 \Rightarrow A = B = 0$

$$y'' = 2C + 6Dx$$

$$W = \frac{1}{2} EI \int_0^L (y'')^2 dx - Py(L) - Gy'(L) =$$

$$= \frac{1}{2} EI \int_0^L (4C^2 + 24CDx + 36D^2x^2) dx - P(CL^2 + DL^3) - G(2CL + 3DL^2)$$

$$W = 2EI(C^2L + 3CDL^2 + 3D^2L^3) - P(CL^2 + DL^3) - G(2CL + 3DL^2)$$

$$\frac{\partial W}{\partial C} = 0 = 2EIL(2C + 3DL) - PL^2 - 2GL$$

$$\frac{\partial W}{\partial D} = 0 = 2EIL(3CL + 6DL^2) - PL^3 - 3GL^2$$

$$\boxed{C = \frac{PL + Pa}{2EI} \quad D = -\frac{P}{6EI}}$$

Exact solution obtained, since sufficient series terms.

Natural b.c's:

$$M(L) = EIy''(L) = EI(2C + 6DL) = PL + Pa - PL = Pa$$

OK.

(b) $y = A\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)$

$y' = \frac{A\pi}{2L}\sin\left(\frac{\pi x}{2L}\right)$

$y'' = A\left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right)$

$y'''' = -A\left(\frac{\pi}{2L}\right)^4\cos\left(\frac{\pi x}{2L}\right)$

$$W = \frac{1}{2}EIA^2\left(\frac{\pi}{2L}\right)^4\int_0^L \cos^2\left(\frac{\pi x}{2L}\right)dx - PA - G\frac{A\pi}{2L} =$$

$$= \frac{L}{4}EIA^2\left(\frac{\pi}{2L}\right)^4 - PA - G\frac{A\pi}{2L}$$

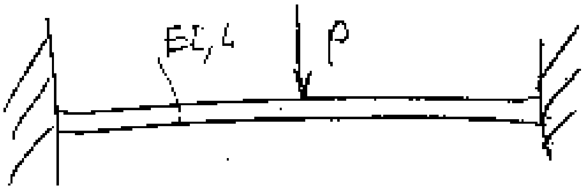
$$\frac{\partial W}{\partial A} = 0 = \frac{2AEI}{4}\left(\frac{\pi}{2L}\right)^4 - P - \frac{G\pi}{2L}\ ; \quad \boxed{A = \frac{32PL^3}{\pi^4 EI}\left(1 + \frac{a\pi}{2L}\right)}$$

NB: $M(L) = EIy''(L) = 0$ — Natural b.c. not satisfied.

Cf. $y(L)$ (a) Polynomial (exact) $\frac{1}{EI}\{0.333\,PL^3 + 0.5\,PaL^2\}$

(b) Cosine $\frac{1}{EI}\{0.323\,PL^3 + 0.516\,PaL^2\}$

②.



Let $y = A\left(1-\cos\frac{2\pi x}{L}\right)$

$$y' = \left(\frac{2\pi}{L}\right) A \sin\left(\frac{2\pi x}{L}\right)$$

$$y'' = \left(\frac{2\pi}{L}\right)^2 A \cos\frac{2\pi x}{L} \qquad y'''' = -\left(\frac{2\pi}{L}\right)^4 A \cos\frac{2\pi x}{L}$$

$$W = \frac{1}{2} EI \left(\frac{2\pi}{L}\right)^4 A^2 \int_0^L \cos^2\left(\frac{2\pi x}{L}\right) dx - PA(1-\cos\pi) =$$

$$= \frac{4\pi^4 A^2}{L^3} EI - 2PA$$

$$\frac{\partial W}{\partial A} = \frac{8\pi^4 A EI}{L^3} - 2P = 0 \quad \mapsto \quad \boxed{A = \frac{PL^3}{4\pi^4 EI}}$$

$$y(L/2) = 2A = \frac{PL^3}{2\pi^4 EI}$$

$$M(L/2) = EI\, y''(L/2) = \frac{PL}{\pi^2}$$

NB: GALERKIN gives the same result, since EULER is:

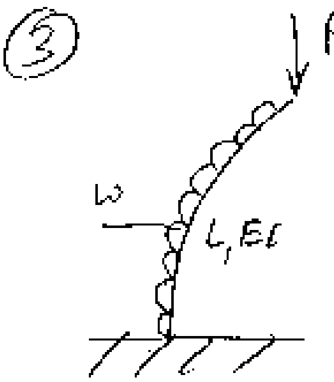
$$EI y'''' - P\delta(x-L/2) = 0$$

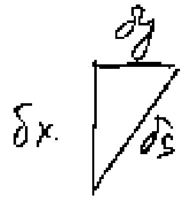
GAL: $\int_0^L \left[EI y'''' - P\delta(x-L/2)\right]\left(1-\cos\frac{2\pi x}{L}\right) dx = 0 =$

$$= -\left(\frac{2\pi}{L}\right)^4 EI \int_0^L \cos\frac{2\pi x}{L}\left(1-\cos\frac{2\pi x}{L}\right) dx - P(1-\cos\pi)$$

$\mapsto A = \frac{PL^3}{4\pi^4 EI}$ again.

③





$$\delta s^2 = \delta x^2 + \delta y^2$$

$$\delta s = dx\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

$$\Delta = \int_0^L ds - L = \int_0^L \left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] dx - L = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

$$W = \frac{1}{2} EI \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{1}{2} P \int_0^L \left(\frac{dy}{dx}\right)^2 dx - \int_0^L w y \, dx$$

(a) Take $y = A\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)$

$$y' = A \frac{\pi}{2L} \sin \frac{\pi x}{2L}$$

$$y'' = A \left(\frac{\pi}{2L}\right)^2 \cos \frac{\pi x}{2L}$$

$$W = \frac{1}{2} EI \int_0^L A^2 \left(\frac{\pi}{2L}\right)^2 \cos^2 \frac{\pi x}{2L} \, dx - \frac{1}{2} P \int_0^L A^2 \left(\frac{\pi}{2L}\right)^2 \sin^2 \frac{\pi x}{2L} \, dx$$
$$- w \int_0^L A\left(1 - \cos \frac{\pi x}{2L}\right) dx$$

$$W = \frac{EI}{64L^3} A^2 \pi^4 - \frac{A^2 \pi^2 P}{16L} - \frac{2AwL}{\pi}$$

$$\frac{\partial W}{\partial A} = \frac{EI\pi^4}{32L^3} A - \frac{\pi^2 P}{8L} A - \frac{2wL}{\pi} = 0$$

$$A = \frac{2wL}{\pi} \Big/ \left[\frac{EI\pi^4}{32L^3} - \frac{\pi^2 P}{8L}\right]$$

(b) $y = A + Bx + Cx^2 + Dx^3$

B.C's: $x = 0,\ y = y' = 0 \Rightarrow A = B = 0$

$$y = Cx^2 + Dx^3$$
$$y' = 2Cx + 3Dx^2$$
$$y'' = 2C + 6Dx$$

$$W = 2EIL\left[C^2 + 3CDL + 3D^2L^2\right] - \frac{P}{2}\left[\frac{4}{3}C^2L^3 + 3CDL^4 + \frac{9}{5}D^2L^5\right] - w\left(\frac{CL^3}{3} + \frac{DL^4}{4}\right)$$

Eq. A: $$\frac{\partial W}{\partial C} = 0 = 4EILC + 6EIDL^2 - \frac{4}{3}PCL^3 - \frac{3}{2}PDL^4 - \frac{w}{3}L^3$$

Eq. B: $$\frac{\partial W}{\partial D} = 0 = 2EIL\left(3CL + 6DL^2\right) - \frac{P}{2}\left(3CL^4 + \frac{18}{5}DL^5\right) - \frac{wL^4}{4}$$

From Eq. A $\rightarrow$ $$C = \frac{\frac{wL^3}{3} + D\left[\frac{3}{2}PL^4 - 6EIL^2\right]}{4EIL - \frac{4}{3}PL^3}$$

From Eq. B $\rightarrow$ $$2EIL\left[\frac{3L\left(\frac{wL^3}{3} + D\left(\frac{3}{2}PL^4 - 6EIL^2\right)\right)}{4EIL - \frac{4}{3}PL^3} + 6DL^2\right] - \frac{P}{2}\left[\frac{3L^4\left(\frac{wL^3}{3} + D\left(\frac{3}{2}PL^4 - 6EIL^2\right)\right)}{4EIL - \frac{4}{3}PL^3}\right] + \frac{18}{5}DL^5 - \frac{wL^4}{4} = 0$$

Gives C, D

(c) Use natural b.c.:

$$M(L) = EIy''(L) = 2C + 6DL = 0\ ;\ D = -\frac{C}{3L}$$

$$W = 2EIC^2L\left[1 + 3L\left(-\frac{1}{3L}\right) + \frac{3L^2}{9L^2}\right] - \frac{PC^2}{2}\left[\frac{4}{3}L^3 - \frac{3L^4}{3L} + \frac{9L^5}{5(3L)^2}\right] -$$

$$- wCL^3\left(\frac{1}{3} - \frac{1}{12}\right) = \frac{2EILC^2}{3} - \frac{4}{15}PC^2L^3 - \frac{wCL^3}{4}$$

$$\frac{\partial W}{\partial C} = 0 = \frac{4EICL}{3} - \frac{8}{15}PCL^3 - \frac{wL^3}{4}\ ;$$

$$\underline{\underline{C = \frac{wL^3}{4}\Big/\left[\frac{4EIL}{3} - \frac{8}{15}PL^3\right]}}$$

(4)

$$W = \frac{1}{2} D \iint \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 dx\,dy - w(\xi, \eta) P$$

$$w = \sum \sum a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$w_{xx} = -\sum \sum a_{mn} \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$w_{yy} = -\sum \sum a_{mn} \left(\frac{n\pi}{b} \right)^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

<u>Rayleigh-Ritz:</u>

$$\frac{\partial W}{\partial a_{ij}} = 0 = D \iint (w_{xx} + w_{yy}) \left(\frac{\partial w_{xx}}{\partial a_{ij}} + \frac{\partial w_{yy}}{\partial a_{ij}} \right) dx\,dy - P \frac{\partial w(\xi, \eta)}{\partial a_{ij}}$$

$$0 = D \iint \left[\sum \sum a_{mn} \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \sum \sum a_{mn} \left(\frac{\pi n}{b} \right)^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right] \times$$

$$\times \left[\left(\frac{i\pi}{a} \right)^2 \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} + \left(\frac{j\pi}{b} \right)^2 \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \right] dx\,dy -$$

$$- P \sin \frac{i\pi \xi}{a} \sin \frac{j\pi \eta}{b} = 0$$

Due to orthogonality:

$$0 = D \iint a_{ij} \sin^2 \frac{i\pi x}{a} \sin^2 \frac{j\pi y}{b} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right]^2 dx\,dy - P \sin \frac{i\pi \xi}{a} \sin \frac{j\pi \eta}{b}$$

If $P = 1$, then we find

$$\boxed{a_{mn} = \frac{4 \sin \frac{m\pi \xi}{a} \sin \frac{n\pi \eta}{b}}{D a b \pi^4 \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^2}}$$

(5)

$$W = \int_0^r \frac{D}{2} 2\pi \left[w_{rr}^2 + \frac{1}{r^2} w_r^2 + \frac{2\nu}{r} w_r w_{rr} \right] r\,dr - \int_0^a 2\pi r p_0 w\,dr$$

Given that $D = D_0 (1 - r/2a)$

and using $w = w_0 (1 - (r/a)^2)$

$w' = w_r = -2 w_0 r / a^2$

$w_{rr} = -2 w_0 / a^2$

$$W = \pi \int_0^a D_0 \left(1 - \frac{r}{2a}\right)\left(\frac{4w_0^2}{a^4} + \frac{4w_0^2}{a^4} + \frac{8\nu w_0^2}{a^4}\right) r\,dr - 2\pi p_0 w_0 \int_0^a \left[1 - \left(\frac{r}{a}\right)^2\right] r\,dr$$

$$W = \frac{8}{3} \frac{\pi D_0 w_0^2}{a^2} (1+\nu) - \frac{1}{2} \pi p_0 w_0 a^2$$

$$\frac{\partial W}{\partial w_0} = \frac{16}{3} \frac{\pi D_0 w_0}{a^2} (1+\nu) - \frac{1}{2} \pi p_0 a^2 = 0$$

$$\Rightarrow \boxed{w_0 = \frac{3}{32} \frac{p_0 a^4}{D_0}}$$

(6) Equilibrium: $\sigma_{xx,x} + \tau_{xy,y} + \rho_x = 0 \quad (x)$

$$\sigma_{yy,y} + \tau_{xy,x} + \rho_y = 0 \quad (y)$$

Compatibility: $\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$

Hooke:
$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$$
$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$
$$\varepsilon_{xy} = \frac{1}{2G}\tau_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

From equil. by diff:
$$\sigma_{xx,xx} + \tau_{xy,xy} + \rho_{x,x} = 0$$
$$\sigma_{yy,yy} + \tau_{xy,xy} + \rho_{y,y} = 0$$

Into compat: $\frac{1}{E}\left[\sigma_{xx,yy} + \sigma_{yy,xx} - \nu\sigma_{yy,yy} - \nu\sigma_{xx,xx}\right] +$

$$+ \frac{1+\nu}{E}\left[\rho_{x,x} + \rho_{y,y} + \sigma_{xx,xx} + \sigma_{yy,yy}\right] = 0$$

$$\frac{1}{1+\nu}\left[\sigma_{xx,yy} + \sigma_{yy,xx} - \nu(\sigma_{xx,xx} + \sigma_{yy,yy}) + (1+\nu)(\sigma_{xx,xx} + \sigma_{yy,yy}\right) = -\rho_{x,x} - \rho_{y,y}$$

Hence $\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)(\rho_{x,x} + \rho_{y,y})$.

(7) Postulate displacement field:

(i) $u = \frac{\delta}{H} \nu \frac{r}{R} f(z/H)$, $v = 0$, $w = -\frac{\delta}{H} z$

Let $z/H = s$; $f(s)$ – barrelling function ; $e = \frac{H}{R}$

$u = \frac{e \nu \delta r}{H^2} f(s)$; $v = 0$; $w = -\frac{\delta}{H} z$; $\Rightarrow$ need $f(\pm 1) = 0$

(ii) $\varepsilon_{rr} = \frac{\partial u}{\partial r} = \frac{e \nu \delta}{H^2} f(s)$; $\varepsilon_{\theta\theta} = \frac{u}{r} = \varepsilon_{rr}$; $\varepsilon_{zz} = -\frac{\delta}{H}$; $\varepsilon_{rz} = \frac{\delta \nu e r f'(s)}{2H^3}$

Note also ε_{rz} is non-zero $\Rightarrow$ shear at platens.

(iii) $\sigma_{rr} = -\frac{\delta \nu (H - e f(s)) E}{H^2 (1+\nu)(1-2\nu)}$; $\sigma_{zz} = -\frac{\delta (H - H\nu - 2e\nu^2 f(s)) E}{H^2 (1+\nu)(1-2\nu)}$

Traction at top/bottom surface : note $f(\pm 1) = 0$!

$\sigma_{zz}\big|_{z=\pm H} = -\frac{\delta (1-\nu) E R^2}{H (1+\nu)(1-2\nu)}$; $W_{ext} = \frac{2\delta^2 (1-\nu) E H}{e^2 (1+\nu)(1-2\nu)}$

(iv) Use strain energy : soln based on KA disp field!

Need Mathematica to perform calcs! see cyl2.nb