The Somigliana ring dislocation revisited

1. Papkovich potential solutions for dislocations in an infinite space

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1. Introduction

The applications of potential theory in solid mechanics often rely on the concept of *nuclei of strain*. This term was introduced by Mindlin [1] to denote fundamental singular solutions of the Lamé equations, such as those for a concentrated force or couple, a dislocation or a dislocation loop (dipole). Within the framework of elasticity these entities play a rôle similar to that of point charges, dipoles and multipoles in electrostatics. The corresponding elastic fields then represent single layer potentials, double layer potentials, etc., and the stresses and displacements resulting from a continuous distribution of strain nuclei are given by an integral expression. If the problem is reversed and the strain nuclei *distribution* is regarded as an unknown, but the stresses are given, an integral equation formulation of elastic problems is obtained.

Distributions of certain types of strain nuclei, in particular, dislocations in plane problems, have found wide application in elastic and plastic analysis. In fracture mechanics, for example, dislocation distributions have been used with great success to model cracks (e.g.[2]). Also, in modelling inclusions and inhomogeneities [3], the jump in plastic distortion (which is related to eigenstrain) across the boundary of the domain may be expressed in terms of the surface dislocation density [4], enabling the stress state in the matrix to be computed.

The technique has been extended to include some geometries other than plane. Problems possessing axial symmetry are an obvious next choice, since they are essentially two dimensional. Examples include that of an annular crack [5], a cylindrical crack [6], and general axisymmetric crack problems [7]. In all cases the most important requirement is that the fundamental solution for the corresponding strain nucleus must be found in a concise and tractable form.

The solution to the problem of circular prismatic *Volterra* dislocation loop was given by Kroupa [8]. Salamon [9, 10, 11] considered the cases of shear and prismatic loops in a two-phase material. The problem of the circular *Somigliana dislocation* has been addressed in great detail in an excellent paper by Demir et al [12]. The solutions were sought in the form of the (biharmonic) Love stress function, and the Fourier transform with respect to the axial coordinate was employed. The resulting stress fields were expressed in terms of complete elliptic integrals of the first, second and third kind. The final solution was obtained using a superposition of an 'innhomogeneous' field (i.e. one containing a jump in the radial or axial component of displacement) over a 'homogeneous' Lamé solution, corresponding to radial compression/tension of an infinite cylinder and an infinite elastic body containing a cylindrical hole. In this paper the following developments of the solution are sought:

- The boundary conditions are imposed in terms of the displacement discontinuities, in accordance with the definition of the Somigliana dislocation [2].

- It is desirable to obtain concise and explicit expressions for the unknown potential functions before proceeding to determine the stress and displacement fields. This requirement is important; otherwise the final form of the solution precludes further modifications.

- It is desirable to extend the result to include Somigliana ring dislocations in an elastic half space or bonded dissimilar half spaces. This task may only be accomplished provided a suitable form of the displacement functions is found. This result would widely increase possible applications of the solution in hand.

In this paper it is shown that the solutions may be found in terms of a single harmonic function, related to the Papkovich-Neuber potentials. In each case this function is given by a Lipschitz-Hankel type integral potential. These potentials are, of course, expressible in terms of the complete elliptic integrals of the first, second and third type, and thus the original solution of Demir et al[12] may be recovered. However, an important improvement may be seen in the fact that the asymptotic behaviour of these potentials has been analysed in much detail [7, 13, 14, 15], which facilitates accurate analysis of the corresponding elastic fields. Moreover, it is shown how the resulting solution may be readily generalised to include the case of an elastic half space or two perfectly bonded dissimilar half spaces, with the interface parallel to the plane of the dislocation. These results open the way for analysing the interaction between a Somigliana ring dislocation and a free surface or an interface with a different material, and also provide the kernels for integral equation formulations of axisymmetric crack problems.

2. Potential functions

Papkovich [16] introduced a complete, general solution to the equations of the theory of elasticity in terms of four *harmonic* potential functions: the scalar potential ψ and a vector potential $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$. Displacements and stresses due to these potentials are given by

$$2\mu u_i = (\kappa+1)\phi_i - (x_j\phi_j + \psi)_{,i} \tag{1}$$
$$\sigma_{ij} = \frac{1}{2}(\kappa-1)(\phi_{j,i}+\phi_{i,j}) + \frac{1}{2}(3-\kappa)\phi_{k,k}\delta_{ij} - x_k\phi_{k,ij} - \psi_{,ij}, \tag{2}$$

where a comma preceding an index denotes differentiation with respect to the relevant coordinate, $\kappa = 3-4\nu$, ν is the Poisson's ratio, and μ is the shear modulus.

The four functions ψ and ϕ_j are not defined uniquely. Sternberg [17] has shown that for an arbitrary three dimensional convex domain it is possible to reduce the number of unknown functions to three.

In problems possessing torsionless axial symmetry a further reduction is possible. Consider a cylindrical system of coordinates r, z (Fig.1). In searching for a solution it suffices to allow only the scalar potential $\psi(r,z)$ and the axial component of the vector potential $\boldsymbol{\phi}(r,z) = \phi(r,z)\,\mathbf{e}_z$ to be non-zero (where $\mathbf{e}_z$ is the unit vector in the axial direction, and the scalar function ϕ has now been relieved of all indices to simplify notation). The resulting expressions for displacements and stresses are [18]

$$2\mu u_r = -z\phi_{,r} - \psi_{,r}, \tag{3}$$
$$2\mu u_z = \kappa\phi - z\phi_{,z} - \psi_{,z}, \tag{4}$$
$$2\mu u_\theta = 0.$$

$$\sigma_{rr} = \frac{1}{2}(3-\kappa)\phi_{,z} - z\phi_{,rr} - \psi_{,rr} \tag{5}$$
$$= \frac{1}{2}(3-\kappa)\phi_{,z} + z\phi_{,zz} + \frac{z}{r}\phi_{,r} + \psi_{,zz} + \frac{1}{r}\psi_{,r}, \tag{6}$$
$$\sigma_{zz} = \frac{1}{2}(\kappa+1)\phi_{,z} - z\phi_{,zz} - \psi_{,zz}, \tag{7}$$
$$\sigma_{rz} = \frac{1}{2}(\kappa-1)\phi_{,r} - z\phi_{,rz} - \psi_{,rz}, \tag{8}$$
$$\sigma_{\theta\theta} = \frac{1}{2}(3-\kappa)\phi_{,z}. \tag{9}$$

Here both the scalar functions are harmonic, i.e. satisfy

$$\nabla^2\psi = 0, \qquad \nabla^2\phi = 0. \tag{10}$$

3. The Fourier transform and associated functions

It is our intention now to seek potential functions for the two types of dislocation, the edge (radial Burgers' vector b_r) and the glide (axial Burgers' vector b_z), in the form of Fourier transforms of unknown 'kernel' functions, to be determined in the course of solution. This approach is similar to that of Demir et al [12].

Let each of the Papkovich-Neuber potentials be sought in the form of a Fourier transform

$$\psi(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}(r,\xi)\exp(-i\xi z)d\xi, \qquad (11)$$

$$\phi(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(r,\xi)\exp(-i\xi z)d\xi. \qquad (12)$$

Since these potential functions are harmonic,

$$\phi_{,zz} + \frac{1}{r}\phi_{,r} + \phi_{,rr} = 0, \qquad (13)$$

and the same equation holds for ψ. Their dependence on the radial coordinate r may now be determined from the modified Bessel equation of zero order (see e.g. [19])

$$\hat{\phi}_{,rr} + \frac{1}{r}\hat{\phi}_{,r} - \xi^2\hat{\phi} = 0 \qquad (14)$$

which has solutions $F(\xi)I_o(\xi r)$ and $F(\xi)K_o(\xi r)$, bounded at the origin and infinity respectively.

Consider a dislocation line which is a circle of radius a in the plane $z = 0$, depicted in Fig.1. Let the inside of the right circular cylinder, $r < a$, be called domain 1, and the outside, $r > a$, be called domain 2. We may now record the unknown potentials $\psi^{1,2}(r,z)$ and $\phi^{1,2}(r,z)$, where the superscript denotes the corresponding domain, in the following form

$$\phi^1(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\xi)I_o(\xi r)\exp(-i\xi z)d\xi, \qquad (15)$$

$$\psi^1(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\xi)I_o(\xi r)\exp(-i\xi z)d\xi, \qquad (16)$$

$$\phi^2(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\xi)K_o(\xi r)\exp(-i\xi z)d\xi, \qquad (17)$$

$$\psi^2(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\xi)K_o(\xi r)\exp(-i\xi z)d\xi, \qquad (18)$$

thereby reducing our task to that of determining four 'kernel' functions of the transform variable ξ.

Note the following important properties of the modified Bessel functions

$$\frac{d}{da} I_o(\xi a) = \xi I_1(\xi a), \qquad \frac{d}{da} K_o(\xi a) = -\xi K_1(\xi a), \tag{19}$$

and

$$I_o(\xi a) K_1(\xi a) + I_1(\xi a) K_o(\xi a) = \frac{1}{\xi a} \tag{20}$$

Integrals containing pairs of Bessel functions, exponentials (or trigonometric functions) and powers, which are known as the Lipschitz-Hankel potentials, have been considered in great detail [7, 10, 13] due to their frequent occurrence in continuum mechanics problems, those with axial symmetry in particular. We will be concerned with their asymptotic properties in later sections dedicated to the analysis of the stress and displacement fields. Here we note some of their properties relevant to the derivation of the fundamental solutions. Eason et al [13] have established the following identities

$J(\mu, \nu; \lambda) =$

$$\int_0^\infty J_\mu(\xi) J_\nu(\rho\xi) e^{-\zeta\xi} \xi^\lambda d\xi = \frac{2}{\pi} \int_0^\infty K_\mu(\eta) I_\nu(\rho\eta) \cos\{\zeta\eta + \frac{1}{2}(\mu - \nu + \lambda)\pi\} \eta^\lambda d\eta \tag{21}$$

$$\int_0^\infty J_\mu(at) J_\nu(bt) e^{-ct} t^\lambda dt = \frac{2}{\pi} \int_0^\infty K_\mu(as) I_\nu(bs) \cos\{cs + \frac{1}{2}(\mu - \nu + \lambda)\pi\} s^\lambda ds \tag{22}$$

Here $J(\mu, \nu; \lambda)$ denotes the canonical form of the Lipschitz-Hankel integral; it may be obtained from the second equation, which is a more general expression, by introducing the normalisation

$$\eta = as, \ \ \xi = at, \qquad \rho = \frac{b}{a}, \ \ \zeta = \frac{c}{a}.$$

These results will be useful in further analysis. The integrals $J_{\mu\nu\lambda}$ may be evaluated in terms of the complete elliptic integrals of the first, second and third kinds. The expression for the Lipschitz-Hankel integrals needed in this paper are given in the Appendix.

We now return to the problem of determining the unknown 'kernel' functions $F_1(\xi)$, $G_1(\xi)$, $F_2(\xi)$, $G_2(\xi)$. These will be found using the problem's boundary conditions, which are prescribed over the cylindrical surface $r = a$ and consist of two parts. The *stress continuity* conditions

$$\sigma_{rz}^2 - \sigma_{rz}^1 = 0, \ \ -\infty < z < \infty \tag{23}$$

$$\sigma_{rr}^2 - \sigma_{rr}^1 = 0, \ \ |z| > 0. \tag{24}$$

are common for both dislocation types (edge and glide). The nature of the *displacement discontinuity* conditions depends on the nature of the dislocation. These will be treated separately in the following sections.

4. Axial glide Somigliana ring dislocation

In this case the displacement boundary conditions require

$$2\mu(u_z^2 - u_z^1) \; = \; 2\mu H(z), \;\; -\infty < z < \infty \tag{25}$$
$$2\mu(u_r^2 - u_r^1) \; = \; 0, \;\; -\infty < z < \infty \tag{26}$$

where $H(z)$ is the Heaviside step function. Equation (26) leads to

$$\int_{-\infty}^{\infty} \exp(-i\xi z) \left\{ \left[\xi G_2(\xi) + z\xi F_2(\xi)\right] K_1(\xi a) + \left[\xi G_1(\xi) + z\xi F_1(\xi)\right] I_1(\xi a) \right\} d\xi = 0,$$

which may only be satisfied provided

$$\begin{array}{l} F_1(\xi) = -f(\xi)K_1(\xi a), \;\; F_2(\xi) = f(\xi)I_1(\xi a), \\ G_1(\xi) = -g(\xi)K_1(\xi a), \;\; G_2(\xi) = g(\xi)I_1(\xi a). \end{array} \tag{27}$$

The boundary condition (25) for the displacement component u_z yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\xi z) \; \times$$
$$\left\{ \left(i\xi g(\xi) + z\, i\xi f(\xi) + kf(\xi)\, \right)\left[I_o(\xi a)K_1(\xi a) + I_1(\xi a)K_o(\xi a)\right] \right\} d\xi = 2\mu H(z).$$

Making use of the relationship (20) we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ ig(\xi) + izf(\xi) + k\xi^{-1}f(\xi) \right\} \exp(-i\xi z) d\xi = 2\mu a H(z). \tag{28}$$

The following results (see e.g. [20]) concerning Fourier transforms of generalised functions will be needed:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\xi z) d\xi \; = \; \delta(z), \tag{29}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \xi^{-1}\exp(-i\xi z) d\xi \; = \; -iH(z). \tag{30}$$

Also note that (e.g. [21])

$$z\delta(z) \; = \; 0, \tag{31}$$
$$z\delta'(z) \; = \; -\delta(z). \tag{32}$$

In order to satisfy equation (28) we set $g(\xi) = g^* \, \xi^{-1}$, $f(\xi) = if^*$, where g^* and f^* are constants. Then

$$(g^* + kf^*)\delta(z) - f^* z\delta(z) = 2\mu a\delta(z),$$

so that with (31)

$$g^* + kf^* = 2\mu a. \tag{33}$$

Boundary condition (24) for the stress component σ_{rr} becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\xi z)\left\{-g^* - z\, i\xi f^* + \frac{(3-\kappa)}{2} f^*\right\} d\xi = 0,$$

and leads to the result

$$-g^*\delta(z) + f^* z\delta'(z) + \frac{(3-\kappa)}{2} f^*\delta(z) = 0,$$

so that using (32)

$$g^* = -\frac{(\kappa - 1)}{2} f^*. \tag{34}$$

Together with the equation (33) this leads to

$$f^* = \frac{4\mu a}{(\kappa+1)}, \tag{35}$$

$$g^* = -\frac{2\mu a(\kappa - 1)}{(\kappa+1)}. \tag{36}$$

Thus the unknown Papkovich potentials ϕ and ψ are found in the form

$$\phi^1(r,z) = -\frac{4\mu a}{2\pi(\kappa+1)}\int_{-\infty}^{\infty} iK_1(\xi a)I_o(\xi r)\exp(-i\xi z)d\xi, \tag{37}$$

$$\psi^1(r,z) = \frac{2\mu a(\kappa-1)}{2\pi(\kappa+1)}\int_{-\infty}^{\infty} \xi^{-1}K_1(\xi a)I_o(\xi r)\exp(-i\xi z)d\xi, \tag{38}$$

$$\phi^2(r,z) = \frac{4\mu a}{2\pi(\kappa+1)}\int_{-\infty}^{\infty} iI_1(\xi a)K_o(\xi r)\exp(-i\xi z)d\xi, \tag{39}$$

$$\psi^2(r,z) = -\frac{2\mu a(\kappa-1)}{2\pi(\kappa+1)}\int_{-\infty}^{\infty} \xi^{-1}I_o(\xi a)K_o(\xi r)\exp(-i\xi z)d\xi. \tag{40}$$

The remaining boundary condition (23) for the stress component σ_{rz} is also enforced, as may be verified by considering the Fourier integral expressions for $\sigma^2_{rz} - \sigma^1_{rz}$. All terms contain the factor

$$[I_1(\xi r)K_1(\xi a) - I_1(\xi a)K_1(\xi r)]$$

which vanishes at $r = a$, as required.

With the use of equation (21) the Papkovich potentials may be rewritten in the form (for $z \geq 0$)

$$\psi = \frac{(\kappa-1)}{2}\frac{2\mu}{(\kappa+1)}J(1,0;-1) \tag{41}$$

$$\phi = -\frac{2\mu a}{(\kappa+1)}J(1,0;0) \tag{42}$$

Here the parameters appearing in the definition of the Lipschitz-Hankel integral (21) assume the values

$$\rho = \frac{r}{a} \qquad \zeta = \frac{z}{a}$$

and represent the radial and axial coordinates of the observation point normalised with respect to the radius of the dislocation ring.

Note that these expressions are valid both in domain 1 and domain 2 (this follows from the analysis involving the transformation $\rho \to 1/\rho;\ \zeta \to \zeta/\rho$, [7]).

One final observation should be made here. Strictly speaking, in order for the integral expression in the definition of $J(\mu,\nu;\lambda)$ to converge, ζ must be non-negative. As in practice both positive and negative values of z (and therefore ζ) are required, and the resulting functions must display certain odd/even properties in z, the following notation has been introduced [7]:

$$J_{\mu\nu\lambda} = [\mathrm{sign}(\zeta)]^{(\mu+\nu+\lambda)}\int_0^\infty J_\mu(\xi\rho)J_\nu(\xi)e^{-\xi|\zeta|}\,\xi^\lambda d\xi, \tag{43}$$

which leads to the necessary behaviour.

Note that the found Papkovich-Neuber potentials may be expressed in terms of a single harmonic function,

$$\psi(r,z) = \frac{(\kappa-1)}{2}\Omega, \tag{44}$$

$$\phi(r,z) = \Omega_{,z}, \tag{45}$$

where

$$\Omega(r,z) = \frac{2\mu a}{(\kappa+1)}J_{10;-1} \tag{46}$$

5. Radial edge Somigliana ring dislocation

In this case the displacement boundary conditions require

$$2\mu(u_z^2 - u_z^1) \;=\; 0, \;\; -\infty < z < \infty \tag{47}$$
$$2\mu(u_r^2 - u_r^1) \;=\; 2\mu H(z), \;\; -\infty < z < \infty \tag{48}$$

Equation (47) leads to

$$\int_{-\infty}^{\infty} \exp(-i\xi z) \;\times \{[-i\xi G_2(\xi) - zi\xi F_2(\xi) + \kappa F_2(\xi)]\, K_o(\xi a) - [-i\xi G_1(\xi) - zi\xi F_1(\xi) + \kappa F_1(\xi)]\, I_o(\xi a)\}\, d\xi = 0,$$

which may only be satisfied provided

$$\begin{array}{l} F_1(\xi) = f(\xi)K_o(\xi a), \;\; F_2(\xi) = f(\xi)I_o(\xi a), \\ G_1(\xi) = g(\xi)K_o(\xi a), \;\; G_2(\xi) = g(\xi)I_o(\xi a). \end{array} \tag{49}$$

The boundary condition (48) for the displacement component u_r yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\xi z) \;\times \{\xi g(\xi)[I_o(\xi a)K_1(\xi a) + I_1(\xi a)K_o(\xi a)] + z\xi f(\xi)[I_o(\xi a)K_1(\xi a) + I_1(\xi a)K_o(\xi a)]\}\, d\xi = 2\mu H(z).$$

Again using the relationship (20) we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \{g(\xi) + z f(\xi)\} \exp(-i\xi z) d\xi = 2\mu a H(z) \tag{50}$$

In order to satisfy equation (50) we set $g(\xi) = ig^*\,\xi^{-1}$, $f(\xi) = f^*$, where g^* and f^* are constants. Once againg using (31) we obtain

$$g^* = 2\mu a. \tag{51}$$

Boundary condition (23) for the stress component σ_{rz} yields, with the use of (32)

$$g^* = -\frac{(\kappa+1)}{2}\, f^*. \tag{52}$$

Together with the equation (51) this leads to

$$f^* \;=\; \frac{4\mu a}{(\kappa+1)}, \tag{53}$$
$$g^* \;=\; 2\mu a. \tag{54}$$

Substituting these values into the final boundary condition (24) gives an identity.

Using a procedure entirely similar to that presented in the previous section, the final form of the Papkovich potentials follows

$$\psi(r,z) = \frac{(\kappa+1)}{2}\,\Omega, \tag{55}$$
$$\phi(r,z) = \Omega_{,z}, \tag{56}$$

where now

$$\Omega(r,z) = \frac{2\mu a}{(\kappa+1)} J_{00;-1} \tag{57}$$

6. Analysis of the elastic fields

In this section an analysis of the components of stress and displacement in the vicinity of axial glide and radial edge Somigliana ring dislocations is given. Formulae for the displacements and stresses in terms of the Lipschitz-Hankel potentials are first derived using equations (3)-(9). The following basic differentiation formulae are used [14]

$$\frac{\partial}{\partial z} J_{\mu\nu\lambda} = -a^{-1} J_{\mu\,\nu;(\lambda+1)} \tag{58}$$
$$\frac{\partial}{\partial r} J_{\mu\nu\lambda} = -a^{-1} J_{\mu\,(\nu+1);(\lambda+1)} \tag{59}$$

The resulting stress expressions are then expanded in series in powers of δ, where $\delta^2 = \zeta^2 + (\rho-1)^2$, $\rho = r/a$, $\zeta = z/a$, and β denotes the angle between the line connecting the source point and the observation point, and plane of the ring dislocation (Fig.2). The expansions are based on the asymptotic properties of the Lipschitz-Hankel potentials, [13, 7, 14, 15]. Ellipsis in the formulae denotes the terms which are regular or may contain singularities not stronger than logarithmic in δ. Note that no singularity is present in the displacement terms.

The magnitude of the Burgers vector (b_z for the axial case and b_r for the radial case) is assumed unity.

Axial glide dislocation

$$u_z = -\frac{1}{(\kappa+1)}\left[\frac{(\kappa+1)}{2}J_{100} + \zeta J_{101}\right] \tag{60}$$
$$u_r = \frac{1}{(\kappa+1)}\left[\frac{(\kappa-1)}{2}J_{110} - \zeta J_{111}\right] \tag{61}$$

$$\sigma_{zz} = \frac{2\mu}{(\kappa+1)a}\left[J_{101} + \zeta J_{102}\right] \tag{62}$$

$$\simeq -\frac{2\mu}{(\kappa+1)\pi a}\frac{\cos\beta(1+2\sin^2\beta)}{\delta}+\ldots \tag{63}$$

$$\sigma_{rr} = \frac{2\mu}{(\kappa+1)a}\left[J_{101}-\zeta J_{102}+\frac{\zeta}{\rho}J_{111}-\frac{(\kappa-1)}{2\rho}J_{110}\right] \tag{64}$$

$$\simeq \frac{2\mu}{(\kappa+1)\pi a}\frac{\cos\beta\cos 2\beta}{\delta}+\ldots \tag{65}$$

$$\sigma_{rz} = \frac{2\mu}{(\kappa+1)a}\left[\zeta J_{112}\right] \tag{66}$$

$$\simeq -\frac{2\mu}{(\kappa+1)\pi a}\frac{\sin\beta\cos 2\beta}{\delta}+\ldots \tag{67}$$

Radial edge dislocation

$$u_z = -\frac{1}{(\kappa+1)}\left[\frac{(\kappa-1)}{2}J_{000}+\zeta J_{001}\right] \tag{68}$$

$$u_r = \frac{1}{(\kappa+1)}\left[\frac{(\kappa+1)}{2}J_{010}-\zeta J_{011}\right] \tag{69}$$

$$\sigma_{zz} = \frac{2\mu}{(\kappa+1)a}\left[\zeta J_{002}\right] \tag{70}$$

$$\simeq -\frac{2\mu}{(\kappa+1)\pi a}\frac{\sin\beta\cos 2\beta}{\delta}+\ldots \tag{71}$$

$$\sigma_{rr} = \frac{2\mu}{(\kappa+1)a}\left[2J_{001}-\zeta J_{002}+\frac{\zeta}{\rho}J_{011}-\frac{(\kappa+1)}{2\rho}J_{010}\right] \tag{72}$$

$$\simeq \frac{2\mu}{(\kappa+1)\pi a}\frac{\sin\beta(1+2\cos^2\beta)}{\delta}+\ldots \tag{73}$$

$$\sigma_{rz} = \frac{2\mu}{(\kappa+1)a}\left[-J_{011}+\zeta J_{012}\right] \tag{74}$$

$$\simeq \frac{2\mu}{(\kappa+1)\pi a}\frac{\cos\beta\cos 2\beta}{\delta}+\ldots \tag{75}$$

A number of useful observations can be made with the use of these formulae. For example, it is immediately apparent that as the radius of the ring dislocation tends to infinity, the corresponding stress fields approach those of a glide or edge dislocation (depending on the orientation) in the plane problem. This observation may be interpreted in a different manner: as the distance between the source and observation points becomes very small ($\delta \to 0$), i.e. the radius of the ring becomes very large, the above expressions provide the leading terms in the expansion of stresses both in plane and axisymmetric problems.

The difference between the two cases becomes apparent only from the subsequent, less singular terms.

7. Discussion

In this paper concise and effective solutions for glide and edge Somigliana ring dislocations were derived. It is believed that both the method of derivation and the form of the final results lead to remarkable elucidation of the solution, making it easier to follow, verify and use in application to the analysis of axisymmetric problems in elasticity.

The asymptotic behaviour of the stresses in the vicinity of the dislocation ring was analysed and presented in a simple form. This allows effective comparison to be made with the limiting case of plane geometry, which is approached as the ring radius becomes large.

An interesting observation could be made here. The value of *strain nuclei* solutions in elasticity problems generally lies in the fact that they provide a basis for generating controlled states of stress/strain, which could be used satisfy the required conditions over part of the boundary (e.g. crack surface in fracture problems). However, in various problems there may also exist additional boundaries, where 'homogeneous' conditions must be maintained (e.g. stress-free surfaces or regions). The corresponding strain nuclei would be required to possess the property that they do not disturb this 'homogeneity' over the remainder of the boundary. In the specific case of ring dislocations in an infinite elastic space, considered here, there are no free surfaces present. However, in many elasticity problems it is usual to impose the requirement that stresses vanish (sometimes according to a certain law) at infinity. It is interesting to analyse our solutions in this respect.

As an example, choose the radial edge dislocation. It is fairly obvious that stresses vanish with increasing radial distance r from the axis as it becomes much greater than the ring radius a, $\rho \gg 1$. However, since in generating the edge dislocation a cut has been introduced over the cylindrical surface $r = a$ from the point $z = 0$ to infinity, one might expect to find a radial stress σ_{rr} persisting to infinitely large values of ζ.

Consider a point on $\rho = 1$ and, letting $\zeta \gg 1$, and determine the radial component of stress

$$\sigma_{rr} = \frac{2\mu}{(\kappa+1)a}\left[2J_{001} - \zeta J_{002} + \zeta J_{011} - \frac{(\kappa+1)}{2} J_{010}\right] \qquad (76)$$

Note that the result would also give a good estimate of the same stress on the axis, since stress variation is slow far away from the source, i.e. the ring dislocation itself.

In order to evaluate the above expression for large ζ and $\rho = 1$ the values of the $J-$integrals (and therefore the complete elliptic integrals) must be found for small values of the elliptic parameter $k \simeq 2/|\zeta| \ll 1$. For such k it holds that

$$\mathbf{K}(k) \simeq \frac{\pi}{2}\left\{1 + \frac{1}{\zeta^2} + \frac{9}{4\zeta^4} + \dots\right\}, \tag{77}$$

$$\mathbf{E}(k) \simeq \frac{\pi}{2}\left\{1 - \frac{1}{\zeta^2} - \frac{3}{4\zeta^4} + \dots\right\} \tag{78}$$

When these values of the elliptic integrals are substituted into the expressions for $J_{\mu\nu\lambda}$, it is found that each of the individual $J-$terms in the above expression for the radial stress component vanishes with $\zeta \to +\infty$ at least as fast as $1/\zeta^2$. Similar condition for $\zeta \to +\infty$ may be verified for the remaining stress components σ_{zz}, σ_{rz}, and $\sigma_{\theta\theta}$.

If, however, the limit $\zeta \to -\infty$ is considered, stresses persist to infinitely large values of ζ (z).

This result is different from that given in [12], where radial stress at $r = a$ for values of $z \to +\infty$ is reported to approach the value of $4\mu/(\kappa+1)a$ (equation (20) in [12], magnitude of the Burgers vector b_1 is assumed unity), and zero stresses for $z \to -\infty$. This difference with the present results may be explained as follows.

In section 3 of their paper, Demir et al [12] introduce the Somigliana ring dislocation by a 'cut and paste' exercise, whereby the two faces of a certain surface S (which is given by $r = a$, $z > 0$ in the present problem) are displaced by a given amount (the Burgers vector b_r) relative to each other. If needed, additional material may then be introduced into the resulting slit, and the whole assembly be pasted together again. Let us call the state of stress arising due to this procedure Solution 1.

Now consider an alternative procedure, whereby another surface, $S' : r = a, z < 0$, is chosen, a cut is introduced and the amount of material equal to the Burgers vector b_r, is *removed*, and the faces are again glued together. Let us call the corresponding state of stress Solution 2.

Note that the Burgers vectors are equal in both cases, as may be verified by appying the definition, i.e. considering closed paths linked with the ring of dislocation. However, the stresses are not equal in the two problems, as Lamé problem 'homogeneous' stress terms can be seen to give the difference between the two stress states considered. Note that in [12] boundary conditions for the displacements on the surface of the right circular cylinder $r = a$ are given in terms of the *derivatives* of displacement jumps (equations (10) in section 3.1). These in fact give rise to a whole family of solutions corresponding to a multitude of boundary conditions given by

$$2\mu(u_r^2 - u_r^1) = H(z) + C, \tag{79}$$

where the choice of an arbitrary constant C does not affect the value of the Burgers vector. In fact, it is fairly obvious that it is the parameter C controls the strength of the 'homogeneous' Lamé term. It also seems most natural to choose the 'fundamental' solution so that $C = 0$, as in the present formulation). Then, if needed, the entire family of solutions may be obtained by adding Lamé terms of intensity determined by C.

Depending on the nature of the problem where the fundamental solutions for ring dislocations are intended to be used, the appropriate member of the above family must be chosen. For example, if the problem of a finite cylindrical crack in an infinite elastic space is considered, any solution will do, since the crack closure condition will ensure that net dislocations (and hence resultant stresses at infinity) are zero. If a cylindrical crack in an elastic half space $z > z_0$ is considered [22], it is necessary to choose the solution where stresses vanish for $z \to +\infty$.

In summary, the approach presented in this paper provides further insight and clarification of the Somigliana ring dislocation solutions. It also opens the way for the development of the solutions for an elastic half space and for bonded dissimilar elastic half spaces, which are given in the companion paper [22].

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Appendix

A. Expressions for $J_{\mu\nu\lambda}$ in terms of elliptic functions

In the formulae below

$$J_{\mu\nu\lambda} = \{\text{sign}(\zeta)\}^{(\mu+\nu+\lambda} \int_0^\infty J_\mu(t) J_\nu(t\rho) e^{-|\zeta|t} t^\lambda dt$$

The complete elliptic integrals

$$\mathbf{E} \ = \ \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{1/2} d\phi$$

$$\begin{aligned}\mathbf{K} &= \int_0^{\pi/2} \frac{d\phi}{(1-k^2\sin^2\phi)^{1/2}} \\ \mathbf{\Pi} &= \int_0^{\pi/2} \frac{d\phi}{(1-h\sin^2\phi)(1-k^2\sin^2\phi)^{1/2}}\end{aligned}$$

where the elliptic parameters

$$k^2 = \frac{4\rho}{(1+\rho)^2+\xi^2};\; h = \frac{4\rho}{(1+\rho)^2};\; k'^2 = 1-k^2.$$

$$J_{000} = \frac{k}{\pi\rho^{1/2}}\mathbf{K} \tag{80}$$

$$J_{001} = \frac{k^3\zeta}{4\pi\rho^{3/2}k'^2}\mathbf{E} \tag{81}$$

$$J_{002} = \frac{k^5}{16\pi\rho^{5/2}k'^2}\{[\frac{2(1+k'^2)\zeta^2}{k'^2} - \frac{4\rho}{k^2}]\mathbf{E} - \zeta^2\mathbf{K}\} \tag{82}$$

$$J_{01;-1} = \begin{cases} \frac{2}{\pi k\rho^{1/2}}\mathbf{E} - \frac{(1-\rho^2)k}{2\pi\rho^{3/2}}\mathbf{K} + \frac{k^2\zeta^3}{4\pi\rho^2}\frac{1-\rho}{1+\rho}\mathbf{\Pi}, & \rho < 1; \\ \frac{2}{\pi k}\mathbf{E} - \frac{\zeta}{2\rho}; & \rho = 1, \\ \frac{2}{\pi k\rho^{1/2}}\mathbf{E} - \frac{(1-\rho^2)k}{2\pi\rho^{3/2}}\mathbf{K} - \frac{k^2\zeta^3}{4\pi\rho^2}\frac{1-\rho}{1+\rho}\mathbf{\Pi} - \frac{\zeta}{\rho}, & \rho > 1. \end{cases} \tag{83}$$

$$J_{010} = \begin{cases} -\frac{k\zeta\rho^{1/2}}{2\pi}\mathbf{K} - \frac{k^2\zeta^2}{4\pi}\frac{1-\rho}{1+\rho}\mathbf{\Pi}, & \rho < 1; \\ -\frac{k\zeta}{2\pi}\mathbf{K} + \frac{1}{2}\rho; & \rho = 1, \\ -\frac{k\zeta\rho^{1/2}}{2\pi}\mathbf{K} + \frac{k^2\zeta^2}{4\pi}\frac{1-\rho}{1+\rho}\mathbf{\Pi} + \rho, & \rho > 1. \end{cases} \tag{84}$$

$$J_{011} = \frac{k^3(\rho^2-1-\zeta^2)}{8\pi k'^2\rho^{5/2}}\mathbf{E} + \frac{k}{2\pi\rho^{3/2}}\mathbf{K} \tag{85}$$

$$J_{012} = \frac{k^3\zeta}{8\pi k'^2\rho^{5/2}}\left\{\left[\frac{k^4[\rho^4-(1+\zeta^2)^2]}{4\rho^2k'^2} + 3\right]\mathbf{E} + \frac{k^2(1+\zeta^2-\rho^2)}{4\rho}\mathbf{K}\right\} \tag{86}$$

$$J_{10;-1} = \begin{cases} \frac{2\rho^{1/2}}{\pi k}\mathbf{E} + \frac{(1-\rho^2)k}{2\pi\rho^{1/2}}\mathbf{K} + \frac{k^2\zeta^3}{4\pi\rho}\frac{1-\rho}{1+\rho}\mathbf{\Pi} - \zeta, & \rho < 1; \\ \frac{2}{\pi k}\mathbf{E} - \frac{\zeta}{2}; & \rho = 1, \\ \frac{2\rho^{1/2}}{\pi k}\mathbf{E} + \frac{(1-\rho^2)k}{2\pi\rho^{\frac{1}{2}}}\mathbf{K} - \frac{k^2\zeta^3}{4\pi\rho}\frac{1-\rho}{1+\rho}\mathbf{\Pi}, & \rho > 1. \end{cases} \tag{87}$$

$$J_{100} = \begin{cases} -\frac{k\zeta}{2\pi\rho^{1/2}}\mathbf{K} - \frac{k^2\zeta^2}{4\pi\rho}\frac{1-\rho}{1+\rho}\mathbf{\Pi} + 1, & \rho < 1; \\ -\frac{k\zeta}{2\pi}\mathbf{K} + \frac{1}{2}; & \rho = 1, \\ -\frac{k\zeta}{2\pi\rho^{1/2}}\mathbf{K} + \frac{k^2\zeta^2}{4\pi\rho}\frac{1-\rho}{1+\rho}\mathbf{\Pi}, & \rho > 1. \end{cases} \tag{88}$$

$$J_{101} = \frac{k^3(1-\rho^2-\zeta^2)}{8\pi k'^2\rho^{3/2}}\mathbf{E} + \frac{k}{2\pi\rho^{1/2}}\mathbf{K} \tag{89}$$

$$J_{102} = \frac{k^3\zeta}{8\pi k'^2\rho^{3/2}}\left\{\left[\frac{k^4[1-(\rho^2+\zeta^2)^2]}{4\rho^2k'^2}+3\right]\mathbf{E} + \frac{k^2(\rho^2+\zeta^2-1)}{4\rho}\mathbf{K}\right\} \tag{90}$$

$$J_{11;-1} = \begin{cases} \frac{\zeta}{\pi k\rho^{1/2}}\mathbf{E} - \frac{k\zeta(1+\rho^2+\zeta^2/2)}{2\pi\rho^{3/2}}\mathbf{K} + \frac{k^2\zeta^2(1-\rho)^2}{8\pi\rho^2}\mathbf{\Pi} + \frac{\rho}{2}, & \rho < 1; \\ \frac{\zeta}{\pi k}\mathbf{E} - \frac{k\zeta}{2\pi}(2+\frac{\zeta^2}{2})\mathbf{K} + \frac{1}{2}; & \rho = 1, \\ \frac{\zeta}{\pi k\rho^{1/2}}\mathbf{E} - \frac{k\zeta(1+\rho^2+\zeta^2/2)}{2\pi\rho^{3/2}}\mathbf{K} - \frac{k^2\zeta^2(1-\rho)^2}{8\pi\rho^2}\mathbf{\Pi} + \frac{\rho}{2}, & \rho > 1. \end{cases} \tag{91}$$

$$J_{110} = \frac{2}{\pi k\rho^{1/2}}\left(\frac{2-k^2}{2}\mathbf{K} - \mathbf{E}\right) \tag{92}$$

$$J_{111} = \frac{k\zeta}{2\pi\rho^{3/2}}\left(\frac{2-k^2}{2k'^2}\mathbf{E} - \mathbf{K}\right) \tag{93}$$

$$J_{112} = \frac{k}{2\pi\rho^{3/2}}\left\{\frac{k^2}{4\rho k'^2}\left[\frac{k^4\zeta^2}{k'^2} - 1 - \rho^2\right]\mathbf{E} + \left[1 - \frac{k^2\zeta^2(2-k^2)}{8\rho k'^2}\right]\mathbf{K}\right\} \tag{94}$$

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