4ME06 Stress Analysis
Lecture 4:
Further energy principles and variational formulations

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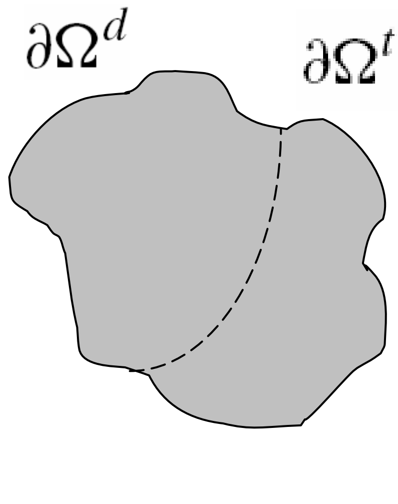
http://users.ox.ac.uk/~engs0161/4me6.html

KA and SA solutions



Kinematically admissible displacements (KA):

$$\mathcal{K}(\boldsymbol{u}^D, \partial\Omega^d) = \{ \quad \boldsymbol{v} | \quad \boldsymbol{v} = \boldsymbol{u}^D \quad \text{on } \partial\Omega^d \}.$$

Statically admissible stresses (SA):

$$\mathcal{S}(\boldsymbol{b}^D, \boldsymbol{t}^D, \partial\Omega^t) = \{ \quad \boldsymbol{s} | \quad \operatorname{div} \boldsymbol{s} + \boldsymbol{b}^D = 0 \quad \text{in } \Omega, \quad \boldsymbol{s} \cdot \boldsymbol{n} = \boldsymbol{t}^D \quad \text{on } \partial\Omega^t \}$$

Strain energy

$$\mathcal{U}_i(\boldsymbol{v}) = \int_\Omega \left(\boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}[\boldsymbol{v}] + \frac{1}{2}\boldsymbol{\varepsilon}[\boldsymbol{v}] : \boldsymbol{C} : \boldsymbol{\varepsilon}[\boldsymbol{v}] - \boldsymbol{\varepsilon}[\boldsymbol{v}] : \boldsymbol{C} : \boldsymbol{A}\theta^D \right) \, dv$$

Can be defined over arbitrary kinematically admissible (KA) field of displacements (and corresponding strains and stresses).

$$\boldsymbol{v} \in \mathcal{K}(\boldsymbol{u}^D, \partial\Omega^d)$$

Work of external forces
(over *virtual* KA displacement field by given external forces):

$$\mathcal{U}_e(\boldsymbol{v}) = -\int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}^D \, dv - \int_{\partial\Omega^t} \boldsymbol{v} \cdot \boldsymbol{t}^D \, ds.$$

Strain energy potential:

$$\mathcal{U}_p(\boldsymbol{v}) = \mathcal{U}_e(\boldsymbol{v}) + \mathcal{U}_i(\boldsymbol{v})$$

$$\int_\Omega \left(\boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}[\boldsymbol{v}] + \frac{1}{2}\boldsymbol{\varepsilon}[\boldsymbol{v}] : \boldsymbol{C} : \boldsymbol{\varepsilon}[\boldsymbol{v}] - \boldsymbol{\varepsilon}[\boldsymbol{v}] : \boldsymbol{C} : \boldsymbol{A}\theta^D \right) \, dv - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}^D \, dv - \int_{\partial\Omega^t} \boldsymbol{v} \cdot \boldsymbol{t}^D \, ds$$

Complementary energy

$$\mathcal{U}_i^*(\boldsymbol{s}) = -\frac{1}{2}\int_\Omega (\boldsymbol{s} - \boldsymbol{\sigma}_0 + \boldsymbol{C} : \boldsymbol{A}\theta^D) : \boldsymbol{C}^{-1} : (\boldsymbol{s} - \boldsymbol{\sigma}_0 + \boldsymbol{C} : \boldsymbol{A}\theta^D)\, dv$$

Can be defined over arbitrary statically admissible (SA) field of stresses (and corresponding strains and displacements).

$$\boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{b}^D, \boldsymbol{t}^D, \partial\Omega^t)$$

Work of external forces
(*trial* tractions due to SA stress field
over <u>given</u> external displacements):

$$\mathcal{U}_e^*(\boldsymbol{s}) = \int_{\partial\Omega} \boldsymbol{u}^D \cdot (\boldsymbol{s} \cdot \boldsymbol{n})\, ds$$

Complementary energy potential:

$$\mathcal{U}_p^*(\boldsymbol{s}) = \mathcal{U}_e^*(\boldsymbol{s}) + \mathcal{U}_i^*(\boldsymbol{s})$$

$$-\frac{1}{2}\int_\Omega (\boldsymbol{s} - \boldsymbol{\sigma}_0 + \boldsymbol{C} : \boldsymbol{A}\theta^D) : \boldsymbol{C}^{-1} : (\boldsymbol{s} - \boldsymbol{\sigma}_0 + \boldsymbol{C} : \boldsymbol{A}\theta^D)\, dv + \int_{\partial\Omega} \boldsymbol{u}^D \cdot (\boldsymbol{s} \cdot \boldsymbol{n})\, ds$$

Equality of potentials

Theorem: Equality of potentials If $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is the solution of a regular elastic problem with vanishing residual stresses and temperature change field,

$$\boldsymbol{\sigma}_0 = 0, \qquad \theta^D = 0,$$

then the following equality between the strain energy potential and the complementary energy potential holds:

$$\mathcal{U}_p(\boldsymbol{u}) = \mathcal{U}_p^*(\boldsymbol{\sigma}).$$

Proof

$$\mathcal{U}_p(\boldsymbol{u}) - \mathcal{U}_p^*(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon}\, dv + \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{C}^{-1} : \boldsymbol{\sigma}\, dv - \int_\Omega \boldsymbol{u} \cdot \boldsymbol{b}^D\, dv - \int_{\partial\Omega^t} \boldsymbol{u} \cdot \boldsymbol{t}^D\, ds - \int_{\partial\Omega^d} \boldsymbol{u}^d \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, ds$$

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}$$

$$\boldsymbol{u} \in \mathcal{K}(\boldsymbol{u}^D, \partial\Omega^d) \quad \text{and} \quad \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{b}^D, \boldsymbol{t}^D, \partial\Omega^t)$$

From Stokes:

$$\mathcal{U}_p(\boldsymbol{u}) - \mathcal{U}_p^*(\boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon} - \int_\Omega \boldsymbol{u} \cdot \boldsymbol{b}^D\, dv - \int_{\partial\Omega} \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, ds = 0$$

Extremum theorems

Theorem: The minimum property of the strain energy potential The displacement field $\boldsymbol{u}$ of the solution $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ of a regular thermoelatic problem *minimises* the strain energy potential over all kinematically admissible displacement fields:

$$\mathcal{U}_p(\boldsymbol{v}) \geq \mathcal{U}_p(\boldsymbol{u}) \qquad \forall \boldsymbol{v} \in \mathcal{K}(\boldsymbol{u}^D, \partial\Omega^d).$$

Theorem: The maximum property of the complementary energy potential The stress field $\boldsymbol{\sigma}$ of the elastic solution $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ of a regular thermoelatic problem *maximises* the complementary energy potential over all statically admissible stress fields:

$$\mathcal{U}_p^*(\boldsymbol{\sigma}) \geq \mathcal{U}_p^*(\boldsymbol{s}) \qquad \forall \boldsymbol{s} \in \mathcal{S}(\boldsymbol{b}^D, \boldsymbol{t}^D, \partial\Omega^t)$$

Theorem: The inequality of strain energy potential and complementary energy potential Let $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ be the solution of a regular elastic problem with vanishing residual stresses and temperature change field:

$$\boldsymbol{\sigma}_0 = 0 \qquad \theta^D = 0.$$

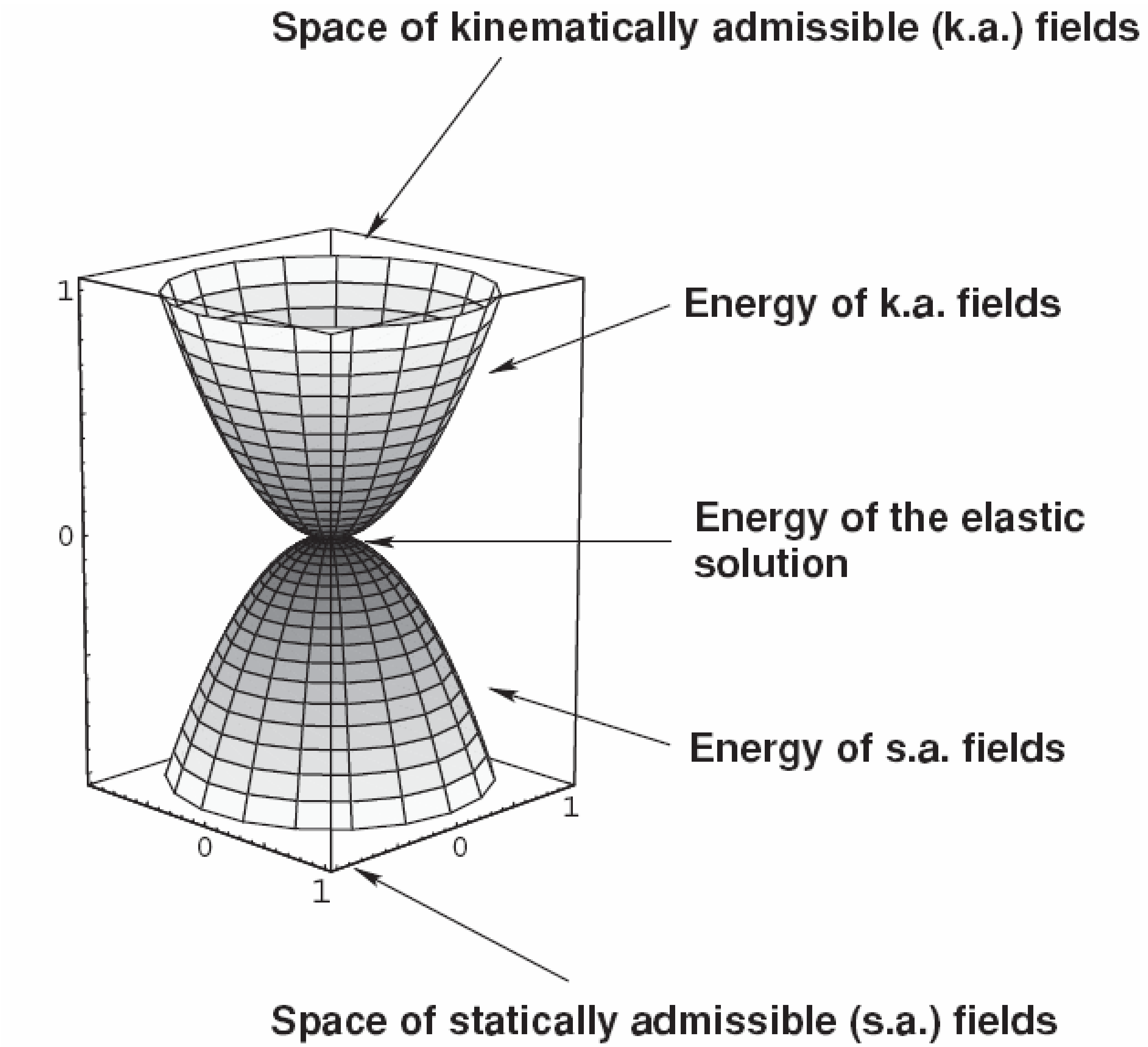
Then the following inequalities between the strain energy potential and the complementary energy potentials hold,

$$\mathcal{U}_p(\boldsymbol{v}) \geq \mathcal{U}_p(\boldsymbol{u}) = \mathcal{U}_p^*(\boldsymbol{\sigma}) \geq \mathcal{U}_p^*(\boldsymbol{s}),$$

where $\boldsymbol{v}$ and $\boldsymbol{s}$ are respectively the kinematically admissible displacement and statically admissible stress fields (see Figure 7.2 for illustration),

$$\boldsymbol{v} \in \mathcal{K}(\boldsymbol{u}^D, \partial\Omega^d) \qquad \boldsymbol{s} \in \mathcal{S}(\boldsymbol{b}^D, \boldsymbol{t}^D, \partial\Omega^t).$$

Potential inequalities



Properties of solutions

Theorem: Properties of solution at extremum Consider the solution $(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ of a regular thermoelastic problem. Then the equalities below hold for the derivatives of the energy potentials:

- Strain energy potential:

$$\mathrm{D}\mathcal{U}_p[\boldsymbol{u}](\boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \mathcal{K}(\mathbf{0}, \partial\Omega^d). \tag{7.26}$$

  The above equality can also be expressed as an extended integral formula:

$$\int_\Omega \boldsymbol{\sigma}[\boldsymbol{u}] : \boldsymbol{\varepsilon}[\boldsymbol{v}]\, dv - \int_\Omega \boldsymbol{b}^D \cdot \boldsymbol{v}\, dv - \int_{\partial\Omega^t} \boldsymbol{t}^D \cdot \boldsymbol{v}\, ds = 0 \qquad \forall \boldsymbol{v} \in \mathcal{K}(\mathbf{0}, \partial\Omega^d). \tag{7.27}$$

- Complementary energy potential:

$$\mathrm{D}\mathcal{U}_p^*[\boldsymbol{\sigma}](\boldsymbol{s}) = 0 \qquad \forall \boldsymbol{s} \in \mathcal{S}(\mathbf{0}, \mathbf{0}, \partial\Omega^t). \tag{7.28}$$

  The above equality can also be expressed as an extended integral formula:

$$-\int_\Omega (\boldsymbol{\sigma}[\boldsymbol{u}] - \boldsymbol{\sigma}_0 - \boldsymbol{C} : \boldsymbol{A}\theta^D) : \boldsymbol{\varepsilon}[\boldsymbol{v}]\, dv + \int_{\partial\Omega^d} \boldsymbol{u}^D \cdot (\boldsymbol{s} \cdot \boldsymbol{n})\, ds = 0 \quad \forall \boldsymbol{s} \in \mathcal{S}(\mathbf{0}, \mathbf{0}, \partial\Omega^t). \tag{7.29}$$

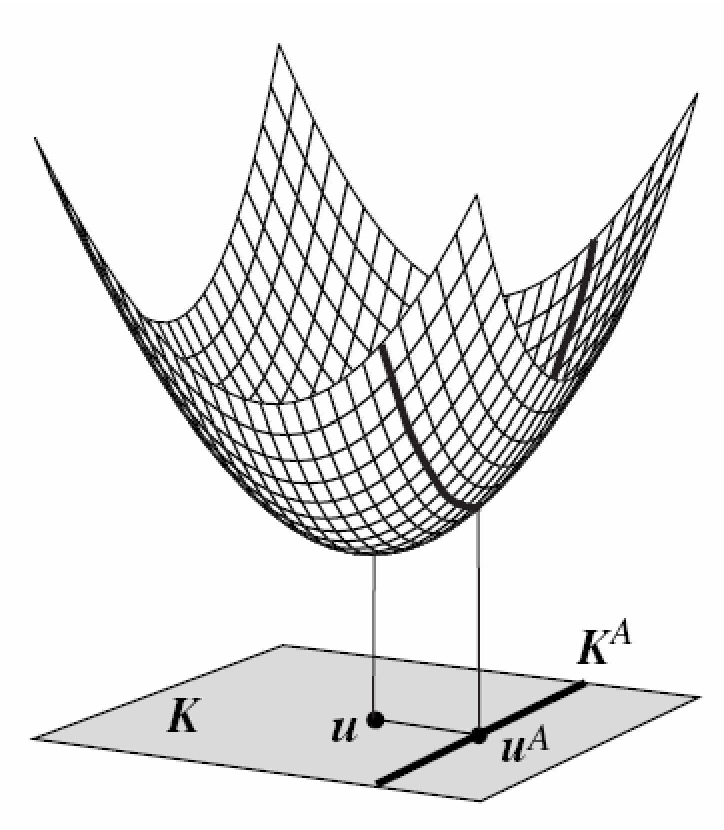
Approximate solutions

The approximate displacement solution $\boldsymbol{u}^A$ within the subset of admissible fields $\mathcal{K}_A \subset \mathcal{K}(\boldsymbol{u}^D, \partial\Omega^d)$ is the solution that minimises the strain energy potential over the subset $\mathcal{K}_A$ of admissible displacement fields; that is,

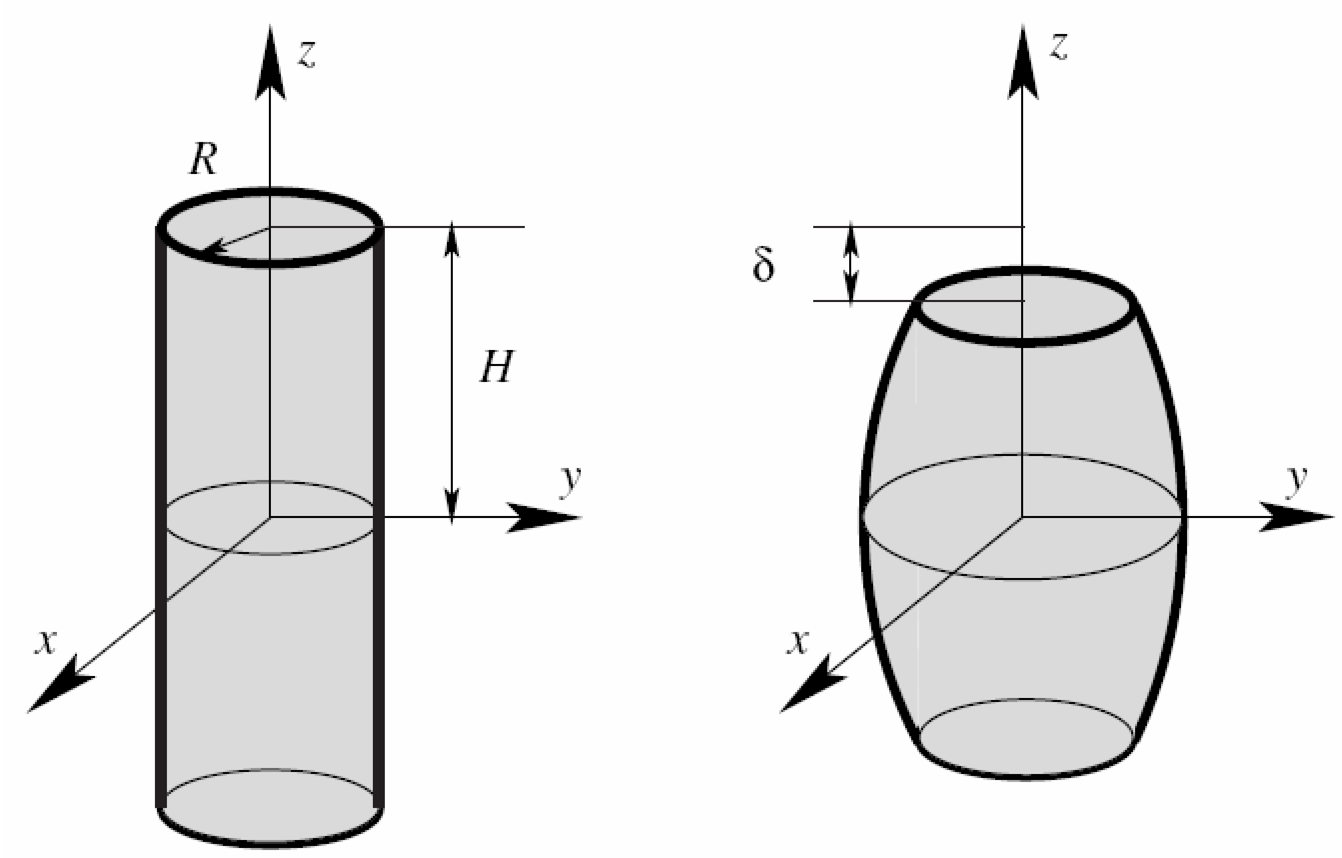
$$\boldsymbol{u}^A = \arg \min_{v \in \mathcal{K}_A} \mathcal{U}_p[\boldsymbol{v}].$$

The approximate stress solution $\boldsymbol{\sigma}^A$ within the subset of statically admissible stress fields $\mathcal{S}_A \subset \mathcal{S}(\boldsymbol{b}^D, \boldsymbol{t}^D, \partial\Omega^t)$ is the solution that maximises the complementary energy potential over the subset $\mathcal{S}_A$ of admissible stress fields; that is,

$$\boldsymbol{\sigma}^A = \arg \max_{s \in \mathcal{S}_A} \mathcal{U}_p^*[\boldsymbol{s}].$$



Compression of a cylinder



Conclusions

In this course we have covered a range of topics from continuum mechanics:

- energy minimisation and variational methods
- direct numerical solution methods based on trial functions and energy minimisation (Rayleigh-Ritz)
- connection between the variational approach and the FEM
- fundamentals equations elasticity and methods of solution
- fundamental singular solutions of elasticity, and the connection with the boundary element method
- energy principles and approximate solutions

Although the topics presented in this course were quite varied, my central purpose has been to help you see the way in which they are all connected in a logical way, through common principles, approaches and results.

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