The Somigliana ring dislocation revisited

2. Solutions for dislocations in a half space and in one of two perfectly bonded dissimilar half spaces

Alexander M. Korsunsky
Department of Materials Science and Metallurgy, University of Cambridge, Pembroke St, Cambridge, England CB2 3QZ

1. Introduction

The problem of the circular prismatic *Volterra* dislocation loop was first solved by Kroupa [1]. Solutions for the circular Somigliana radial edge dislocation and the circular Volterra glide dislocation were given by Demir *et al.* [2] in terms of the Love stress function.

In the companion paper [3] the solutions for circular dislocations were derived anew in terms of the Papkovich potential functions and the Lipschitz-Hankel integral potentials [4]. This approach (see also [5]) leads to a concise formulation of the solution, which is amenable to further treatment and modifications. In particular, in this paper it will be shown how some results concerning the Somigliana ring dislocations embedded in an elastic half space or in one of two perfectly bonded dissimilar half spaces may be obtained.

The derivation relies on some properties of the Papkovich potentials of various eigenstrains in an infinite elastic space and in bonded dissimilar half-spaces. These properties were expressed by Aderogba [6] in the form of a general theorem relating the full space and the half space solutions.

In this paper, some specific consequences of Aderogba's theorem required in the present analysis are first obtained, together with a simple transformation rule for the Papkovich potentials under translation of the coordinate frame. These results are then applied to the Papkovich potential solutions for circular Somigliana dislocations [3]. The resulting elastic fields are analysed with respect to their asymtotic behaviour at large distances from the dislocation and at large dislocation radii, and energies of circular dislocations embedded in an elastic half space are found.

2. Dislocation solutions in the infinite space

Expressed concisely, the results of [3] are as follows:

The Papkovich-Neuber potential solutions for the Somigliana ring dislocations lying on the plane $z = 0$ are given by the two harmonic functions: the scalar potential ψ and the axial component of the vector potential ϕ. These two functions are related to a single harmonic potential, given by a Lipschitz-Hankel integral, in the following way

$$\psi(r,z) = \frac{2\mu a}{(\kappa+1)}\,\eta\, J_{n0;-1}(r/a, z/a), \tag{1}$$

$$\phi(r,z) = \frac{2\mu a}{(\kappa+1)}\,\frac{\partial}{\partial z}\, J_{n0;-1}(r/a, z/a), \tag{2}$$

where

for the axial glide dislocation $n = 1$, $\eta = (\kappa - 1)/2$, and

for the radial edge dislocation $n = 0$, $\eta = (\kappa + 1)/2$.

In the above expressions μ is the shear modulus, and $\kappa = 3 - 4\nu$, where ν is the Poisson's ratio; a denotes the ring radius.

3. Preliminaries

In order for the full space solutions given in the previous section to be generalised to the case of dislocations in elastic half spaces, two results concerning the transformation rules for the Papkovich potentials must be available.

3.1. Translation of the coordinate system

The transformation rule for the Papkovich potentials under translation of the coordinate system is given by the following Lemma.

Let the state of stress in an isotropic homogeneous infinite elastic solid be characterized by the Papkovich potentials ϕ_i^o and ψ^o defined with respect to a Cartesian coordinate system (y_1, y_2, y_3). Then with respect to another system of coordinates (x_1, x_2, x_3) given by

$$\begin{aligned} y_1 &= x_1 - x_1', \\ y_2 &= x_2 - x_2', \\ y_3 &= x_3 - x_3', \end{aligned} \tag{3}$$

the Papkovich potentials have the form

$$\begin{aligned} \phi_1(x_1,x_2,x_3) &= \phi_1^o(x_1 - x_1', x_2 - x_2', x_3 - x_3'), \\ \phi_2(x_1,x_2,x_3) &= \phi_2^o(x_1 - x_1', x_2 - x_2', x_3 - x_3'), \\ \phi_3(x_1,x_2,x_3) &= \phi_3^o(x_1 - x_1', x_2 - x_2', x_3 - x_3'), \\ \psi(x_1,x_2,x_3) &= \psi^o(x_1 - x_1', x_2 - x_2', x_3 - x_3') - x_i'\,\phi_i^o(x_1 - x_1', x_2 - x_2', x_3 - x_3'). \end{aligned} \tag{4}$$

This lemma is verified by the substitution of both sides of equations (4) into equations for the displacements in terms of the Papkovich potentials ([3], equation (1)).

Applying the Lemma, the Somigliana ring dislocation solutions for a ring lying on the plane $z = z'$ are found from the results of [3]:

$$\psi(r,z) = \frac{2\mu a}{(\kappa+1)}\Big[\eta\, J_{n0;-1}(\rho,\zeta-\zeta') + \zeta'\, J_{n00}(\rho,\zeta-\zeta')\Big], \tag{5}$$

$$\phi(r,z) = -\frac{2\mu}{(\kappa+1)}\, J_{n00}(\rho,\zeta-\zeta'), \tag{6}$$

where the notation has been introduced $\rho = r/a$, $\zeta = z/a$, $\zeta' = z'/a$.

3.2. Dissimilar elastic half spaces

The second result required concerns the transformation rule for the Papkovich-Neuber potentials when a half space of the original material is replaced with elastically dissimilar medium. A general theorem concerning this operation was given by Aderogba [6].

The following notation is used. In the coordinate system (x_1, x_2, x_3), the nucleus of strain is assumed to lie within the half space $x_3 > 0$. The cylindrical coordinates (r, z, θ) are introduced so that $r = (x_1^2 + x_2^2)^{1/2}$, $z = x_3$, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. A 'reflected' coordinate system will be used as well, where the axial coordinate is y such that $y = -z = -x_3$.

The Dundurs parameters α and β are conventionally used to describe materials mismatch. They are

$$\alpha = \frac{\mu_2(\kappa_1+1) - \mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1) + \mu_1(\kappa_2+1)} \tag{7}$$

$$\beta = \frac{\mu_2(\kappa_1-1) - \mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1) + \mu_1(\kappa_2+1)} \tag{8}$$

$$\tag{9}$$

Certain combinations of material elastic constants appear in the formulation of Aderogba's theorem. Some of them may be related to the Dundurs parameters. In particular, we will need the following

$$\Gamma = \frac{\mu_2}{\mu_1}, \tag{10}$$

$$A = \frac{(\Gamma-1)}{(\Gamma\kappa_1+1)} = \frac{\alpha-\beta}{1+\beta}, \tag{11}$$

$$B = \frac{(\Gamma\kappa_1-\kappa_2)}{(\Gamma+\kappa_2)} = \frac{\alpha+\beta}{1-\beta}. \tag{12}$$

Here we specialise Aderogba's theorem to the particular case when only two of the Papkovich-Neuber potential functions, ψ and $\phi = \phi_3$ are non-zero [7]. The following corollary ensues:

Let a subregion of an isotropic homogeneous infinite elastic solid be subjected to any admissible loading condition characterized by axisymmetric Papkovich potentials ψ^o *and* ϕ^o *(i.e. such that* $\phi_1^o = \phi_2^o = 0$, $\phi^o = \phi_3^o$*), where all the singularities of these functions lie in the half space* $x_3 > 0$*. Let an elastically dissimilar material be introduced into the half space* $z = x_3 (= -y) < 0$ *so that the perfect bond continuity conditions are satisfied at the interface. Then the new displacements and stresses are generated by the following potentials:*

For material 1:

$$\begin{aligned}
\psi^{(1)} &= \psi^o(r,z) - A\kappa_1\psi^o(r,-z) - \frac{(A\kappa_1^2 - B)}{2}\int \phi^o(r,-z)dz \\
\phi^{(1)} &= \phi^o(r,z) - A\kappa_1\phi^o(r,-z) - 2A\frac{\partial}{\partial z}\psi^o(r,-z)
\end{aligned} \tag{13}$$

and for material 2:

$$\begin{aligned}
\psi^{(2)} &= (A+1)\psi^o(r,-y) - \frac{(A\kappa_1^2 - B)}{2}\Gamma\int \phi^o(r,-y)dy \\
\phi^{(2)} &= -(B+1)\phi^o(r,-y)
\end{aligned} \tag{14}$$

Now consider a particular case when material 2 is void, $\mu_2 = \nu_2 = 0$. Then the bimaterial constants assume the values $\Gamma = 0$, $A = B = -1$, and one further result is obtained:

In the conditions of Corollary 1, let material 2 be void, $\mu_2 = 0$, $\kappa_2 = 3$*. Then the new displacements and stresses in the elastic half space filled with material 1 are generated by the following potentials:*

$$\begin{aligned}
\psi^{(1)} &= \psi^o(r,z) + \kappa_1\psi^o(r,-z) + \frac{(\kappa_1^2 - 1)}{2}\int \phi^o(r,-z)dz \\
\phi^{(1)} &= \phi^o(r,z) + \kappa_1\phi^o(r,-z) + 2\frac{\partial}{\partial z}\psi^o(r,-z)
\end{aligned} \tag{15}$$

4. Dislocation solutions for half spaces

In order to specialise these result further to the problems considered in this paper, let the Papkovich potentials for a given strain nucleus in an infinite elastic half space be related to a single function $\Omega(\rho, \zeta)$ so that

$$\psi^o(\rho, \zeta) = \eta\,\Omega - \zeta'\Omega_{,\zeta}\,, \tag{16}$$
$$\phi^o(\rho, \zeta) = \Omega_{,\zeta}, \tag{17}$$

Here we use the coordinates normalised with respect to the dislocation ring radius a: $\rho = r/a$, $\zeta = z/a$, and $\zeta' = z'/a$, where z' is the axial coordinate of the plane of the dislocation. A comma in the subscript denotes differentiation, $\Omega_{,\zeta} = \partial\Omega/\partial\zeta$. From equations (1-2) and (5-6), the infinite space dislocation solutions are obtained when the function $\Omega(\rho, \zeta)$ is given by a multiple of the modified Lipschitz-Hankel integral [3, 7]

$$\Omega(\rho, \zeta) = \frac{2\mu a}{(\kappa+1)}\, J_{n0;-1}(\rho, \zeta - \zeta'). \tag{18}$$

Let us introduce an 'image' function

$$\bar{\Omega}(\rho, \zeta) = \Omega(\rho, -\zeta), \tag{19}$$

where the overbar denotes the substitution of $-\zeta$ instead of ζ in the function argument.

From the properties of the modified Lipschitz-Hankel potentials ([3], equation (43)) it follows that

$$\bar{\Omega}(\rho, \zeta) = \frac{2\mu a}{(\kappa+1)}\, \bar{J}_{n0;-1} = (-1)^{(n+1)} \frac{2\mu a}{(\kappa+1)}\, J_{n0;-1}(\rho, \zeta + \zeta'), \tag{20}$$

As a consequence of this convention the following relationship holds

$$\overline{(\Omega_{,\zeta})} = -\bar{\Omega}_{,\zeta}. \tag{21}$$

Using the notation introduced, the solution for the elastic fields of a circular dislocation embedded in one of two bonded half spaces may be recorded, so that in material 1:

$$\psi^{(1)} = \eta\Omega - \zeta'\Omega_{,\zeta} + \frac{A\kappa_1(\kappa_1 - 2\eta) - B}{2}\bar{\Omega} - A\kappa_1\zeta'\bar{\Omega}_{,\zeta}, \tag{22}$$
$$\phi^{(1)} = \Omega_{,\zeta} + A(\kappa_1 - 2\eta)\bar{\Omega}_{,\zeta} - 2A\zeta'\bar{\Omega}_{,\zeta\zeta}, \tag{23}$$

and in material 2:

$$\psi^{(2)} = (A+1)(\eta\Omega + \zeta'\Omega_{,\zeta}) + \frac{(A\kappa_1^2 - B)\Gamma}{2}\Omega, \tag{24}$$
$$\phi^{(2)} = (B+1)\Omega_{,\zeta}. \tag{25}$$

In the above expressions the values of η and Ω must be chosen accordingly with the type of dislocation considered, as in equations (1-2).

Again considering the case of a single elastic half space separately, we record the potentials in material 1 in the form

$$\psi^{(1)} = \eta\Omega - \zeta'\Omega_{,\zeta} + \frac{1-\kappa_1(\kappa_1 - 2\eta)}{2}\bar{\Omega} + \kappa_1\zeta'\bar{\Omega}_{,\zeta}, \tag{26}$$

$$\phi^{(1)} = \Omega_{,\zeta} - (\kappa_1 - 2\eta)\bar{\Omega}_{,\zeta} + 2\zeta'\bar{\Omega}_{,\zeta\zeta}. \tag{27}$$

5. Displacements and stresses due to dislocations in bonded elastic half spaces

In this section we give the expressions for stresses and displacements due to radial edge and axial glide circular Somigliana dislocations. In order to make the expressions as concise as possible, we introduce one further notation convention. We drop the arguments of Hankel-Lipschitz integral potentials to write

$$J_{npq} = J_{npq}(\rho, \zeta - \zeta'), \tag{28}$$

$$I_{npq} = J_{npq}(\rho, \zeta + \zeta') \tag{29}$$

Here I_{npq} is obtained from $J_{npq}(\rho, \zeta - \zeta')$ by replacing the *whole* combination $(\zeta - \zeta')$ with $(\zeta + \zeta')$.

Note that this operation is not identical with replacing ζ with $-\zeta$, which yields

$$\bar{J}_{npq} = J_{npq}(\rho, -(\zeta + \zeta')), \tag{30}$$

(cf. equation (20)), i.e. a result that differs by a factor of $(-1)^{(n+p+q)}$.

From the above discussion it is evident that given the values of the potential functions $\Omega(\rho,\zeta)$ and $\bar{\Omega}(\rho,\zeta)$ and of the parameter η in equations (1-2), full elastic fields may be determined. In the following sections we record the displacements and stresses arising due to the two types of dislocations.

5.1. Axial glide dislocation

Parameter $\eta = \dfrac{(\kappa_1 - 1)}{2}$.

Potential functions:

$$\Omega(\rho,\zeta) = \frac{2\mu_1 a}{(\kappa_1 + 1)}\, J_{10;-1}\,, \qquad \Omega(\rho,\zeta) = \frac{2\mu_1 a}{(\kappa_1 + 1)}\, I_{10;-1}\,. \tag{31}$$

Displacements and stresses in material 1:

$$(\kappa_1 + 1)u_r = \frac{\kappa_1 - 1}{2} J_{110} - (\zeta - \zeta')J_{111} \tag{32}$$

$$+\frac{A\kappa_1 - B}{2} I_{110} - A(\zeta - \kappa_1\zeta')I_{111} - 2A\zeta\zeta' I_{112},$$

$$(\kappa_1 + 1)u_z = -\frac{\kappa_1 + 1}{2} J_{100} - (\zeta - \zeta')J_{101} \tag{33}$$

$$-\frac{A\kappa_1 + B}{2} I_{100} - A(\zeta + \kappa_1\zeta')I_{101} - 2A\zeta\zeta' I_{102},$$

$$\sigma_{zz} = \frac{2\mu}{(\kappa_1 + 1)a} \Big[J_{101} + (\zeta - \zeta')J_{102} \tag{34}$$

$$+\frac{A + B}{2} I_{101} + A(\zeta + \zeta')I_{102} + 2A\zeta\zeta' I_{103} \Big],$$

$$\sigma_{rr} = \frac{2\mu}{(\kappa_1 + 1)a} \Big[J_{101} - (\zeta - \zeta')J_{102} - \frac{\kappa_1 - 1}{2\rho} J_{110} + \frac{\zeta - \zeta'}{\rho} J_{111} \tag{35}$$

$$+\frac{3A - B}{2} I_{101} - A(\zeta - 3\zeta')I_{102} - 2A\zeta\zeta' I_{103}$$

$$-\frac{A\kappa_1 - B}{2\rho} I_{110} + \frac{A(\zeta - \kappa_1\zeta')}{\rho} I_{111} + \frac{2A\zeta\zeta'}{\rho} I_{112} \Big],$$

$$\sigma_{rz} = \frac{2\mu}{(\kappa_1 + 1)a} \Big[(\zeta - \zeta')J_{112} \tag{36}$$

$$-\frac{A - B}{2} I_{111} + A(\zeta - \zeta')I_{112} + 2A\zeta\zeta' I_{113} \Big].$$

Displacements and stresses in material 2:

$$\frac{(\kappa_1 + 1)}{\Gamma} u_z = -[(B + 1)\zeta + (A + 1)\zeta']J_{101} \tag{37}$$

$$\frac{-(\kappa_1 + 1) + A(\kappa_1 - 1) + \Gamma(A\kappa_1^2 - B)}{2} J_{100},$$

$$\frac{(\kappa_1 + 1)}{\Gamma} u_r = -[(B + 1)\zeta + (A + 1)\zeta']J_{111} \tag{38}$$

$$\frac{(\kappa_1 - 1) + A(\kappa_1 - 1) + \Gamma(A\kappa_1^2 - B)}{2} J_{110}.$$

$$\sigma_{zz} = \frac{2\mu}{(\kappa_1 + 1)a} \Big[\frac{B(\kappa_1 + 1) - A(\kappa_1 - 1) - \Gamma(A\kappa_1^2 - B) + 2}{2} J_{101} \tag{39}$$

$$+[(B + 1)\zeta + (A + 1)\zeta']J_{102} \Big],$$

$$\sigma_{rr} = \frac{2\mu}{(\kappa_1 + 1)a} \Big[\frac{B(3 - \kappa_1) + A(\kappa_1 - 1) + \Gamma(A\kappa_1^2 - B) + 2}{2} J_{101} \tag{40}$$

$$-[(B + 1)\zeta + (A + 1)\zeta']J_{102} + \frac{(B + 1)\zeta + (A + 1)\zeta'}{\rho} J_{111}$$

$$
\begin{aligned}
& \quad - \frac{(\kappa_1 - 1) + A(\kappa_1 - 1) + \Gamma(A\kappa_1^2 - B)}{2\rho} J_{110} \Big], \\
\sigma_{rz} &= \frac{2\mu}{(\kappa_1 + 1)a} \Big[\frac{B(\kappa_1 - 1) - A(\kappa_1 - 1) - \Gamma(A\kappa_1^2 - B) + 2}{2} J_{111} \qquad (41) \\
& \quad + [(B+1)\zeta + (A+1)\zeta'] J_{112} \Big].
\end{aligned}
$$

5.2. Radial edge dislocation

Parameter $\eta = \dfrac{(\kappa_1 + 1)}{2}$.

Potential functions:

$$
\Omega(\rho, \zeta) = \frac{2\mu_1 a}{(\kappa_1 + 1)} J_{00;-1} , \qquad \Omega(\rho, \zeta) = -\frac{2\mu_1 a}{(\kappa_1 + 1)} I_{00;-1} . \qquad (42)
$$

Displacements and stresses in material 1:

$$
\begin{aligned}
(\kappa_1 + 1)u_z &= -\frac{\kappa_1 + 1}{2} J_{000} - (\zeta - \zeta') J_{001} \qquad (43) \\
& \quad - \frac{A\kappa_1 - B}{2} I_{000} - A(\zeta - \kappa_1 \zeta') I_{001} + 2A\zeta\zeta' I_{002}, \\
(\kappa_1 + 1)u_r &= \frac{\kappa_1 + 1}{2} J_{010} - (\zeta - \zeta') J_{011} \qquad (44) \\
& \quad + \frac{A\kappa_1 + B}{2} I_{010} - A(\zeta + \kappa_1 \zeta') I_{011} + 2A\zeta\zeta' I_{012}.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{zz} &= \frac{2\mu}{(\kappa_1 + 1)a} \Big[(\zeta - \zeta') J_{002} \qquad (45) \\
& \quad + \frac{A - B}{2} I_{001} + A(\zeta - \zeta') I_{002} - 2A\zeta\zeta' I_{003} \Big], \\
\sigma_{rr} &= \frac{2\mu}{(\kappa_1 + 1)a} \Big[2J_{011} - (\zeta - \zeta') J_{002} - \frac{\kappa_1 + 1}{2\rho} J_{010} + \frac{\zeta - \zeta'}{\rho} J_{011} \qquad (46) \\
& \quad + \frac{3A + B}{2} I_{001} - A(\zeta + 3\zeta') I_{002} + 2A\zeta\zeta' I_{003} \\
& \quad - \frac{A\kappa_1 + B}{2\rho} I_{010} + \frac{A(\zeta + \kappa_1 \zeta')}{\rho} I_{011} - \frac{2A\zeta\zeta'}{\rho} I_{012} \Big], \\
\sigma_{rz} &= \frac{2\mu}{(\kappa_1 + 1)a} \Big[-J_{011} + (\zeta - \zeta') J_{012} \qquad (47) \\
& \quad - \frac{A + B}{2} I_{011} + A(\zeta + \zeta') I_{012} - 2A\zeta\zeta' I_{013} \Big].
\end{aligned}
$$

Displacements and stresses in material 2:

$$\frac{(\kappa_1+1)}{\Gamma}u_z = -[(B+1)\zeta+(A+1)\zeta']J_{001} + \frac{(1-\kappa_1)+A(\kappa_1+1)+\Gamma(A\kappa_1^2-B)}{2}J_{000}, \tag{48}$$

$$\frac{(\kappa_1+1)}{\Gamma}u_r = -[(B+1)\zeta+(A+1)\zeta']J_{011} + \frac{(\kappa_1+1)+A(\kappa_1+1)+\Gamma(A\kappa_1^2-B)}{2}J_{010}. \tag{49}$$

$$\sigma_{zz} = \frac{2\mu}{(\kappa_1+1)a}\left[\frac{B(\kappa_1+1)-A(\kappa_1+1)-\Gamma(A\kappa_1^2-B)}{2}J_{001} + [(B+1)\zeta+(A+1)\zeta']J_{002}\right], \tag{50}$$

$$\sigma_{rr} = \frac{2\mu}{(\kappa_1+1)a}\left[\frac{B(3-\kappa_1)+A(\kappa_1+1)+\Gamma(A\kappa_1^2-B)+4}{2}J_{001} - [(B+1)\zeta+(A+1)\zeta']J_{002} + \frac{(B+1)\zeta+(A+1)\zeta'}{\rho}J_{011} - \frac{(\kappa_1+1)+A(\kappa_1+1)+\Gamma(A\kappa_1^2-B)+1}{2\rho}J_{010}\right], \tag{51}$$

$$\sigma_{rz} = \frac{2\mu}{(\kappa_1+1)a}\left[\frac{B(\kappa_1-1)-A(\kappa_1+1)-\Gamma(A\kappa_1^2-B)-2}{2}J_{011} + [(B+1)\zeta+(A+1)\zeta']J_{012}\right]. \tag{52}$$

As a final note we mention that the elastic fields due to a Somigliana ring dislocation in an elastic half space are obtained from the above expressions by substituting the special values of the bimaterial parameters $A = B = -1$, $\Gamma = 0$.

6. Deformation energy and the force acting on a dislocation

As an example application of the solutions developed in the preceding sections we calculate the elastic energy of a circular Somigliana dislocation embedded in an elastic half space.

Let the half space occupy the domain $z > 0$. Consider an axial glide dislocation with the Burgers vector b and radius a, lying on the plane $z = z' > 0$.

It is well known that the deformation energy associated with a *straight* dislocation, which is formally calculated using classical elasticity principles, becomes unbounded. This happens because the stresses are inversely proportional to the distance from the dislocation line, so that the corresponding integral diverges at the two extremes: in the immediate vicinity of the dislocation (the 'core' region) and at infinity.

The latter problem does not arise in the axisymmetric case, since the stresses due to a circular dislocation decay more rapidly with distance. This makes the energy integral converge at large distances, as will be confirmed by the analysis below. Also, if a dislocation in an elastic half space is considered, then the elastic body is bounded by the free surface. From the discussion in the last section of [3] it is evident that the dislocation may be created by introducing a displacement discontinuity which is confined to the region between the plane of the dislocation line and the free surface. The corresponding upper bound in the energy integral is therefore finite.

The problem of the dislocation core energy may be addressed in the usual way, whereby a separate 'line tension' term is introduced, which is proportional to the length of the dislocation line and independent of its curvature. This may be estimated from atomic considerations, so that the dislocation core energy term is given by

$$W_c = \frac{\mu b^2}{(\kappa + 1)}. \tag{53}$$

The full expression for the deformation energy is composed of two terms: the 'core' term W_c and the term accounting for the elastic energy outside the core region W_e,

$$W = W_c + W_e. \tag{54}$$

Let us evaluate the term W_e as follows. Introduce a cut over the cylindrical surface $r = a$ between the plane $z = z'$ and the free surface. Displace the opposite faces of the cut with respect to each other by the length b in the axial direction. The work of external shear forces needed to perform this operation is given by the following integral

$$W_e = 2\pi a \int_0^{z'-\epsilon/a} [b\sigma_{rz}(a,z)] \times [-b/2] \; dz. \tag{55}$$

Here $\epsilon << a, z'$ is thedislocation core radius.

As before, σ_{rz} denotes the shear stress of the axial glide dislocation with Burgers vector unity, and in terms of the Papkovich potentials is given by

$$\sigma_{rz} = \frac{(\kappa - 1)}{2}\phi_{,r} - z\phi_{,rz} - \psi_{,rz}. \tag{56}$$

The expression for the energy may then be rewritten in the form

$$W_e = \pi a b^2 \left[\frac{-(\kappa+1)}{2} \Phi_{,r} + \psi_{,r} + z\phi_{,r} \right]_0^{z'-\epsilon/a}, \tag{57}$$

where Φ denotes the original of ϕ, $\Phi_{,z} = \phi$, and the potential functions are found from equations (26-27) with $\eta = (\kappa - 1)/2$,

$$\psi = \frac{(\kappa-1)}{2}(\Omega - \bar{\Omega}) - z'\Omega_{,z} + kz'\bar{\Omega}_{,z}, \tag{58}$$

$$\phi = \Omega_{,z} - \bar{\Omega}_{,z} + 2z'\bar{\Omega}_{,zz}, \tag{59}$$

$$\Phi = \Omega - \bar{\Omega} + 2z'\bar{\Omega}_{,z}, \tag{60}$$

where

$$\Omega = \frac{2\mu a}{(\kappa+1)} J_{10;-1}(\rho, \zeta - \zeta'), \ \ \bar{\Omega} = \frac{2\mu a}{(\kappa+1)} J_{10;-1}(\rho, \zeta + \zeta'). \tag{61}$$

Substitution then leads to

$$W_e = W_e^\infty + W_e^i = \frac{2\mu\pi a b^2}{(\kappa+1)} \{ J_{110}(1, \epsilon/a)$$
$$- J_{110}(1, 2\zeta') + 2\zeta' \left[J_{111}(1, \zeta') - J_{111}(1, 2\zeta') - \zeta' J_{112}(1, 2\zeta') \right] \}.$$

The dislocation energy is given here as a sum of two terms, that of a dislocation in an infinite space, and the *interaction* term, which arose due to the presence of the free surface. Note that the small parameter ϵ has been dropped everywhere in the argument of $J-$integrals except where it appears on its own, i.e. in the term $J_{110}(1, \epsilon)$.

For the dislocation energy in an infinite space we obtain

$$W_e^\infty = \frac{2\mu\pi a b^2}{(\kappa+1)} J_{110}(1, \epsilon/a). \tag{62}$$

This result may be verified by letting ζ' become very large in the expression for W_e. The values of $J-$integrals then may be found in terms of the complete elliptic integrals of the first and second kind $\mathbf{K}(k)$ and $\mathbf{E}(k)$ (see Appendix in [3, 5]). For large ζ' the elliptic modulus assumes the value $k \simeq 2/\zeta'$. Using the series expansions for $\mathbf{K}$ and $\mathbf{E}$, given in [3], equations (77-78), it is readily verified that all the terms in W_e vanish, except the term containing $J_{110}(1, \epsilon)$.

This contribution to the dislocation energy is evaluated as follows. Use

$$J_{110}(\rho, \xi) = \frac{2}{\pi k \rho^{1/2}} \left[\frac{2-k^2}{2} \mathbf{K} - \mathbf{E} \right], \tag{63}$$

where $k^2 = 4\rho/[(1+\rho)^2+\xi^2]$. For small $\xi = \epsilon/a$ we have $k \simeq (1-\xi^2/4)$ and $k' \simeq \xi/2$. Also, for $k \to 1$

$$\mathbf{K}(k) \simeq \ln(4/k') \simeq \ln(8a/\epsilon), \quad \mathbf{E}(k) \simeq 1. \tag{64}$$

Thus finally

$$W_e^\infty \simeq \frac{2\mu a b^2}{(\kappa+1)}\left[\ln(8a/\epsilon)\right], \tag{65}$$

a result which agrees with the expression given by Kroupa [1]. It is curious to note that although the *dislocation surfaces* are different in the two problems, the *dislocation lines* and the *Burgers vectors* are identical, leading to the same deformation energies.

We may now turn our attention to the *interaction* term W_e^i

$$W_e^i = \frac{2\mu\pi a b^2}{(\kappa+1)}\left\{-J_{110}(1,2\zeta') + 2\zeta'\left[J_{111}(1,\zeta') - J_{111}(1,2\zeta') - \zeta' J_{112}(1,2\zeta')\right]\right\}. \tag{66}$$

By calculating the derivative of the dislocation energy with respect to its axial coordinate z' the force on the dislocation may be determined. Since W_c and W_e^∞ do not depend on z' we have

$$F(\zeta') = -\frac{dW}{dz'} = -\frac{1}{a}\frac{dW_e^i}{d\zeta'} =$$

$$= \frac{4\mu\pi b^2}{(\kappa+1)}\Big\{-J_{111}(\zeta') + \zeta' J_{112}(\zeta') - \zeta' J_{112}(2\zeta') - 2\zeta'^2 J_{113}(2\zeta')\Big\}. \tag{67}$$

The dependence of the interaction energy and the force acting on the dislocation on the distance from the free surface is shown in Fig.1. Note that the dislocation is attracted to the surface if it lies at depths below approximately $1.1a$, and is repelled at larger depths.

7. Discussion

In this paper the solutions for the circular Somigliana dislocations in one of two elastically dissimilar half spaces were obtained. The derivation is based on the Papkovich potential solutions for dislocations in an infinite space, given in terms of the Lipschitz-Hankel integrals in [3], and the application of Aderogba's theorem [6].

The resulting elastic fields are composed of two parts: the infinite space solution, identical with that given in [3], and the 'image' part, arising due to the interaction between the dislocation loop and the interface or free surface. This effect was analysed by evaluating the interaction energy and the force, acting on an axial circular dislocation in an elastic half space. It was shown that the dislocation is repelled from the surface for depths exceeding 1.1 dislocation line radii, and attracted to the surface at lower depths. [1]

Further results concerning the effect of bimaterial parameters on the dislocation properties may be readily obtained from the general formulae given in this paper. Further investigations may include the problems of dislocations lying at planar or cylindrical bimaterial interfaces.

8. Acknowledgement

The author wishes to acknowledg the support of EPSRC during the period when this paper was prepared. The author would like to express his gratitude to Prof. John Willis and Dr. David Hills for their helpful comments about the manuscript.

References

1. F. Kroupa, Circular Edge Dislocation Loop. *Chech. J. Phys.* B10 (1960) 284-293.
2. I. Demir, J.P. Hirth and H.M. Zbib, The Somigliana ring dislocation. *J. Elasticity* 28 (1992) 223-246.
3. A.M. Korsunsky, The Somigliana ring dislocation revisited. *1. Papkovich potential solutions for dislocations in an infinite space.* This volume.
4. G. Eason, B. Noble and I.N. Sneddon, On certain integrals of Lipschitz-Hankel type involving products of Bessel functions. *Trans. Roy. Soc.* 247 (1955) 529-551.
5. A.M. Korsunsky, Fundamental eigenstrain solutions for axisymmetric crack problems. *J. Mech. Phys. Solids* 43 (1995) 1221-1241.
6. K. Aderogba, On eigenstresses in dissimilar media. *Phil. Mag.* 35 (1977) 281-292.
7. A.M. Korsunsky, *The solution of axisymmetric crack problems in inhomogeneous media.* D.Phil. Thesis, University of Oxford (1994).
8. A.M. Korsunsky and D.A. Hills, The solution of plane crack problems by dislocation dipole procedures. *J. Strain Anal.* 30 (1995) 21-27.

[1] Tthe limit of straight dislocation corresponds to very large radii (or very low depths).

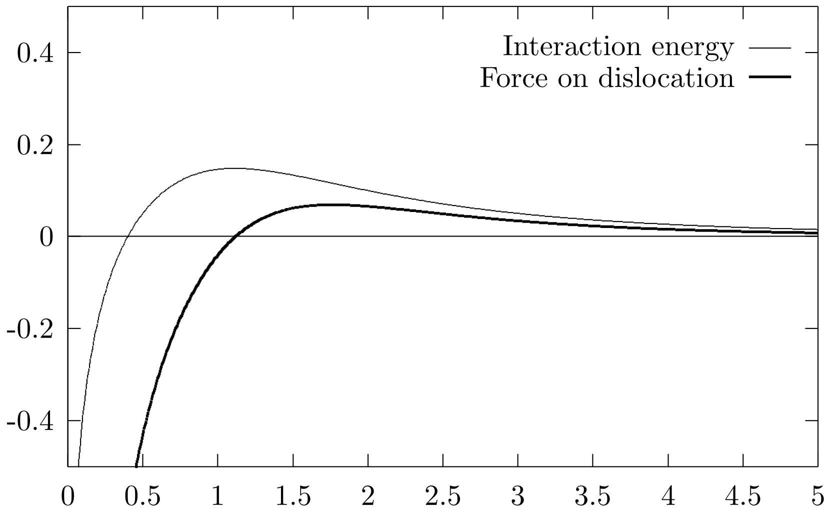

Figure 1. Normalized interaction energy $W_e^i * (\kappa + 1)/(2\mu\pi ab^2)$ and the force $F(\kappa + 1)/(2\mu\pi b^2)$ acting on a circular glide dislocation in an elastic half space, versus normalized dislocation depths z'/a