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Derivation of the Born Rule from Operational Assumptions

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The Born rule is derived from operational assumptions, together with assumptions of quantum mechanics that concern only the deterministic development of the state. Unlike Gleason's theorem, the argument applies even if probabilities are defined for only a single resolution of the identity, so it applies to a variety of foundational approaches to quantum mechanics. It also provides a probability rule for state spaces that are not Hilbert spaces.

1 Introduction

Whence the Born rule? It is fundamental to quantum mechanics; it is the essential link between probability and a formalism which is otherwise deterministic; it encapsulates the measurement postulates. Gleason's theorem [4] is mathematically informative, but its premises are too strong to have any direct operational meaning: in what follows the Born rule is derived more simply, given certain operational assumptions, along with the elements of the deterministic formalism of quantum mechanics.

The argument we shall present is based on Deutsch's derivation of the Born rule from decision theory [2]. The latter was criticized by Barnum *et al* [1], but their objections hinged on ambiguities in Deutsch's notation that have since been resolved by Wallace [12]; here we follow Wallace's formulation. The argument is not quite the same as his, however: Wallace draws heavily on the Everett interpretation, as well as on decision theory; like Deutsch, he is concerned to derive constraints on "subjective" probability, rather than any objective counterpart to it. In contrast, the derivation of the Born rule that follows is independent of decision theory, independent of the interpretation of probability, and independent of the Everett interpretation. As such it applies to a variety of foundational approaches to quantum mechanics. But it does assume from the outset that there *is* a probability rule for the outcomes (pointer readings) of experiments, determined by the state and observable measured. Any further details of the measuring process or the experimental design we suppose concern only the question of *which* state, *which* observable, and *which* associated pointer readings are relevant to the experiment.[1] Approaches to foundations that require additional information - values of hidden variables for instance - to define

[1] This assumption plays much the same role as does "measurement neutrality", in Wallace's treatment. (My thanks to Jerry Finkelstein on this point.)

the probabilities of the observational outcomes of experiments will be immune to it (in which case, of course, the values of these variables will not exactly be *hidden*).

Assuming there is a general algorithm of this sort it will be sufficient to establish it in the case of a particular class of experiments. The argument then takes the following form: there is one class of experiments for which there is a clear operational procedure for associating with a given experiment a particular state, observable and associated set of pointer readings (a model for the experiment). But in general *different* models of the same experiment can be provided in this way; since the probabilities for the observable outcomes of the experiment must be the same, independent of the particular model used in interpreting it, the algorithm for determining them must give the same output in each case. This puts a constraint on the algorithm if it is to be consistent with the operational procedure for modeling experiments of this kind. In the case where the ratios in the modulus squares of the relevant amplitudes are rational numbers, this constraint is in fact sufficient to force the Born rule.

To derive the Born rule quite generally, a continuity assumption is required: sufficiently small variations in the state-preparation device, and hence of the initial state, should yield small variations in expectation value. But this is a natural assumption from an operational point of view (with respect to whatever topology is used to define a notion of closeness in the state space of a physical theory); the same applies to sufficiently small variations in the other quantities on which the algorithm depends (the observable measured and pointer readings); likewise for any algorithm for determining physically measurable quantities.

2 Multiple-Channel Experiments

The kinds of experiments we shall consider are limited in the following respects: they are repeatable; there is a clear distinction between the state preparation device and the detection and registration device; and - this the most important limitation - we assume that for a given state-preparation device, preparing the system to be measured in a definite initial state, the state can be resolved into *channels,* each of which can be independently blocked, in such a way that when only one channel is open the outcome of the experiment is *deterministic* - in the sense that if there is any registered outcome at all (on repetition of the experiment) it is always the *same* outcome.

Two other assumptions are made for convenience, to assist in the argument that follows. We suppose that for every outcome there is at least one channel for which it is deterministic, and - in order to associate a definite initial state with a particular region of the apparatus - we suppose that all the channels are recombined in some particular region of the apparatus, prior to the detection process proper.

For an example of such an experiment that measures spin, consider a neutron interferometer in which a beam-splitter produces orthogonal states of spin (with respect to a given axis) in each of the two arms of the interferometer; these are then recombined at some point prior to the measurement of that particular component of spin. For an example that measures position, consider an optical two-slit experiment, adapted so that the lensing system after the slits first brings the light into coincidence at some point, and then focuses the light on two detectors in such a way that each detector can receive light from only one of the two slits. It is not too hard to specify an analogous procedure in the case of momentum (the conventional method for preparing a beam of charged partciles of definite momentum - by selecting for deflection in a magnetic field - can be adapted quite simply). Any number of familiar experiments can be converted into an experiment of this kind.

We introduce the following notation. Let there be d channels in all, with $D \leq d$ possible outcomes u_j in some set of outcomes U, $j = 1, ..., D$. These outcomes are macroscopic events (e.g. positions of pointers). Let M denote the experiment that is performed when all the channels are open, and M_k, $k = 1, ..., d$ the (deterministic) experiment that is performed when only the k^{th} channel is open. Let there be identifiable regions $r_1, r_2, ...$ of the state-preparation device through which the system to be measured must pass (if it is to be subsequently detected at all - regardless of which channels are open). Call an experiment performed using such an apparatus a *multiple-channel experiment.* Whether or not an experiment falls into this category is clearly decidable on purely operational grounds.

3 Experimental Models

One could go further, and provide operational definitions of the initial states in each case, but we are looking for a probability algorithm that can be applied to states that are mathematically defined (so any operational definition of the initial state, *e.g.* in terms of the specification of the state-preparation device, would eventually have to be converted into a mathematical one). We therefore assume we already have the mathematical specification of states $\psi_1, \psi_2, ...$[2] at given regions $r_1, r_2, ..$ of the apparatus, as elements of a complex Hilbert space H (which for convenience we take to be finite dimensional[3]). We further assume the conventional, schematic description of experiments in quantum mechanics: we suppose an experiment is designed to measure some self-adjoint operator $\widehat{X}$ on H, and that on measurement one of a finite number of probabilistic outcomes labelled by real numbers $\lambda_k \in Sp(\widehat{X})$ results, $k = 1, ..., d$ (we allow for repetitions, i.e. for some $j \neq k$ we may have $\lambda_j = \lambda_k$). (In the case of the Everett interpretation, we say rather that *all* of the macroscopic outcomes result, but that each of them is in a different branch, with a given amplitude. We will consider the interpretation of probabilty in the Everett interpretation in due

[2] "State" is used loosely, and applies to vectors in Hilbert space as well as rays. (We will eventually deduce the independence of probabilities from both the overall phase *and* norm.)

[3] It would be just as easy to work with Hilbert spaces of countably inifnite dimension, and restrict instead the observables to self-adjoint operators with purely point spectra. (The difficulty with observables with continuous spectra is purely technical, however.)

course.) We suppose that these events are reliably amplified and recorded by some non-probabilistic physical process $\Omega : Sp(\widehat{X}) \rightarrow U$, yielding one or other of D possible macroscopic outcomes $u_j \in U$. The probabilities p_j, $j = 1, .., D$ of the latter are the quantities reflected in the observable statistical data.

We take it that all parties agree that in an experiment there is some set of probabilistic events labelled by the λ_k's, and some reliable amplification and recording process Ω resulting in observational records, independent of questions of foundations - although just what the λ_k's lable, and just what probability means, may vary from one approach to the next. But it may be that Ω is many-one, so will not assume (as is usual) the numerical equality of $\Omega(\lambda_k)$ with λ_k. We do assume the records $u_j \in U$ are numerals, so that addition and multiplication operations can be defined on them. (For convenience we assume none of them is the zero.)

Call the triple $\left\langle \psi, \widehat{X}, \Omega \right\rangle$ an *experimental model*, denote g. This scheme extends without any modification to experiments where there are inefficiencies in the detection and registration devices, so long as they are the same for every channel; but a more sophisticated scheme will be needed if the efficiencies differ from one channel to the next (we neglect this complication here).

This scheme applies to a much wider variety of experiments than multiple-channel experiments; the Born rule is conventionally stated in just these terms. We are seeking an algorithm that assigns expectation values to models, i.e. weighted averages of the quantities u_j, with weights given by the probabilities p_j of each u_j, $j = 1, ..., D$. We are therefore looking for a map $V : g \rightarrow R$ of the form:

$$V[\psi, \widehat{X}, \Omega] = \sum_{j=1}^{D} p_j u_j, \quad \sum_{j=1}^{D} p_j = 1. \tag{1}$$

If $D = d$ we can write the u_j's directly in terms of the $\Omega(\lambda_k)$'s. Otherwise define $\lambda^{-1}(u_j) = \{k : \Omega(\lambda_k) = u_j\}$, $j = 1, ..., D$, and real numbers $w_k \in [0, 1]$, $k = 1, ..., d$ such that $\sum_{k \in \lambda^{-1}(u_j)} w_k = p_j$. From Eq.(1) we obtain:

$$V[\psi, \widehat{X}, \Omega] = \sum_{k=1}^{d} w_k \Omega(\lambda_k), \quad \sum_{k=1}^{d} w_k = 1. \tag{2}$$

Conversely, given any d real numbers $w_k \in [0, 1]$ satisfying Eq.(2), define the D numbers $p_j = \sum_{k \in \lambda^{-1}(u_j)} w_k$; from Eq.(2) we obtain Eq.(1).

Evidently for fixed ψ and $\widehat{X}$ (and events λ_k) the amplification and registration process Ω (assumed to be reliable) only affects expectation values V in the way shown in Eq.(2); that is, the w_k's are independent of Ω.[4] But they are otherwise model-dependent; the p_j's, in contrast, are directly measurable. The latter are the quantities that matter for the consistency condition.

[4] My thanks on this point to Michael Dickson.

4 The Consistency Condition

As we shall see, in the special case of multiple-channel experiments there is a clear operational criterion for assigning models to experiments. But in general it is many-one; the same experiment may be modelled in many different ways. There follows an important constraint: for if M is assigned models g and g', and if there is to be any general algorithm V for mapping models to real numbers, then these expectation values had better agree, i.e. $V(g) = V(g')$. We view this as a *consistency condition* on V and the operational criterion for model assignments.

That a constraint of this kind played a tacit role in Deutsch's derivation was recognized by Wallace; it was used explicitly in Wallace's deduction [12] of the Born rule, although there it was cast in a slightly different form, and the conditions for its use were stated in terms of the Everett theory of measurement (including the theory of the detection and registration process). Here we make do with operational criteria, and with assumptions about the behavior of the state *prior* to any detection events. We suppose the prior evolution of the state is purely deterministic, and governed by the unitary formalism of quantum mechanics.[5]

Consider a multiple-channel experiment M. By assumption, there are d deterministic experiments M_k, $k = 1, ..., d$ that can also be performed with this apparatus, on blocking every channel save the k^{th}, each yielding one of the D macroscopic outcomes $u_j \in U$. Given that the initial state in region r for M_k is φ_k, and that M_k is deterministic, it is clear enough, on operational grounds, as to what can be counted as a model of M_k: the experiment measures any $\widehat{X}$ such that $\widehat{X}\varphi_k = \lambda_k \varphi$, for any λ_k and any Ω such that $\Omega(\lambda_k) \in U$ is the observable outcome of M_k.

Now consider the indeterministic experiment M performed with every channel open. We suppose that the state of M at r is $\psi = \sum_{k=1}^{d} c_k \varphi_k$; then the observable measured is any $\widehat{X}$ such that $\widehat{X}\varphi_k = \lambda_k \varphi_k$ for $k = 1, ..., d$, and any Ω such that $\Omega(\lambda_k) \in U$ is the observable outcome of each M_k.

This criterion can be stated as a definition that applies equally to the deterministic case ($d = D = 1$):

Definition 1 *Let M have d channels and D outcomes. Then M realizes $\left\langle \psi, \widehat{X}, \Omega \right\rangle$ if and only if:*

(i) for some region r and orthogonal states $\{\varphi_k\}$, φ_k is the state of M_k in r, $k = 1, ..., d \geq D$, and $\psi = \sum_{k=1}^{d} c_k \varphi_k$ is the state of M in r,

(ii)$\widehat{X}\varphi_k = \lambda_k \varphi_k$, $k = 1, ..., d$ where at least D of the λ_k's are distinct.[6]

[5] Of course in its initial phases the process of state preparation will involve probabilistic events, if only in collimating particles produced from the source, or in blocking particular channels. But it does not matter what these probabilities are; all that matters is that *if* a particle is located in a given region of the apparatus, *then* it is in a definite state, and unitarily develops in a definite way (prior to any detection or registration process).

[6] This follows from our assumption that for any multiple-channel experiment M, for each $u_j \in U$, $j = 1, .., D$, there is at least one M_k for which u_j is deterministic.

(iii) $\Omega(\lambda_k)$ is the outcome of M_k, $k = 1, ..., d$.

Why is it right to model experiments in this way and not some other? The deterministic case speaks for itself; in the indeterministic case, the short answer is that it is underwritten by the linearity (prior to any measurement) of the equations of motion. An apparatus that *deterministically* measures each eigenvalue λ_k of $\widehat{X}$, when the state in a given region of the apparatus is φ_k, will *indeterministically* measure the eigenvalues λ_k of $\widehat{X}$, when the state in that region is in a superposition of the φ_k's. This principle is implicit in standard laboratory procedures; this is how measuring devices are standardly calibrated, and how their functioning is checked.

The consistency condition now reads:

Definition 2 *V is consistent if and only if $V(g) = V(g')$ whenever g and g' can be realized by the same experiment.*

In the deterministic case evidently:

$$V[\varphi_k, \lambda_k \widehat{P}_{\varphi_k}, \Omega] = \Omega(\lambda_k). \tag{3}$$

We will show that if V is consistent, with $\langle , \rangle$ the inner product on H, then[7]

$$V[\psi, \widehat{X}, \Omega] = \frac{\left\langle \psi, \Omega(\widehat{X})\psi \right\rangle}{\langle \psi, \psi \rangle}. \tag{4}$$

Eq.(4) is the Born rule.

We begin with some simple consequences of the consistency condition. The Born rule is then derived in stages: first for equal norms in the simplest possible case of a spin half system; then for the general case of equal norms; and then for rational norms. The general case of irrational norms is handled by a continuity condition. We shall also derive a probability rule for more general vector spaces than Hilbert spaces.

5 Consequences of the Consistency Condition

We prove four general constraints on V that follow from consistency. (Eqs.(5),(8) may be found in Wallace [12], derived from somewhat different assumptions; Eqs.(6), (7) are special cases of more general equivalences derived by Wallace.) In each case an equality is derived from the fact that there exists an experiment that realizes two *different* models: by consistency, each must be assigned the same expectation value.

We assume it is not in doubt that there do exist such multiple-channel experiments, in which the initial state (prior to any detection or amplification process) evolves unitarily in the manner stated.

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[7] Since $\Omega : R \to U$, $\Omega(\widehat{X})$ is not an operator on H; but since numerical operations are defined on the outcomes U we may define $< \psi, \Omega(\widehat{X})\psi >$ by the spectral decomposition of $\widehat{X}$ (as $\sum_k \Omega(\lambda_k) < \psi, \widehat{P}_{\varphi_k} \psi >$)

Proposition 3 *Let V be consistent. It follows*

(i) for invertible $f : R \rightarrow R$:

$$V[\psi, \widehat{X}, \Omega] = V[\psi, f(\widehat{X}), \Omega \circ f^{-1}]. \tag{5}$$

(ii) For orthogonal projectors $\{\widehat{P}_k\}$, $k = 1, .., d$, such that $\widehat{P}_k \varphi_j = \delta_{kj} \varphi_j$

$$V[\sum_{k=1}^{d} c_k \varphi_k, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega] = V[\sum_{k=1}^{d} c_k \varphi_k, \sum_{k=1}^{d} \lambda_k \widehat{P}_k, \Omega]. \tag{6}$$

(iii) For $\widehat{U}_\theta : \varphi_k \rightarrow e^{i\theta_k} \varphi_k$, $k = 1, ..., d$, for arbitrary $\theta_k \in [0, 2\pi]$

$$V[\psi, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega] = V[\widehat{U}_\theta \psi, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega]. \tag{7}$$

(iv) For $\widehat{U}_\pi : \varphi_k \rightarrow \varphi_{\pi(k)}$, where π is any permutation of $< 1, ..., d >$

$$V[\psi, \widehat{X}, \Omega] = V[\widehat{U}_\pi \psi, \pi^{-1}(\widehat{X}), \Omega]. \tag{8}$$

Proof. Let $g = \left\langle \psi, \widehat{X}, \Omega \right\rangle$ be realized by M with d channels. Then for some region r_1 the state of M_k is φ_k, $k = 1, .., d$, that of M is $\sum_{k=1}^{d} c_k \varphi_k$, and there exist (not necessarily distinct) real numbers $\lambda_1, ..., \lambda_k$ such that $\widehat{X} \varphi_k = \lambda_k \varphi_k$, $\Omega(\{\lambda_k\}) = U$. Since for invertible f, $\Omega[f^{-1}(f(\lambda_k)] = \Omega(\lambda_k)$, $f(\widehat{X})\varphi_k = f(\lambda_k)\varphi_k$, M realizes $\left\langle \psi, f(\widehat{X}), \Omega \circ f^{-1} \right\rangle$, and (i) follows from consistency. Further, M realizes any other model $\left\langle \psi, \widehat{X'}, \Omega \right\rangle$ such that $\widehat{X'} \varphi_k = \lambda_k \varphi_k$; $\sum_{k=1}^{d} \lambda_k \widehat{P}_k$ is such a $\widehat{X'}$, so (ii) follows from consistency. Let now M be such that ψ evolves unitarily to the state $\widehat{U}_\theta \psi$ in region r_2. Then at r_2 the state of each M_k is $e^{i\theta_k} \varphi_k$, and since $\widehat{P}_{e^{i\theta_k} \varphi_k} = \widehat{P}_{\varphi_k}$, M also realizes $\left\langle \widehat{U}_\theta \psi, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega \right\rangle$, and (iii) follows from consistency. Let ψ subsequently evolve to the state $\widehat{U}_\pi \psi$ in region r_3. Then at r_3 the state of each M_k is $\varphi_{\pi(k)}$, and the state of M is $\sum_{k=1}^{d} c_k \varphi_{\pi(k)}$. Without loss of generality, we may write $\widehat{X}$ as $\sum_{k=1}^{d} \lambda_k \ \widehat{P}_{\varphi_k}$; then $\pi^{-1}(\widehat{X}) = \sum_{k=1}^{d} \lambda_k \ \widehat{P}_{\varphi_{\pi(k)}}$ satisfies $\pi^{-1}(\widehat{X}) \varphi_{\pi(k)} = \lambda_k \varphi_{\pi(k)}$, so M also realizes $\left\langle \widehat{U}_\pi \psi, \pi^{-1}(\widehat{X}), \Omega \right\rangle$, and (iv) follows from consistency ■

Eqs.(5)-(8) are of course trivial consequences of the Born rule, Eq.(4). Note further that in each case the observables whose expectation values are identified *commute* - these are constraints among probability assignments to projectors belonging to a *single* resolution of the identity. Finally, note that the normalization of the initial state ψ played no role in the proofs.

6 Case 1: The Stern-Gerlach Experiment for Equal Norms

Consider the Stern-Gerlach experiment with $d = D = 2$. Let $\widehat{X} = \frac{1}{2}\widehat{P}_+ - \frac{1}{2}\widehat{P}_- = \widehat{\sigma}_z$ (in conventional notation), the observable for the z-component of spin with eigenstates $\varphi_\pm$, and let $\psi = c_+\varphi_+ + c_-\varphi_-$. Let $\widehat{U}_\pi$ interchange φ_+ and φ_-, so $\widehat{U}_\pi\widehat{\sigma}_z\widehat{U}_\pi^{-1} = -\widehat{\sigma}_z$. From Prop. 3(iv) it follows that:

$$V[c_+\varphi_+ + c_-\varphi_-, \widehat{\sigma}_z, \Omega] = V[c_+\varphi_- + c_-\varphi_+, -\widehat{\sigma}_z, \Omega]. \tag{9}$$

From Eq.(9) and Prop. 3(i):

$$V[c_+\varphi_+ + c_-\varphi_-, \widehat{\sigma}_z, \Omega] = V[c_+\varphi_- + c_-\varphi_+, \widehat{\sigma}_z, \Omega \circ -I] \tag{10}$$

(where $(\Omega \circ -I)(x) = \Omega(-x)$). From Eq.(10), in the special case that $|c_+|^2 = |c_-|^2$, and using Prop.3(iii) to compensate for any differences in phase:

$$V[c_+\varphi_+ + c_-\varphi_-, \widehat{\sigma}_z, \Omega] = V[c_+\varphi_+ + c_-\varphi_-, \widehat{\sigma}_z, \Omega \circ -I]. \tag{11}$$

Consider the LHS of this equality. From Eq.(2), writing $w_1 = w$, $w_2 = 1 - w$, $\Omega(\pm\frac{1}{2}) = \Omega(\pm)$ (so that $\Omega(+)$ results with probability w, and $\Omega(-)$ results with probability $1-w$) we obtain the expectation value $x = w\Omega(+) + (1-w)\Omega(-)$. But by similar reasoning, the RHS yields $w\Omega(-) + (1-w)\Omega(+) = -x + \Omega(+) + \Omega(-)$ (note the initial state and observable measured are the same, so w is independent of Ω). Equating the two, $x = \frac{1}{2}[\Omega(+) + \Omega(-)]$.

We have shown, for $|c_+|^2 = |c_-|^2$:

$$V[c_+\varphi_+ + c_-\varphi_-, \widehat{\sigma}_z, \Omega] = \frac{1}{2}\Omega(+) + \frac{1}{2}\Omega(-) \tag{12}$$

in accordance with the Born rule. Note that here we have derived an expectation values in a situation (dimension 2) where Gleason's theorem does *not* apply. (Note also that the normalization of the initial state ψ is again irrelevant to the result.)

7 Case 2: General Superpositions of Equal Norms

Consider an arbitrary observable on any $d-$dimensional subspace H^d of Hilbert space. By the spectral theorem, we may write $\widehat{X} = \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}$, for some set of orthogonal vectors $\{\varphi_k\}$, $k = 1, ..., d$ spanning H^d, where there may be repetitions among the λ_k's. Let ψ be a (not-necessarily normalized) vector in H^d; then for some d-tuple of complex numbers $< c_1, ..., c_d >$, $\psi = \sum_{k=1}^{d} c_k\varphi_k$. For any permutation π, we have from Prop.3(iv), (i):

$$V[\sum_{k=1}^{d} c_k\varphi_k, \widehat{X}, \Omega] = V[\sum_{k=1}^{d} c_k\varphi_{\pi(k)}, \pi^{-1}(\widehat{X}), \Omega] = V[\sum_{k=1}^{d} c_k\varphi_{\pi(k)}, \widehat{X}, \Omega \circ \pi]. \tag{13}$$

If $|c_k|^2 = |c_j|^2$, $j, k = 1, ..., d$, using Prop.3(iii) as before to adjust for any phase differences, it follows:

$$V[\psi, \widehat{X}, \Omega] = V[\psi, \widehat{X}, \Omega \circ \pi]. \tag{14}$$

Since ψ and $\widehat{X}$ are the same on both sides we obtain from Eq.(2):

$$\sum_{k=1}^{d} w_k \Omega(\lambda_k) = \sum_{k=1}^{d} w_k \Omega(\lambda_{\pi(k)}). \tag{15}$$

Eq.(15) holds for any permutation; let π interchange j and k, and otherwise act as the identity. There follows:

$$w_j \Omega(\lambda_j) + w_k \Omega(\lambda_k) = w_k \Omega(\lambda_j) + w_j \Omega(\lambda_k). \tag{16}$$

Conclude that if $\Omega(\lambda_j) \neq \Omega(\lambda_k)$ then $w_k = w_j$ (recall that by convention the $\Omega(\lambda_k)$'s are non-zero).

If $D = d$, evidently $w_k = w_j$ for all j,$k = 1, ..., d$. Since $\sum_k^d w_k = 1$, $w_k = \frac{1}{d}$, $k = 1, ..., d$. Therefore

$$V[\psi, \widehat{X}, \Omega] = \frac{1}{d} \sum_{k=1}^{d} \Omega(\lambda_k). \tag{17}$$

If $D < d$, suppose $\Omega(\lambda_j) = \Omega(\lambda_k)$ for $j, k = 1, ..., b < d$. (If $b = d$ Eq.(17) follows trivially.) For any j, k such that $b < j \leq d$, $k \leq b$, $\Omega(\lambda_k) \neq \Omega(\lambda_j)$, from which we conclude as before that $w_k = w_j$. Note further that under the stated conditions, $1/d = |c_k|^2 (\sum_{j=1}^d |c_j|^2)^{-1}$. We have proved:

Theorem 4 *Let $\psi = \sum_{k=1}^d c_k \varphi_k$, where $|c_k|^2 = |c_j|^2$ for all $j, k = 1, ..., d$. Let V be consistent. Then:*

$$V[\sum_{k=1}^{d} c_k \varphi_k, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega] = \sum_{k=1}^{d} \frac{|c_k|^2}{\sum_{j=1}^d |c_j|^2} \Omega(\lambda_k). \tag{18}$$

Like Prop.3, Th.4 is independent of the normalization of ψ.

8 Case 3: d=2 Normalized Superpositions with Rational Norms

The idea for extending these methods to treat the case of unequal but rational norms is as follows: consider an experiment in which the initial state ψ evolves deterministically so that each component φ_k entering into the initial superposition with amplitude c_k evolves into a superposition of $z_k \in N$ orthogonal states of equal norm $1/\sqrt{z_k}$, such that $|c_k/\sqrt{z_k}|^2$ is constant for all k. One can then show that the experiment has a model in which the initial state is a superposition of states of equal norms, so Th.4 can be applied. (Evidently for this to work the $|c_k|^2$'s will have to be rational numbers.)

For simplicity, consider first the case $d = 2$ for real amplitudes. Let $\psi = \frac{\sqrt{m}}{\sqrt{m+n}}\varphi_1 + \frac{\sqrt{n}}{\sqrt{m+n}}\varphi_2$, where m and n are integers. Let $\widehat{X} = \lambda_1 \widehat{P}_{\varphi_1} + \lambda_2 \widehat{P}_{\varphi_2}$. We will show that if V is consistent, $V[\psi, \widehat{X}, \Omega] = \frac{m}{m+n}\Omega(\lambda_1) + \frac{n}{m+n}\Omega(\lambda_2)$. Let the deterministic experiments of M be M_1, M_2, with registered outcomes $\Omega(\lambda_1)$, $\Omega(\lambda_2)$ respectively. Let the initial states of M, M_1, M_2 in region r be ψ, φ_1, φ_2 respectively. Then M realizes $g = \left\langle \psi, \widehat{X}, \Omega \right\rangle$. Now let ψ evolve to $\widehat{U}\psi$ in region r', where $\widehat{U}\varphi_1 = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \chi_k$, $\widehat{U}\varphi_2 = \frac{1}{\sqrt{n}} \sum_{k=m+1}^{n+m} \chi_k$, for some orthogonal set of vectors $\{\chi_k\}$, $k = 1, ..., m+n$. Denote $\lambda_1 \widehat{P}_{\widehat{U}\varphi_1} + \lambda_2 \widehat{P}_{\widehat{U}\varphi_2}$ by $\widehat{X'}$. Then the initial state of M_i, $i = 1, 2$ at r' is $\widehat{U}\varphi_i$, whilst that of M at r' is $c_1\widehat{U}\varphi_1 + c_2\widehat{U}\varphi_2$; since $\widehat{X'}\widehat{U}\varphi_i = \lambda_i \widehat{U}\varphi_i$ it follows that M realizes $g' = \left\langle \widehat{U}\psi, \widehat{X'}, \Omega \right\rangle$. By consistency, $V(g) = V(g')$. Now define $\widehat{P}_1 = \lambda_1 \sum_{k=1}^{m} \widehat{P}_{\chi_k}$, $\widehat{P}_2 = \lambda_2 \sum_{k=m+1}^{n+m} \widehat{P}_{\chi_k}$; since $\widehat{P}_k \widehat{U}\varphi_j = \delta_{kj} \widehat{U}\varphi_j$, $k, j = 1, 2$, by Prop.3(ii) $V[\widehat{U}\psi, \widehat{X'}, \Omega] = V[\widehat{U}\psi, \lambda_1 \widehat{P}_1 + \lambda_2 \widehat{P}_2, \Omega]$. But $\widehat{U}\psi = \frac{1}{\sqrt{m+n}} \sum_{k=1}^{n+m} \chi_k$; applying Th.4 for $d = m+n$, and noting that $\Omega(\lambda_k) = \lambda_1$ for $k = 1, ..., m$, and λ_2 otherwise, the result follows.

9 Case 4: General Superpositions with Rational Norms

The argument just given assumed ψ was normalized to one. The standard rational for this is of course based on the probabilistic interpretation of the state, and hence, at least tacitly, on the Born rule. It may be objected that we are only able to derive the dependence of the expectation value on the *squares* of the norms of the initial state, because this is put in by hand from the beginning. But this suspicion is unfounded. Suppose, indeed, only that $\frac{|c_1|^2}{|c_2|^2} = \frac{m}{n}$. As before, define $\widehat{U}\varphi_1 = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \chi_k$, $\widehat{U}\varphi_2 = \frac{1}{\sqrt{n}} \sum_{k=m+1}^{n+m} \chi_k$.The state $\widehat{U}\psi$ at region r_2 will have whatever normalization ψ had in r_1; the states $\widehat{U}\varphi_i$ $i = 1, 2$ will be eigenstates of $\widehat{P}_i$, as before; Def.1,2 will apply as before. Conclude that if V is consistent, $V[\psi, \widehat{X}, \Omega] = V[\widehat{U}\psi, \lambda_1 \widehat{P}_1 + \lambda_2 \widehat{P}_2, \Omega]$, as before. The difference is that now $\widehat{U}\psi = \frac{c_1}{\sqrt{m}} \sum_{k=1}^{m} \chi_k + \frac{c_2}{\sqrt{n}} \sum_{k=m+1}^{n+m} \chi_k = \frac{c_1}{\sqrt{m}} \sum_{k=1}^{n+m} \chi_k = \frac{c_2}{\sqrt{n}} \sum_{k=1}^{n+m} \chi_k$ (adjusting the phases of c_1 and c_2 , using Prop.3(ii), as required). Evidently we have an initial state which is a superposition of $n+m$ components of equal norm, m of which yield outcome $\Omega(\lambda_1)$ and n of which yield outcome $\Omega(\lambda_2)$. Since $\frac{m}{n+m} = \frac{|c_1|^2}{|c_1|^2+|c_2|^2}$, $\frac{n}{n+m} = \frac{|c_2|^2}{|c_1|^2+|c_2|^2}$ there follows:

$$V[\psi, \lambda_1 \widehat{P}_{\varphi_1} + \lambda_2 \widehat{P}_{\varphi_2}, \Omega] = \frac{|c_1|^2}{|c_1|^2 + |c_2|^2}[\Omega(\lambda_1)] + \frac{|c_2|^2}{|c_1|^2 + |c_2|^2}[\Omega(\lambda_2)]. \quad (19)$$

Evidently the normalization of ψ is irrelevant.

This result is worth proving in full generality:

Theorem 5 *Let $c_k \in C$, $k = 1, ..., d$ satisfy $|c_k| > 0$, $\frac{|c_k|^2}{|c_j|^2} \in Z$ and let V be consistent. Then:*

$$V[\sum_{k=1}^{d} c_k \varphi_k, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega] = \sum_{k=1}^{d} \frac{|c_k|^2}{\sum_{j=1}^{d} |c_j|^2} \Omega(\lambda_k). \tag{20}$$

Proof. Choose $c \in C$, $z_k \in Z$, $\theta_k \in [0, 2\pi]$, $k = 1, .., d$ such that $c_k = ce^{i\theta_k}\sqrt{z_k}$, and integers m_k, n such that $z_k = \frac{m_k}{n}$, $k = 1, ..., d$. Let $\{\chi_i\}$, $i = 1, ..., s$ be an orthonormal basis on an $s-$dimensional Hilbert space H^s, where $s = \sum_{k=1}^{d} m_k$ (we may suppose for $i = 1, ..., d$, $\chi_i = \varphi_i$). Let $s_k = \sum_{j=1}^{k} m_j$, $k = 1, ..., d$, $s_0 = 0$, and define $\widehat{U}$ on H^s by the action $\widehat{U}\varphi_k = \frac{1}{\sqrt{m_k}} \sum_{i=s_{k-1}+1}^{s_k} \chi_i$. Let $\widehat{P}_k = \sum_{i=s_{k-1}+1}^{s_k} \widehat{P}_{\chi_i}$, $k = 1, ..., d$. Let $\psi = \sum_{k=1}^{d} c_k \varphi_k$, and suppose that M realizes $g_1 = \left\langle \psi, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega \right\rangle$; then for some region r_1, the initial state of M is ψ and the state of each M_k is φ_k with outcome $\Omega(\lambda_k)$. Let the state of M at r_2 be $\widehat{U}\psi$; since $\widehat{P}_k \widehat{U} \varphi_j = \delta_{kj} \widehat{U} \varphi_j$, $j, k = 1, ..., d$, M also realizes $g_2 = \left\langle \widehat{U}\psi, \sum_{k=1}^{d} \lambda_k \widehat{P}_k, \Omega \right\rangle$, so by consistency $V(g_1) = V(g_2)$. But by construction

$$\widehat{U}\psi = \sum_{k=1}^{d} c_k \widehat{U} \varphi_k = \sum_{k=1}^{d} \frac{ce^{i\theta_k}}{\sqrt{n}} \sum_{j=s_{k-1}+1}^{s_k} \chi_j \tag{21}$$

so $V(g_2) = V[\frac{c}{\sqrt{n}} \sum_{k=1}^{s} \chi_k, \sum_{k=1}^{d} \lambda_k \widehat{P}_k, \Omega]$ (by Prop.3(ii)). Now define $\sigma_i \in R$, $i = 1, ..., s$ by $\sigma_i = \lambda_k$ for $s_{k-1} < i \leq s_k$; then $\sum_{k=1}^{d} \lambda_k \widehat{P}_k = \sum_{i=1}^{s} \sigma_i \widehat{P}_{\chi_i}$ and Th.4 can be applied. Conclude that $V(g_1) = V(g_2) = \frac{1}{s} \sum_{k=1}^{s} \Omega(\sigma_k) = \sum_{k=1}^{d} \frac{m_k}{s} \Omega(\lambda_k)$; since $\frac{m_k}{s} = m_k (\sum_{j=1}^{d} m_j)^{-1} = z_k (\sum_{j=1}^{d} z_j)^{-1} = |c_k|^2 (\sum_{j=1}^{d} |c_j|^2)^{-1}$, the result follows directly. ■

Examination of the proof shows that the dependence of probabilities on the modulus square of the expansion coefficients of the state ultimately derives from the fact that we are concerned with unitary evolutions on Hilbert space, specifically an *inner-product* space, and not some general normed linear topological space. A general class of norms on the latter is of the form $\|x\| = \left(\sum_{k=1}^{d} |\xi_k|^p \right)^{1/p}$, $1 \leq p < \infty$ (d may also be taken as infinite). Such spaces (l^p spaces) are metric spaces and can be completed in norm. The proof as we have developed it would apply equally to a theory of unitary (i.e. invertible norm-preserving) motions on such a space, yielding the probability rule

$$V[\sum_{k=1}^{d} c_k \varphi_k, \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}, \Omega] = \sum_{k=1}^{d} \frac{|c_k|^p}{\sum_{j=1}^{d} |c_j|^p} \Omega(\lambda_k) \tag{22}$$

(assuming that $\frac{|c_i|^p}{|c_j|^p} \in Z$, $j, k = 1, ..., d$). But of the L^p spaces, only $p = 2$ is an inner product space (a Hilbert space).

10 Case 5: Arbitrary States

There are a variety of possible strategies for the treatment of irrational norms, but the one that is most natural, given that we are making use of operational criteria for the interpretation of experiments, is to weaken these criteria in the light of the limitations of *realistic* experiments. In practise, one would not expect precisely the same state to be prepared on each run of the experiment. Properly speaking, the statistics actually obtained will be those for an ensemble of experiments; correspondingly, they should be obtained from a family of models, differing slightly in their initial states. We should therefore speak of *approximate* models (or of models that are *approximately* realized) - where the differences among the models are small.

How small is small? What is the topology on the space of states? The obvious answer, from a theoretical point of view, is the norm topology, since this is the topology used to obtain a complete state space in quantum mechanics (a Hilbert space). We should suppose that for sufficiently small ϵ, so long as $|\psi - \psi'| < \epsilon$, then if $\left\langle \psi, \widehat{X}, \Omega \right\rangle$ is an approximate model for M then so is $\left\langle \psi', \widehat{X}, \Omega \right\rangle$. Indeed, from an operational point of view, $\widehat{X}$ and Ω should likewise be subject to small variations. (Only the outcome set U can be regarded as precisely specified, insofar as the outcomes are numerals.)

In the case of multiple-channel experiments of course in practise each channel will correspond to a very large subspace of the Hilbert space. There will be many (perhaps infinitely many) orthogonal states producable at the source in each channel, any of which (for a given channel) will yield the same macroscopic outcome (if they yield any outcome at all). All that is required, for such an experiment, is that variations in each channel ψ'_k, ψ'_j, $k \neq j$ are sufficiently small such that these states remain orthogonal, and that each leads reliably to the same macroscopic outcome with the same efficiency. But then certainly they should lead to the same outcomes under the much stronger condition that $|\psi - \psi'| < \epsilon$.

With that it is clear that the details are hardly important; any algorithm that applies to families of models of this type, yielding expectation values, will have to be *continuous* in the norm topology. Given that, the extension of Th.5 to the irrational case is trivial. We define:

Definition 6 *Let $g^{(i)}$ be any sequence of models $\left\langle \psi^{(i)}, \widehat{X}, \Omega \right\rangle$, $i = 1, 2, ...$such that $\lim_{i \to \infty} |\psi^{(i)} - \psi| = 0$. Then V is continuous in norm only if $\lim_{i \to \infty} V(\ g^{(i)}) = V(g)$.*

We may finally prove:

Theorem 7 *Let V be consistent and continuous in norm. Then for any model $\left\langle \psi, \widehat{X}, \Omega \right\rangle$*

$$V[\psi, \widehat{X}, \Omega] = \frac{\left\langle \psi, \Omega(\widehat{X})\psi \right\rangle}{\langle \psi, \psi \rangle}. \tag{23}$$

Proof. It is enough to prove that any realizable model satisfies Eq.(23). If realizable, there is some multiple-channel experiment M with d channels and D outcomes that realizes $\left\langle \psi, \widehat{X}, \Omega \right\rangle$. Let $\{\varphi_k\},\ k = 1, .., d$ be any orthogonal family of vectors such that $\widehat{X}\varphi_k = \lambda_k \varphi_k$ (not all the λ_k's need be distinct). Without loss of generality, let $\psi = \sum_{k=1}^{d} c_k \varphi_k$, $\widehat{X} = \sum_{k=1}^{d} \lambda_k \widehat{P}_{\varphi_k}$. Let let $< c^1, c^2, .., c^{(i)}, ... >$ be any sequence of d-tuples of complex numbers such that for each i and each j, k, $\frac{|c_j^{(i)}|^2}{|c_k^{(i)}|^2} \in Z$, and for each $k \lim\limits_{i\to\infty} c_k^{(i)} = c_k$ (such a sequence can always be found). Let $\psi^{(i)} = \sum_{k=1}^{d} c_k^{(i)} \varphi_k$, $g^{(i)} = \left\langle \psi^{(i)}, \widehat{X}, \Omega \right\rangle$. By Th.5, $V[\psi^{(i)}, \widehat{X}, \Omega] = \sum_{k=1}^{d} \frac{|c_k^{(i)}|^2}{\sum_{j=1}^{d} |c_j^{(i)}|^2} \Omega(\lambda_k) = \frac{1}{\sum_{j=1}^{d} |c_j^{(i)}|^2} \sum_{k=1}^{d} \Omega(\lambda_k) \left\langle \psi^{(i)}, \widehat{P}_{\varphi_k} \psi^{(i)} \right\rangle$. The numerator is $\left\langle \psi^{(i)}, \Omega(\widehat{X})\psi^{(i)} \right\rangle$; since the denominator is non-negative with $\lim\limits_{i\to\infty} \sum_{j=1}^{d} |c_j^{(i)}|^2 = \sum_{j=1}^{d} |c_j|^2 = \langle \psi, \psi \rangle$, and since $\lim\limits_{i\to\infty} \left\langle \psi^{(i)}, \Omega(\widehat{X})\psi^{(i)} \right\rangle = \left\langle \psi, \Omega(\widehat{X})\psi \right\rangle$ (by the continuity of the inner product), Eq.(23) follows if V is continuous in norm. ■ The proof evidently extends *mutatis mutandis* to give the probability rule for l^p spaces, $p \neq 2$.

11 A Role for Decision Theory?

Is a continuity assumption permitted in the present context? Gleason's theorem does not require it; if one is going to do better than Gleason's theorem, it would be pleasant to derive the continuity of the probability measure, rather than to assume it. But from an operational point of view continuity is very natural; no algorithm that could ever be used is going to distinguish between states that differ infinitesimally.

Deutsch [2] took a rather different view. He was at pains to establish the Born rule for irrational norms, without assuming continuity. His method, however, was far from operational: along with axioms of decision theory, he assumed that quantum mechanics is *true* (under the Everett interpretation). A hybrid is possible: the present method can be supplemented with axioms of decision theory, yielding the Born rule for irrational norms, without any continuity assumption. But nothing much hangs on this question. One can do without a continuity assumption, but there are just as good reasons to invoke it from a decision theoretic point of view as from an operational one, as Wallace makes clear [12, p.47]. In neither case is there any reason to distinguish between states that differ infinitesimally.

Decision theory is important for a rather different reason: it matters because what Deutsch calls the non-probabilistic parts of decision theory (what Wallace more accurately calls decision theory in the face of uncertainty) *can provide an account of probability in terms of something else* - namely, in terms of the ordering of preferences of rational agents. This is important above all in the case of the Everett interpretation. According to many, the Everett interpretation has no place for probability [7]. Given Everett, any concept of probability has to be *justified*; it cannot be taken as *primitive*.

So it is clear why Deutsch took the more austere line: if Everett is to be believed, quantum mechanics *is* purely deterministic, even in the face of measurements. Deutsch supposed that the fundamental concept (that *can* be taken as primitive) is rather the *value* or the *utility* that an agent places upon a model - that $V(g)$ is in fact a utility. He argued that experiments should be thought of as games; for each registered outcome in U, we are to associate some utility, fixed in advance. So, in effect, the mapping $\Omega : \lambda_k \rightarrow \Omega(\lambda_k) \in U$ defines the *payoff* for the outcome λ_k.

Decision theory on this approach has a substantial role. If we suppose that the utilities of a rational agent are ordered, and satisfy very general assumptions ("axioms of rationality"), a representation theorem can be derived [10] which *defines* subjective probability in terms of the ordering of an agent's utilities. In effect, one deduces - in accordance with these axioms - that the agent acts *as if* she places such-and-such subjective probabilities on the outcomes of various actions.[8]

It is important that one can still make sense of uncertainty in this context, as Wallace explains. It may be we cannot help ourselves to probabilistic ideas *ab initio*, but that does not mean that one only deals with certainties - that games, in some sense, have only a *single* payoff, as Deutsch at one point suggests [2, p.3132-3]. From a first-person perspective, one does not know what outcome of a quantum game one should anticipate; one cannot anticipate them all, for there is no first-person perspective from which they can all be observed. So it is enough if, in the face of branching, a rational agent still has any anticipation of anything (that she does not expect *oblivion* [9]).

On this line of thought, the proofs of the Born rule just presented make an illegitimate assumption: Eq.(1). We are not entitled to assume that the observational outcomes $u_j \in U$, $j = 1, ..., D$ occur with probabilities p_j, for they all occur; no more that there are probabilities w_k for any underlying events λ_k, satisfying Eq.(2). But the proof of Th.4 (hence 5 and 7) depended on this assumption. Of course we may, with Deutsch and Wallace, eventually be in a position to make statements about the *subjective* probabilities of branches, but if so such statements will have to come at a later stage - *after* establishing the values $V(g)$ of various games. But then how are we to establish these values?

Here Wallace has provided a considerably more detailed derivation than Deutsch, and from weaker premises. But the proofs are correspondingly more complicated; for the sake of simplicity we shall only consider Deutsch's argument, removing the ambiguities of notation in the way shown by Wallace.

First, consider Case 1, the Stern-Gerlach experiment. All is in order up to Eq.(11), but we must do without the assumption subsequently made - that the registered outcome $\Omega(+)$ results with probability w, and outcome $\Omega(-)$ with probability $1 - w$. Here Deutsch invokes a new principle, what he calls *the zero-sum rule*:

[8] This does not mean that subjective probabilities are illusory, and correspond to nothing in reality. (The point is to *legitimate* the concept, not to abolish it.)

$$V[\varphi, \widehat{X}, \Omega] = -V[\varphi, \widehat{X}, -\Omega]. \tag{24}$$

Following Deutsch, let us assume that the numerical value of the utility $\Omega(\lambda_k)$ equals λ_k. Then, in the special case where $\lambda_1 = -\lambda_2$ (true for the measurement of a component of spin), from Eq.(24), applied to Eq.(11), we deduce:

$$V[c_1\varphi_1 + c_2\varphi_2, \widehat{\sigma}_z, \Omega] = -V[c_1\varphi_1 + c_2\varphi_2, \widehat{\sigma}_z, \Omega] \tag{25}$$

and hence that $V[c_1\varphi_1 + c_2\varphi_2, \widehat{\sigma}_z, \Omega] = 0$, in accordance with the Born rule in this special case.

Although evidently of limited generality, the result is illustrative - assuming the zero-sum rule can be independently justified. Here is a justification for it: banking too is a form of gambling; the only difference between acting as the gambler who bets, and as the banker who accepts the bet, is that whereas the gambler *pays* a stake in order to play, and *receives* payoffs according to the outcomes, the banker *receives* the stake in order to play, and *pays* the payoffs according to the outcomes. The zero-sum rule is the statement that the most that one will pay in the hope of gaining a utility is the least that one will accept to take the risk of losing it. We take this as the principle on which two rational agents must agree if they are to share the same utilities and play each other in a zero-sum game. (And evidently any quantum experiment can be used to play such a game.)

What of the general equal-norm case, Case 2? Here the zero-sum rule is not enough. But if we consider only the case $d = 2$, it is enough to supplement it with another rule, what Deutsch calls the *additivity* rule. A payoff function $\Omega : R \to U$ is *additive* if and only if $\Omega(x + y) = \Omega(x) + \Omega(y)$. Let $f_\tau : R \to R$ be the function $f_\tau(x) = x + \tau$; then V is *additive* if and only if

$$V[\psi, \widehat{X}, \Omega \circ f_\tau] = V[\psi, \widehat{X}, \Omega] + \Omega(\tau). \tag{26}$$

Additivity of the payoff function is a standard assumption of elementary decision theory, eminently valid for small bets (but hardly valid for large ones, or for utilities that only work in tandem). Additivity of V then has a clear rational. It is an example of a *sure-thing principle*: given two games, each exactly the same, save that in one of them one receives an additional utility $\Omega(\tau)$ *whatever* the outcome, then one should value that game an having an *additional* utility $\Omega(\tau)$.

To see how additivity can be used in Case 2 (but restricted to $d = 2$), observe that for $\tau = -\lambda_1 - \lambda_2$, the function $-I \circ f_\tau$ is the permutation π. Therefore from Eq.(14) we may conclude:

$$V[\psi, \widehat{X}, \Omega] = V[\psi, \widehat{X}, \Omega \circ -I \circ f_\tau]. \tag{27}$$

By additivity the RHS is $V[\psi, \widehat{X}, \Omega \circ -I] + (\Omega \circ -I)(\tau)$, and since Ω is additive (so $\Omega \circ -I = -\Omega$) we obtain, from the zero-sum rule

$$V[\psi, \widehat{X}, \Omega] = -V[\psi, \widehat{X}, \Omega] - \Omega(\tau). \tag{28}$$

With a further application of payoff additivity there follows

$$V[\psi, \widehat{X}, \Omega] = \frac{1}{2}[\Omega(\lambda_1) + \Omega(\lambda_2)] \tag{29}$$

in accordance with the Born rule.

As Wallace has shown, this, along with the higher dimensional cases ($d > 2$), can be derived from weaker axioms of decision theory, that do not assume additivity. Th.5 then goes through unchanged. As already remarked, one is then in a position to derive the extension to the irrational case without assuming continuity: for the details, I refer to Wallace [12].

Decision theory can evidently play a role in the derivation of the Born rule, but it is only needed if the notion of probability is itself in need of justification. That may well be so, in the context of the Everett interpretation; but on other approaches to quantum mechanics, probability, whatever it is, can be taken as given.

12 Gleason's Theorem

Compare our derivation of the Born rule with Gleason's theorem:

Theorem 8 *Let f be any function from 1-dimensional projections on a Hilbert space of dimension $d > 2$ to the unit interval, such that for each resolution of the identity $\{\widehat{P}_k\}$, $k = 1, ..., d$, $\sum_{k=1}^{d} \widehat{P}_k = I$, $\sum_{k=1}^{d} f(\widehat{P}_k) = 1$. Then there exists a unique density matrix ρ such that $f(\widehat{P}_k) = Tr(\rho \widehat{P}_k)$.*

Proof. Gleason (1967) ■

Our theorem is different: it says nothing about the uniqueness of the state, for assigning probabilities to resolutions of the identity, only about the uniqueness of the algorithm, for assigning values to measurement models. The latter range over Hilbert spaces of arbitrary dimension.

More important, on a variety of approaches to quantum mechanics, nothing so strong as Gleason's premise is really motivated. It is not required that probabilities can be defined for a projector independent of the family of projectors of which it is a member. This requirement, sometimes called *non-contextuality* [8], is a strong one. Very few approaches to quantum mechanics subscribe to it. The theorem has no relevance to any approach that singles out a unique basis once and for all: its premise is not assumed by the GRW theory [5], nor by the de Broglie-Bohm theory [6], which single out the position basis; nor by the Everett interpretation [13], which singles out a basis approximately localized in phase space; nor by the consistent histories approach [3], supposing probabilities are only defined for a unique decoherent history space. All these theories require only that probabilities be defined for projectors associated with the preferred basis (insofar as they can be associated with projectors at all). If probabilities can be assigned to any other projectors, it is only insofar as they can be correlated with projectors in the preferred resolution of the identity, as determined by some particular experimental context.

But so much is entirely compatible with the derivation that we have offered. By all means restrict Def.1 to observables compatible with a unique resolution of the identity (and likewise the consistency condition of Def. 2). Prop.3 proves identities for expectation values for commuting observables; it too can be restricted to a unique resolution of the identity; likewise in Th.4. In Th.5 an auxiliary basis was used, but again this can be taken as the preferred basis. And whilst it is in the spirit of Th.7 that probabilities should also be defined for small variations in projectors, this does not yet amount to the assumption of non-contextuality.

Unlike the premise of Gleason's theorem, the operational criteria that we have used are hardly disputed; they are common ground to all the main schools of foundations of quantum mechanics. But it would be wrong to suggest that they apply to all of them equally: on some approaches - in particular, those that provide a detailed dynamical model of measurements - there is good reason to suppose that an algorithm for expectation values will depend on additional factors (in particular, if any explicit stochastic process is introduced); the Born rule may no longer be forced in consequence. (But we take it that this would be an *unwelcome* consequence of these approaches; the Born rule will have to be *otherwise* justified - presumably, as it is in the GRW theory, as a *hypothesis.*)

Of the main schools, two - the Everett interpretation, and those based on operational assumptions (here we include the Copenhagen interpretation) - offer no such resources. This point is clear enough in the latter case; in the case of the Everett interpretation, the association of models with multiple channel experiments as given in Def.1 follows from the full theory of measurement. Quantum mechanics under the Everett interpretation provides no leeway in this matter. The same is likely to be true of any approach to quantum mechanics that preserves the unitary formalism intact, without any supplement to it.

The principal remaining schools have a rather different status. One, the state-reduction approach, has already been mentioned. The other is the hidden-variable approach, in which the state evolves unitarily even during measurements (but is incomplete). This case deserves further comment.

13 Completeness

The one approach to foundations in which the Born rule has been seriously questioned is an example of this type (the de Broglie-Bohm theory)[11]. Hidden variables certainly make a difference to the argument we have presented. Consider the proof of Th.4. The passage from Eq.(13) to Eq.(14) hinged on the fact that the state on both sides of Eq.(13) is *identical* when the norms of its components are the same. (Likewise the step from Eq.(10) to (11).) But if the state is incomplete, this is not enough to ensure the required identification. Including the state of the hidden variables as well (denote w), we should replace ψ by the pair $< \psi, \omega >$ (w may be the value of the hidden variable, or a probability distribution over its values). Doing this, there is no guarantee that in the case of superpositions of equal norms - e.g. for $\psi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)$, where φ_1, φ_2

are, as in Case 1, eigenstates of the $z-$component of spin - that $\widehat{U}_\pi$ (permuting φ_1 and φ_2) will act as the identity.[9] Although $\widehat{U}_\pi\psi = \psi$, its action on $< \psi, \omega >$ may well be different from the identity; how is the permutation to act on the hidden variables?

The question is clearer when $\widehat{U}_\pi$ implements a spatial transformation. We have an example where it does: the Stern-Gerlach experiment. In this case $\widehat{U}_\pi\widehat{\sigma}_z\widehat{U}_\pi^{-1} = -\widehat{\sigma}_z$, a reflection in the $x-y$ plane. Under the latter, a particle initially with positive z-coordinate ($\omega = +$) is mapped to one with negative z-coordinate ($\omega = -$). Under this same transformation, the superposition $\psi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)$ is unchanged. Therefore $\widehat{U}_\pi :< \psi, + > \to < \psi, - > \neq < \psi, + >$; there is no longer any reason to suppose that Eq.(11) will be satisfied.

This situation is entirely as expected. In the de Broglie-Bohm theory, given such an initial state ψ, it is well known that if the incident particle is located on one side of the plane of symmetry of the Stern-Gerlach apparatus, then it will always remain there. It is obvious that if the particles is always located on the same side of this plane, on repetition of the experiment, the statistics of the outcomes will disagree with the Born rule. It is equally clear that if particles are randomly distributed about this plane of symmetry then the Born rule will be obeyed - but that is only to say that the probability distribution for the hidden variables is determined by the state, in accordance with the Born rule. This is what we are trying to prove.

The deduction we have provided gives *half* of what is required in this case. Our strategy, recall, was to derive constraints on an algorithm - any algorithm - that takes as inputs experimental models and yields as outputs expectation values. The constraints apply even if the state is incomplete, even if there are additional parameters controlling individual measurement outcomes - so long as the state *alone* determines the statistical distribution of the hidden variables. Given that, then any symmetries of the state will also be symmetries of the distribution of hidden variables. In application to the de Broglie-Bohm theory, our result indeed implies that the particle distribution *must* be given by the Born rule - this is no longer an additional postulate of the theory - *so long as the particle distribution is determined only by the state.* The assumption is not that particles must be distributed in accordance with the Born rule, but that they are distributed by any rule at all that is determined by the state - and from that it follows it is the Born rule. (Whether this assumption is a natural one from the point of view of the de Broglie-Bohm theory is another matter.)

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