C1B Stress Analysis
Lecture 2:
Applications and Implications of the Calculus of Variations

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Hilary Term (January 08)

http://users.ox.ac.uk/~engs0161/4me6.html

Recommended reading

Some good books on the topics are available:

1. S.P. Timoshenko, J.N. Goodier, Theory of elasticity, Prentice Hall, 1990.
2. T. Mura, T. Koya, Variational methods in mechanics, OUP 1992.
3. H.M. Westergaard, Theory of elasticity and plasticity, Dover, 1964.
4. S. Wolfram, The Mathematica Book, 4th ed., CUP 1999.

Algebraic and numerical manipulations in this course are carried out using Mathematica. The new version 5 is widely available on departmental machines. Notebooks can be downloaded from the course web page:

http://www.eng.ox.ac.uk/~ftgamk/4me6.html

Euler and R-R

In the first two lectures we have reviewed the composition and complexity of the system of equations of elasticity. We then focused on the energy minimisation principle, and the calculus of variations.

Emphasis must be placed firmly on understanding the basic principles behind the manipulations. We review the detailed derivation of the Euler equation for a bent beam:

$$W[y] = EI/2 \int_0^L (y'')^2 dx - \int_0^L f y \, dx$$

$$\delta W = EI/2 \int_0^L 2y'' \delta y'' dx - \int_0^L f \delta y \, dx$$

$$\int_0^L \underbrace{y''}_{u} \underbrace{\delta y'' dx}_{dv} = [y'' \delta y']_0^L - \int_0^L \underbrace{y'''}_{u} \underbrace{\delta y' dx}_{dv} =$$

$$= [y'' \delta y']_0^L - [y''' \delta y]_0^L + \int_0^L y'''' \delta y \, dx$$

$$\delta W = \int_0^L EI y'''' \delta y \, dx - \int_0^L f \delta y \, dx + [y'' \delta y']_0^L - [y''' \delta y]_0^L =$$

$$= \int_0^L \underbrace{[EI y'''' - f]}_{\text{Euler equation.}} \delta y \, dx + [y'' \delta y']_0^L - [y''' \delta y]_0^L$$

Euler does not assume anything about the solution, and finds the governing differential equation for it by analysis of variation.
In contrast, the R-R assumes the solution is a linear combination of known functions with unknown coefficients A_i, and minimises $W[A_i]$:

$$y = \sum_{i=1}^{N} A_i y_i(x)$$

$$\frac{\partial W}{\partial A_i} = EI \int_0^L y'' \frac{\partial y''}{\partial A_i} dx - \int_0^L f \frac{\partial y}{\partial A_i} dx = 0$$

Since y depends on A_i linearly, this is a system of N linear equations.

Further Euler equation$_1$

If function F(y,y′) does not depend on x explicitly, then Euler is

$$F_y - (F_{y'})' = F_y - y'F_{y'y} = \left(F - y'F_{y'}\right)_y = 0$$

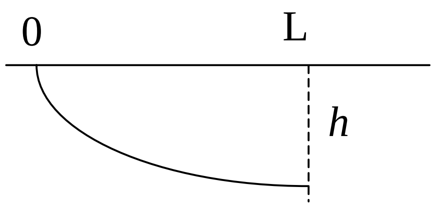
Hence

$$F - y'F_{y'} = \text{const}$$

Fermat's principle

The trajectory of a light ray in inhomogeneous 2D medium with refractive index n(y) and speed of light v(y)=c/n(y) corresponds to the shortest time of propagation between points (0,0) and (L,h), i.e. minimises the integral

$$I[y] = \int_0^L \frac{\sqrt{1+y'^2}}{v(y)}dx = \int_0^L \sqrt{1+y'^2}\,n(y)dx$$



with the boundary conditions y(0)=0, y(L)=h.

$$F(y,y') = \frac{\sqrt{1+y'^2}}{v(y)} = \sqrt{1+y'^2}\,n(y)$$

$$n(y) = 2C\sqrt{1+y'^2}$$

$$\sqrt{1+y'^2} = \frac{n(y)}{2C}$$

$$y' = \frac{dy}{dx} = \sqrt{1-n(y)^2/(4C^2)}$$

$$dx = \int_0^y \frac{dy}{\sqrt{1-n(y)^2/(4C^2)}}$$

The brachistochrone problem

The Fermat problem about light propagation is closely related to the problem about the slope shape between points (0,0) and (L,h) for which the time descent of a point under gravity is a minimum.

The speed of the particle at height y is $v(y) = \sqrt{2gy}$

i.e. a particular case of the above problem is considered.

The solution is the **cycloid** of the form

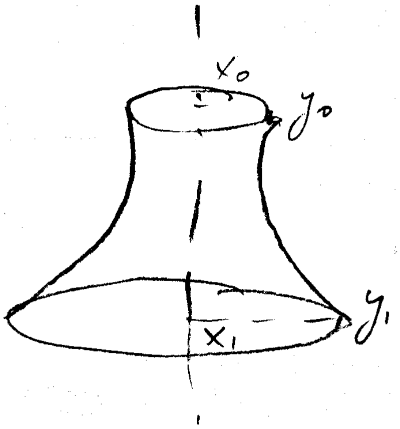
$$x = a(\theta - \sin\theta), x = a(1-\cos\theta), 0 < \theta < \theta_S$$

where

$$h = a(1-\cos\theta_S), l = a(\theta_S - \sin\theta_S).$$

Further Euler equation$_2$

Minimum surface of revolution



$$I[y] = 2\pi \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx$$

$$F(y,y') = y\sqrt{1+y'^2} \times 2\pi$$

$$F_y - F_{yy'}\,y' = 0 = (F - F_{y'}y')_y = 0$$

$$c = F - F_{y'}y' = y\sqrt{1+y'^2} - \frac{y\cdot \cancel{2}y'^2}{\cancel{2}\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}} = c$$

$$y = c\sqrt{1+y'^2}$$

$$\frac{y^2}{c^2} = 1+y'^2;\quad y'^2 = 1-\frac{y^2}{c^2};\quad y' = \frac{\sqrt{c^2-y^2}}{c}$$

$$\frac{dx}{dy} = \frac{c}{\sqrt{c^2-y^2}};\quad dx = \frac{c\,dy}{\sqrt{c^2-y^2}}$$

$$y = c\cos\theta;\quad dx = -\frac{c^{\cancel{2}}\,\cancel{\sin\theta}\,d\theta}{\cancel{c}\,\cancel{\sin\theta}} = c\,d\theta$$

$$x = c\theta + c_1;\quad \theta = \frac{x}{c} - c_1$$

$$\boxed{y = c\cos\left(\frac{x}{c} - c_1\right)}$$ — catenary

(Find c, c_1 from b.c.)

Further Euler equation$_3$

One of the great advantages of the variational approach is that it is easily generalised to different situations and applications. In fact, both classical theoretical mechanics and quantum mechanics use energy-based formalisms to derive the governing equations (equilibrium, boundary conditions etc.).

Consider a membrane at rest, for which the potential energy is proportional to the change of area, with the proportionality factor being the tension T. Let u(x,y) describe the deflection. The change in area is

$$\sqrt{1+u_{,x}^{2}+u_{,y}^{2}}\,\mathrm{d}x\mathrm{d}y-\mathrm{d}x\mathrm{d}y=\left[1+\tfrac{1}{2}\left(u_{,x}^{2}+u_{,y}^{2}\right)\right]\mathrm{d}x\mathrm{d}y-\mathrm{d}x\mathrm{d}y=\tfrac{1}{2}\left(u_{,x}^{2}+u_{,y}^{2}\right)\mathrm{d}x\mathrm{d}y$$

Internal energy U is due to tension T. External energy E is due to applied load f(x,y). Total energy W=U-E, i.e.

$$U=\iint_{\Omega}\tfrac{1}{2}T\left(u_{,x}^{2}+u_{,y}^{2}\right)\mathrm{d}x\mathrm{d}y,\quad E=-\iint_{\Omega}fu\,\mathrm{d}x\mathrm{d}y,\quad W=\iint_{\Omega}\left[\tfrac{1}{2}T\left(u_{,x}^{2}+u_{,y}^{2}\right)-fu\right]\mathrm{d}x\mathrm{d}y.$$

Euler equation is derived by the usual means of perturbing u(x,y), leading to

$$T\left(u_{,xx}+u_{,yy}\right)+f=0.$$

Particularly, of Ω is a circle of radius a under loading f=const, then the above equation assumes the form

$$\frac{T}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+f=0.$$

This can be easily integrated to find the general solution in the form

$$u=-\frac{fr^{2}}{2T}+C.$$

If the edge of the membrane is fixed, u=0 at r=a, so the constant C is found,

$$u=\frac{f\,(a^{2}-r^{2})}{2T}.$$

The deflected shape of the membrane under uniform loading is a paraboloid.

Subsidiary condition/constraint

Minimise functional $I[y] = \int_{x_0}^{x_1} F(x,y,y')\,dx$

under condition $J = \int_{x_0}^{x_1} G(x,y,y')\,dx = C$ (Constraint, or subsidiary cond.)

and b.c. $y(x_0) = y_0$, $y(x_1) = y_1$.

<u>Use Lagrange multiplier λ</u>

Consider $F(x,y,y') + \lambda G(x,y,y') = H(x,y,y')$

Euler equation leads to:

$$F_y - F_{yy'}y' + \lambda G_y - \lambda G_{yy'}y' = 0$$

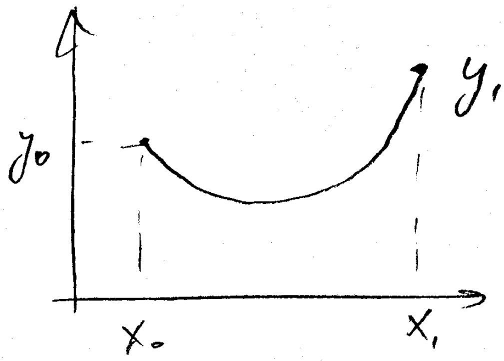
Solve as previously, and find value of λ to satisfy constraint!

The two Phakding funicular bridges.

Funicular function

Funicular function

(this is of Tom's Makin?)



A uniform string of fixed length C and density ρ hangs between (x_0, y_0) and (x_1, y_1).

Find shape.

$$I[y] = \rho g \int_{x_0}^{x_1} \underbrace{y\sqrt{1+y'^2}}_{F(y,y')}\,dx \quad \text{– potential energy to minimise}$$

$$J[y] = \int_{x_0}^{x_1} \overbrace{\sqrt{1+y'^2}}^{G(y')}\,dx = D \quad \text{– constraint.}$$

Euler: $\boxed{F_y - (F_{y'})' + \lambda G_y - \lambda (G_{y'})' = 0}$

$g \times$

$$\rho\sqrt{1+y'^2} - \rho\frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - \lambda\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

or $\boxed{F - y'F_{y'} + \lambda G - \lambda y' G_{y'} = C_1}$

$$\rho y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2} - \rho y\frac{y'^2}{\sqrt{1+y'^2}} - \lambda\frac{y'^2}{\sqrt{1+y'^2}} = C_1$$

Hence $\dfrac{\rho y + \lambda}{\sqrt{1+y'^2}} = C_1$

$$dx = \int \frac{dy}{\sqrt{[(\rho y+\lambda)/C_1]^2 - 1}}$$

Funicular function$_2$

Integrating:

$$\frac{\rho x}{C_1} + C_2 = \cosh^{-1} \frac{\rho y + \lambda}{C_1}$$

$$\boxed{\rho y + \lambda = C_1 \cosh\left[\frac{\rho x}{C_1} + C_2\right]}$$

Three unknown constants: C_1, C_2, λ

Three conditions: $y(x_0) \overset{①}{=} y_0$, $y(x_1) \overset{②}{=} y_1$, $J[y] \overset{③}{=} D$.

① $\rho y_0 + \lambda = C_1 \cosh\left(\frac{\rho x_0}{C_1} + C_2\right)$

② $\rho y_1 + \lambda = C_1 \cosh\left(\frac{\rho x_1}{C_1} + C_2\right)$

③ $\frac{C_1}{\rho}\left[\sinh\left(\frac{\rho x_1}{C_1} + C_2\right) - \sinh\left(\frac{\rho x_0}{C_1} + C_2\right)\right] = D.$

Solution for rope shape:

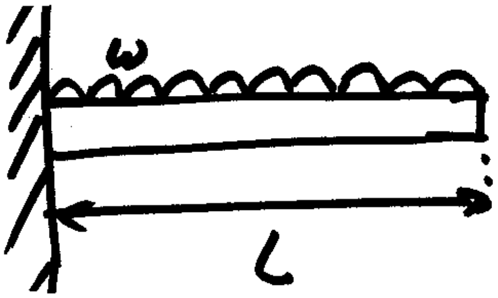
Assume one end is at x0=0, y0=0, so that

$$y = a\left[\cosh\left(\frac{x}{a}\right) - 1\right]$$

To find value of a:

$$y_1 = a\left[\cosh\left(\frac{x_1}{a}\right) - 1\right]$$

Rayleigh-Ritz: cantilever



NB: the Euler equation for this beam was derived in the previous lecture, and you can obtain exact solutions to this problem (under arbitrary loading!) by modifying the input of *Mathematica* notebook `eulerbeam.nb`

For R-R approximation we seek the solution as power series

$$v(x) = x^2 \sum_{i=0..N} A_{i+2} x^i$$

For the one-term solution it is found quickly that $v''(x)=2A_2$, and

$$\frac{\partial W}{\partial A_2} = EI\int_0^L v''\frac{\partial v''}{\partial A_2}\mathrm{d}x - \int_0^L w\frac{\partial v}{\partial A_2}\mathrm{d}x = EI\int_0^L (4A_2 - wx^2)\mathrm{d}x = 0$$

so that $A_2 = \dfrac{wL^2}{12EI}$

This trivial one-term calculation is contained in `ritzlever.nb`

It is interesting to perform the calculations for a series containing an arbitrary number of terms. Using `ritzlevern.nb` we can modify the number of constants Cs={A2,A3} (starting with one) and compare results with the exact solution (e.g. by Euler)

$$v(x) = x^2\left(\frac{wL^2}{4EI} - \frac{wxL}{6EI} + \frac{wx^2}{24EI}\right)$$

```
In[300]:= H Cantilever by Rayleigh-Ritz, n term  L Off@General::spell, General::spell1D
          H Step 1: Define shape in terms of A  L
          Cs = 8A2<; n = Length@CsD;
          v = x^2 Sum@Cs@@iDD x^Hi − 1L, 8i, n<D; v2 = D@v, 8x, 2<D;
          H Step 2: Calculate W in terms of A  L
          W = Integrate@EI v2^2 ê 2 − w v, 8x, 0, L<D
          H Step 3: Minimise dWêdA to find solution  L
          grad = Table@D@W, Cs@@iDDD, 8i, n<D;
          AA = Solve@grad « 0, CsD@@1DD
          v = v ê. AA;
          v2 = v2 ê. AA;
```

Out[303]= $2\,\mathrm{A2}^2\,\mathrm{EI}\,L - \frac{1}{3}\mathrm{A2}\,L^3\,w$

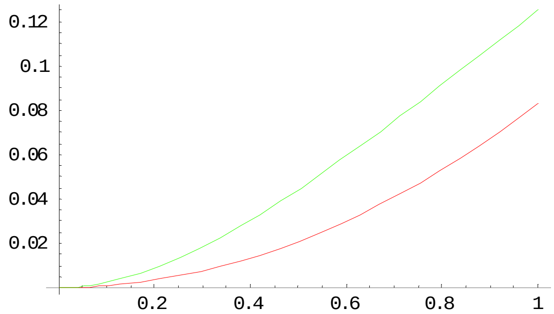
Out[305]= :A2 Ø $\frac{L^2 w}{12\,\mathrm{EI}}$>

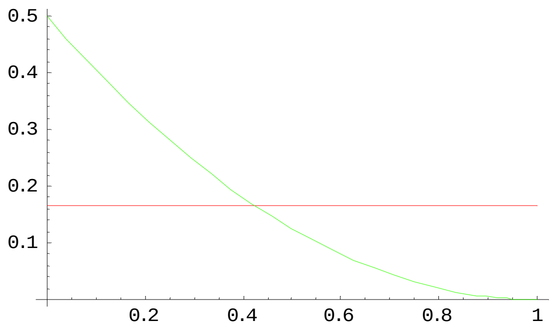
Rayleigh-Ritz: cantilever$_2$

We set up the plotting to compare the results:

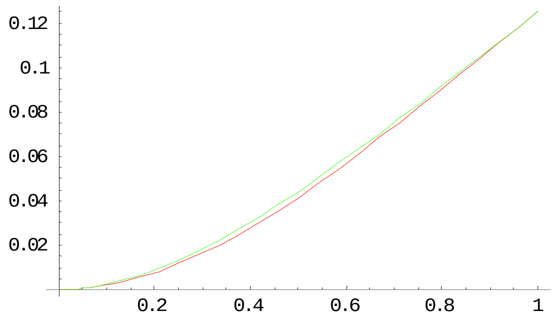
```
In[199]:= H  Exact solution we are working towards - e.g. by Euler  L
          ve = x^2 ( w L^2/(4 EI) - w x L/(6 EI) + w x^2/(24 EI) ); ve2 = D@ve, 8x, 2<D;
          H  Substitutions for plotting  L
          valw = 8EI → 1, L → 1, w → 1<;
          vw = v ê. valw; vew = ve ê. valw;
          v2w = v2 ê. valw; ve2w = ve2 ê. valw;
          H  Plots  L
          Plot@8vw, vew<, 8x, 0, 1<, PlotStyle → 8Hue@1D, Hue@0.3D<D
          Plot@8v2w, ve2w<, 8x, 0, 1<, PlotStyle → 8Hue@1D, Hue@0.3D<D
```

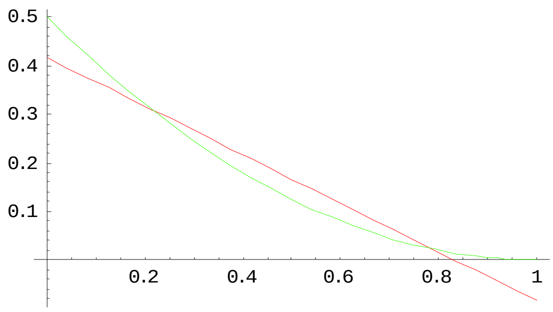
For the 1-term solution the deflection and bending moment are:



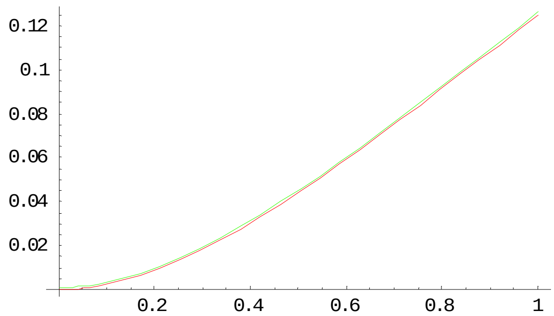


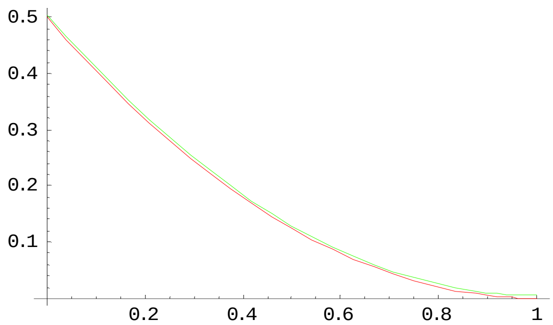
For the 2-term solution the deflection and bending moment are:





For the 3-term solution the deflection and bending moment are:





The 3-term solution is exact. You can check that all constants beyond A_4 evaluate to zero. Thus, the choice of a power series to represent the solution was good - but this may not always possible!

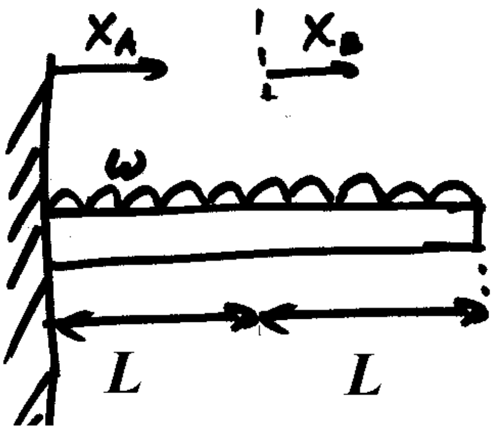
Rayleigh-Ritz: cantilever$_3$

It is interesting to note that in the above example the boundary condition for the bending moment at the free end of the cantilever, M(L)=-Eiv''(L)=0, is not satisfied even in the 2-term solution. (NB: satisfying it for the 1-term solution would zero it!)

The free end condition (natural boundary condition) can be incorporated to improve convergence.

The requirement v''(L)=0 prescribes an expression for A3 in terms of A2, only one degree of freedom remaining, since W=W[A2].

Imposing additional boundary conditions and continuity conditions often accelerates convergence to the correct solution.



In order to illustrate this last point we now develop an application of the Rayleigh-Ritz procedure using piecewise approximating functions.

The **problem** remains the cantilever as shown in the figure, but we re-label the cantilever length as 2L for convenience.

We divide the cantilever into two elements A and B, and introduce 'local' co-ordinates x_A=x and x_B=x-L. We use quadratic functions to represent deflection in the elements, as follows

$$v_A(x) = A_2 x^2 \qquad v_B(x) = B_0 + B_1 x_B + B_2 {x_B}^2 = B_0 + B_1(x-L) + B_2(x-L)^2$$

Note that the choice of $v_A(x)$ automatically satisfies the b.c. at x=0, but **continuity** at x=L must be enforced:

$$v_A(L) = v_B(L) \qquad v_A'(L) = v_B'(L)$$

The procedure can be efficiently implemented in Mathematica – let's have a look at `ritzpiece.nb`

Piecewise Rayleigh-Ritz

```
In[1]:= H Cantilever by Rayleigh-Ritz, n term  L Off@General::spell, General::spell1D
        H Step 1: Define shape in terms of A  L
        Cs = 8A2, B0, B1, B2<; n = Length@CsD;
        H Let's try piecewise  L
        va = A2 x^2; vb = HB0 + B1 Hx - LL + B2 Hx - LL ^2L;
        H Continuity of slope at x=L  L
        eqv1 = HD@va, xD - D@vb, xDL ê. x → L
        vb = vb ê. Solve@eqv1 « 0, B1D@@1DD;
        H Continuity of deflection at x=L  L
        eqv = Hva - vbL ê. x -> L
        vb = vb ê. Solve@eqv « 0, B0D@@1DD
        va2 = D@va, 8x, 2<D ; vb2 = D@vb, 8x, 2<D;
        H Step 2: Calculate W in terms of A  L
        W = Integrate@EI va2^2 ê 2 - w va, 8x, 0, L<D + Integrate@EI vb2^2 ê 2 - w vb, 8x, L, 2 L<D
        H Step 3: Minimise dWêdA to find solution  L
        grad = 8D@W, A2D, D@W, B2D<
        AA = Solve@grad « 0, 8A2, B2<D@@1DD
        va = va ê. AA
        vb = vb ê. AA
        v = va UnitStep@L - xD + vb UnitStep@x - LD;
        v2 = D@va, 8x, 2<D UnitStep@L - xD + D@vb, 8x, 2<D UnitStep@x - LD;
```

Out[4]= $2\,A2\,L - B1$

Out[6]= $A2\,L^2 - B0$

Out[7]= $A2\,L^2 + 2\,A2\,(x - L)\,L + B2\,(x - L)^2$

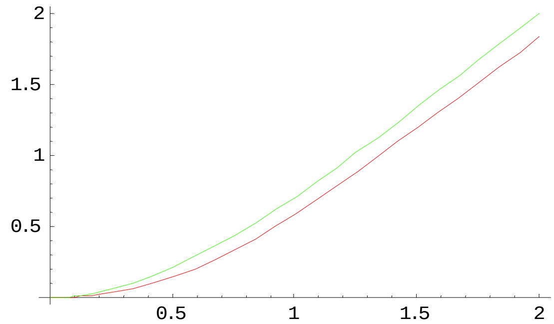
Out[9]= $-\frac{7}{3}\,A2\,w\,L^3 - \frac{1}{3}\,B2\,w\,L^3 + 2\,A2^2\,EI\,L + 2\,B2^2\,EI\,L$

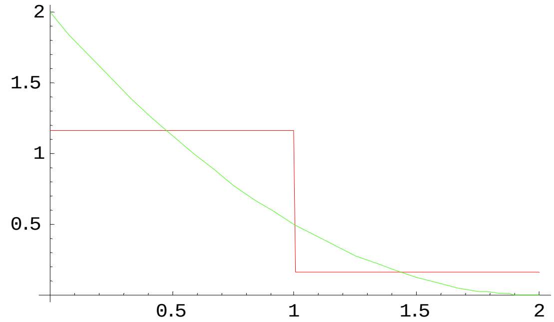
Out[10]= $\left\{4\,A2\,EI\,L - \frac{7\,L^3\,w}{3},\ 4\,B2\,EI\,L - \frac{L^3\,w}{3}\right\}$

Out[11]= $\left\{A2 \to \frac{7\,L^2\,w}{12\,EI},\ B2 \to \frac{L^2\,w}{12\,EI}\right\}$

Out[12]= $\frac{7\,L^2\,w\,x^2}{12\,EI}$

Out[13]= $\frac{7\,w\,L^4}{12\,EI} + \frac{7\,w\,(x - L)\,L^3}{6\,EI} + \frac{w\,(x - L)^2\,L^2}{12\,EI}$





Plot deflection and bending moment:

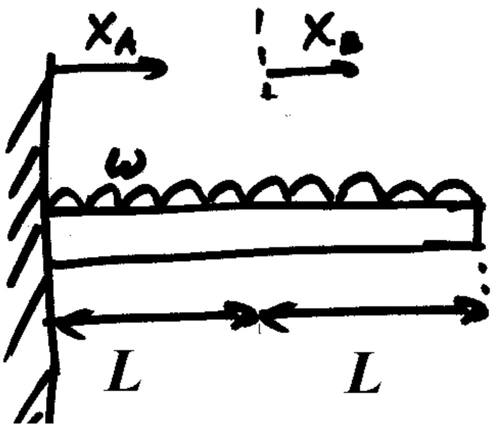
```
In[16]:= H The exact solution to compare with, BUT adjust L→ 2 L  L
```

$$ve = x^2\left(\frac{w\,(2\,L)^2}{4\,EI} - \frac{w\,x\,(2\,L)}{6\,EI} + \frac{w\,x^2}{24\,EI}\right);$$

```
         ve2 = D@ve, 8x, 2<D;
         H Substitutions for plotting  L
         valw = 8EI → 1, L → 1, w → 1<;
         vw = v ê. valw; vew = ve ê. valw;
         v2w = v2 ê. valw; ve2w = ve2 ê. valw;
         H Plots  L
         Plot@8vw, vew<, 8x, 0, 2<, PlotStyle → 8Hue@1D, Hue@0.3D<D
         Plot@8v2w, ve2w<, 8x, 0, 2<, PlotStyle → 8Hue@1D, Hue@0.3D<D
```

Piecewise Rayleigh-Ritz$_2$

We now explore the subject of piecewise R-R a bit further, to reveal the connection with the Finite Element Method.



Example problem: consider the same geometry as before (fig.), but think of the distributed load w as acting along the <u>*tie*</u> (e.g. body force). Find the displacements and stresses in the tie using a two-element approach.

The assumed displacement fields for this problem can be linear:

$$v_A(x) = A_0 + A_1 x \qquad v_B(x) = B_0 + B_1 x$$

In our formulation we clearly have three nodes: x_1=0, x_2=L, x_3=2L. A key requirement of continuity can be enforced by expressing displacements in the elements in terms of their nodal values u_1,u_2,u_3. A tiny script gives the required expressions:

```
ua = A0 + A1 x;
eq1 = u1 - ua ê. x → 0; eq2 = u2 - ua ê. x → L;
ua = ua ê. Solve@8eq1, eq2< « 0, 8A0, A1<D@@1DD
H Shape functions L Coefficient@ua, 8u1, u2<D

Out[135]= u1 - (Hu1 - u2L x)/L

Out[136]= :1 - x/L, x/L>
```

The coefficients in front of the nodal displacements u_1,u_2 are known as 'shape functions' in FEM context. It is easy to derive them for three and four-noded elements (`shapefun.nb`):

```
In[137]:= ua = A0 + A1 x + A2 x^2; eq1 = u1 - ua ê. x → x1; eq2 = u2 - ua ê. x → x2; eq3 = u3 - ua ê. x → x3;
          ua = ua ê. Solve@8eq1, eq2, eq3< « 0, 8A0, A1, A2<D@@1DD;
          H Shape functions L
          Factor@Coefficient@ua, 8u1, u2, u3<DD

Out[139]= :(Hx - x2L Hx - x3L)/(Hx1 - x2L Hx1 - x3L), -(Hx - x1L Hx - x3L)/(Hx1 - x2L Hx2 - x3L), (Hx - x1L Hx - x2L)/(Hx1 - x3L Hx2 - x3L)>

ua = A0 + A1 x + A2 x^2 + A3 x^3; eq1 = u1 - ua ê. x → x1; eq2 = u2 - ua ê. x → x2;
eq3 = u3 - ua ê. x → x3; eq4 = u4 - ua ê. x → x4;
ua = ua ê. Solve@8eq1, eq2, eq3, eq4< « 0, 8A0, A1, A2, A3<D@@1DD;
H Shape functions L
Factor@Coefficient@ua, 8u1, u2, u3, u4<DD

:(Hx - x2L Hx - x3L Hx - x4L)/(Hx1 - x2L Hx1 - x3L Hx1 - x4L), -(Hx - x1L Hx - x3L Hx - x4L)/(Hx1 - x2L Hx2 - x3L Hx2 - x4L), -(Hx - x1L Hx - x2L Hx - x4L)/(Hx1 - x3L Hx2 - x3L Hx4 - x3L), -(Hx - x1L Hx - x2L Hx - x3L)/(Hx1 - x4L Hx2 - x4L Hx3 - x4L)>
```

Piecewise Rayleigh-Ritz$_3$

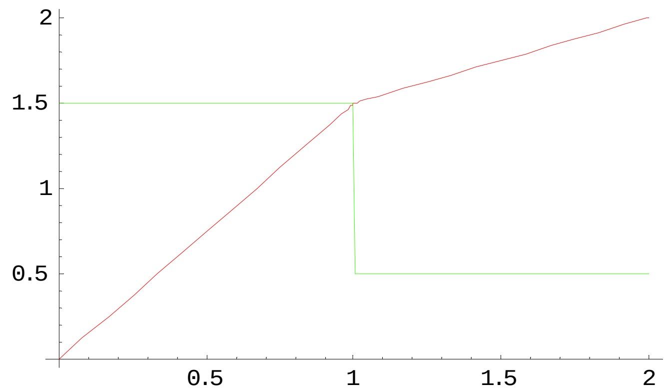
We can use the shape function formulation to solve the problem. (tie.nb)

```
H  Define elements, using built-in b.c.  L u1 = 0;
N1 = 1 - Hx ê LL;  N2 = x ê L;
ua = u1  N1 + u2  N2; ub = u2  N1 + u3  N2;
H  Determine total energy  L
W = Integrate@AE ê 2 HD@ua, xDL^2 - w ua, 8x, 0, L<D +
   Integrate@AE ê 2 HD@ub, xDL^2 - w ub, 8x, 0, L<D
H  Find minimum  L
UU = Solve@8D@W, u2D, D@W, u3D< « 0, 8u2, u3<D@@1DD
H  Find solution  L
ua1 = D@ua, xD; ub1 = D@ub, xD
u = UnitStep@L - xD Hua ê. UUL + UnitStep@x - LD Hub ê. UU ê. x → Hx - LLL;
u1 = UnitStep@L - xD Hua1 ê. UUL + UnitStep@x - LD Hub1 ê. UU ê. x → Hx - LLL;
H  Substitutions for plotting  L
valw = 8AE → 1, L → 1, w → 1<;
H  Plots  L
Plot@8u ê. valw, u1 ê. valw<, 8x, 0, 2<, PlotStyle → 8Hue@1D, Hue@0.3D<D
```

Out[3]= $\frac{AE\,u2^2}{2L} - \frac{L\,w\,u2}{2} + \frac{1}{2} L\,(u2 - u3)\,w + \frac{-2\,u2\,w\,L^2 + AE\,u2^2 + AE\,u3^2 - 2\,AE\,u2\,u3}{2L}$

Out[4]= $\{u2 \to \frac{3L^2 w}{2\,AE},\ u3 \to \frac{2L^2 w}{AE}\}$

Out[5]= $\frac{u3}{L} - \frac{u2}{L}$



As an exercise:

(i) Establish the exact shape and stress profiles. Differentiate the energy functional to find the variation, write down the Euler equation for the tie, and solve it exactly.

(ii) Show that the correct normalised value is 2 for both maximum stress and deflection.

(iii) Modify the Mathematica script to plot the exact solution together with the piecewise R-R result.

(iv) Discuss how the solution could be improved, and what changes you would expect to see. How would you implement this in Mathematica?

R-R and matrix methods

It is interesting to compare the basic procedure following from the R-R method with the stiffness matrix methods you are perhaps more familiar with.
Returning to the example of a beam carrying a UDL w, we started with a family of trial functions

$$v(x) = A_1 v_1(x) + A_2 v_2(x) + A_3 v_3(x) + \ldots = \sum_{i=1..N} A_i v_i(x)$$

We now focus our attention on the coefficients A_i, which can be thought of as **generalised coordinates**. The beam curvature is

$$v''(x) = \sum_{i=1..N} A_i v_i''(x)$$

and the total energy of the system W is a function of A_i

$$W(A_i) = \int_0^L \left[\tfrac{1}{2} EI \sum_{i=1..N} A_i v_i''(x) \sum_{j=1..N} A_j v_j''(x) - w \sum_{i=1..N} A_i v_i''(x) \right] \mathrm{d}x$$

We interchange the order of summation and integration

$$W(A_i) = \tfrac{1}{2} EI \sum_{i=1..N} \sum_{j=1..N} A_i A_j \int_0^L v_i'' v_j'' \mathrm{d}x - \sum_{i=1..N} A_i \int_0^L w v_i'' \mathrm{d}x$$

and introduce the notation

$$k_{ij} = k_{ji} = EI \int_0^L v_i'' v_j'' \mathrm{d}x \qquad P_i = \int_0^L w v_i'' \mathrm{d}x$$

The energy can be written in the form

$$W(A_i) = \tfrac{1}{2} \sum_{i=1..N} \sum_{j=1..N} k_{ij} A_i A_j - \sum_{i=1..N} P_i A_i = \tfrac{1}{2} \mathbf{A}^T \mathbf{K} \mathbf{A} - \mathbf{A}^T \mathbf{P}$$

Minimising it wrt A_i leads to

$$W_{A_m} = \tfrac{1}{2} \left(\sum_{i=1..N} k_{im} A_i + \sum_{j=1..N} k_{mj} A_j \right) - P_m = (\mathbf{KA} - \mathbf{P})_m = 0$$

This is the m-th equation of equilibrium in terms of the generalised coordinates A_i and generalised forces P_i.

2D R-R: plates

The great versatility of the R-R method allows it to be used with great success in problems in two and three dimensions.
We consider an example from plate theory. Plates can be loaded by pressure transverse to the plane of the plate, which results in bending displacement w. If a plate is loaded in its own plane and displacements u,v occur in the median plane, it is called a membrane.
Bending stiffness of a plate is the parameter

$$D = Eh^3/12(1-\nu^2)$$

The total energy of a simply supported plate 0<x<a, 0<y<b is

$$W[w] = \frac{D}{2}\int_0^a\int_0^b \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 \mathrm{d}x\mathrm{d}y - \int_0^a\int_0^b wP(x,y)\mathrm{d}x\mathrm{d}y$$

The R-R approach can be to seek w as a double sine series

$$w(x,y) = \sum_{m=1..M}\sum_{n=1..N} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

We must now proceed to find the derivatives $\partial W/\partial A_{mn}$ and equate them to zero to find A_{mn}.

$$\frac{\partial^2 \omega}{\partial x^2} = \omega_{xx} = -\sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\left(\frac{m\pi}{a}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \text{ etc.}$$

$$\frac{\partial W}{\partial A_{mn}} = D\int_0^a\int_0^b (\omega_{xx}+\omega_{yy})\left(\frac{\partial \omega_{xx}}{\partial A_{mn}} + \frac{\partial \omega_{yy}}{\partial A_{mn}}\right)dxdy - \int_0^a\int_0^b P\frac{\partial \omega}{\partial A_{mn}}dxdy = 0$$

Recall the orthogonality property: $\int_0^a \sin\frac{i\pi x}{a}\sin\frac{j\pi x}{a}\mathrm{d}x = \frac{a}{2}\delta_{ij}$

Hence, only those terms in the expression for $\partial W/\partial A_{mn}$ survive which contain the same indices (m and n) within the brackets, and

$$A_{mn} = \frac{4I_{mn}}{Dab\left[(m\pi/a)^2 + (n\pi/b)^2\right]^2}$$,where $I_{mn} = \int_0^a\int_0^b P\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y$

This works for any P(x,y), but let e.g. $P(x,y) = P_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$
Then A_{mn}=0, except

$$A_{11} = \frac{P_0 a^4 b^4}{D\pi^4\left[a^2+b^2\right]^2} \qquad w(x,y) = A_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

Galerkin's Method$_1$

A minimum problem frequently encountered in classical mechanics is that of

$$W[y]=\int_0^L \left[p(y')^2+qy^2+ry\right]dx$$

subject to the boundary conditions $y(0)=y_0$, $y(L)=y_1$.

The function that provides the solution can therefore be sought as a perturbation of any function that satisfy these conditions, e.g.

$$y=\frac{x}{L}y_1+\frac{L-x}{L}y_0+a_1w_1+a_2w_2+\ldots+a_nw_n.$$

Here the variation of y is described by the functions w_n forming a complete system within which each function satisfies the homogeneous boundary conditions ($\delta y= w_n$, $w_n(0)= w_n(L)=0$).

We can apply the Rayleigh-Ritz method to obtain:

$$\frac{\partial W}{\partial a_n}=2\int_0^L\left[py'\frac{\partial y'}{\partial a_n}+qy\frac{\partial y}{\partial a_n}+r\frac{\partial y}{\partial a_n}\right]dx=2\int_0^L\left[py'w'_n+qyw_n+rw_n\right]dx$$

The above equation can be written as

$$2\int_0^L\left[(-py')'+qy+r\right]w_n dx=0 \quad (n=1,2,\ldots)$$

Note that the integrand of the above equation is the product of the Euler equation by the fundamental function.

The above formulation is called Galerkin's method for the differential equation $(-py')'+qy+r=0.$

Galerkin's method opens interesting possibilities for the analysis of various systems.

Galerkin's Method$_2$

Galerkin proposed method could be viewed as an interesting interpretation of the R-R method.

Consider the steps to solve the minimum problem of W[y] in the bent beam example. We take a complete system of functions satisfying the boundary conditions (e.g. the sine series of the previous example) and find the derivatives $\partial W/\partial A_i$ as

$$\frac{\partial W}{\partial A_i} = \int_0^L [EIv'''' - w - P\delta(x - L/2)] v_i \mathrm{d}x = 0, \quad i = 1..N$$

Here v(x) is the solution we seek, and $v_i(x)$ denotes each of the trial functions in turn, 1..N. Note that the integrand is simply a product of the Euler equation by the trial function. The system can be solved, since the number of equations N is equal to the number of unknowns A_i.

Galerkin approach can be used to solve **any** differential equation, even if the energy functional is not known or not available.
For differential equation L[v]=0 and boundary conditions B[v]=b, seek v(x) in the form

$$v(x) = v_0(x) + A_1 v_1(x) + \ldots + A_N v_N(x)$$

where $v_0(x)$ must satisfy the boundary conditions, $B[v_0]=b$. The coefficients A_i are determined from

$$\int_0^L L[v] v_i \mathrm{d}x = 0$$

It is easy to verify that in the bent beam problem we considered previously using Galerkin method with the one-term trial function from our R-R example leads to the identical result.