INTRODUCTION TO LOGIC

Lecture 6

Natural Deduction

Dr. James Studd

> There's nothing you can't prove
> if your outlook is only sufficiently limited
> *Dorothy L. Sayers*

Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

$$\dfrac{\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa} \qquad \dfrac{\forall z\,(Qz \to Rz)}{Qa \to Ra}}{\dfrac{Ra}{Pa \to Ra}}}{\forall y\,(Py \to Ry)}$$

- The root of the tree is the conclusion
- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
- The rules depend purely on the syntax of the sentences

  ...not on their semantic properties.

Rules for $\wedge$

$\wedge$Intro

The result of appending $\phi \wedge \psi$ to a proof of ϕ and a proof of ψ is a proof of $\phi \wedge \psi$.

$$\dfrac{\begin{matrix}\vdots \\ \phi\end{matrix} \qquad \begin{matrix}\vdots \\ \psi\end{matrix}}{\phi \wedge \psi}\,\wedge\text{Intro}$$

$\wedge$Elim1 and $\wedge$Elim2

(1) *The result of appending ϕ to a proof of $\phi \wedge \psi$ is a proof of ϕ.*

(2) *The result of appending ψ to a proof of $\phi \wedge \psi$ is a proof of ψ.*

$$\dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\phi}\,\wedge\textit{Elim1} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\psi}\,\wedge\textit{Elim2}$$

Example

$(P \wedge Q) \wedge R \vdash P$

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

Rules for $\rightarrow$

$\rightarrow$Elim

The result of appending ψ to a proof of ϕ and a proof of $\phi \rightarrow \psi$ is a proof of ψ.

$$\dfrac{\begin{matrix}\vdots \\ \phi\end{matrix} \qquad \begin{matrix}\vdots \\ \phi \rightarrow \psi\end{matrix}}{\psi} \rightarrow\text{Elim}$$

This rule is often called 'Modus Ponens'.

Example

$\exists y\, Py \rightarrow Qa, \exists y\, Py \vdash Qa$

$\rightarrow$Intro

The result of appending $\phi \rightarrow \psi$ to a proof of ψ and discharging all assumptions of ϕ in the proof of ψ is a proof of $\phi \rightarrow \psi$.

$$\dfrac{\begin{matrix}[\phi] \\ \\ \vdots \\ \psi\end{matrix}}{\phi \rightarrow \psi} \rightarrow\text{Intro}$$

Conditional proof in informal reasoning.

(1) If it's poison and Quintus took it, then he needs to be readmitted.
(2) It's poison
So (C) if Quintus took it, he need to be readmitted.

Informal proof. Suppose Quintus took it.
Then (by 2) It's poison *and* he took it.
Then (by 1 and MP) he needs to be readmitted.

So (by conditional proof) *if* Quintus took it, he needs to be readmitted.

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

We can now define $\Gamma \vdash \phi$.
Let Γ be a set of sentences and ϕ a sentence.

Definition (Provable)

The sentence ϕ is *provable* from Γ if and only if:

- there is a proof of ϕ with only sentences in Γ as non-discharged assumptions.

Notation

- $\Gamma \vdash \phi$ is short for ϕ is provable from Γ
- $\vdash \phi$ is short for $\emptyset \vdash \phi$
- $\psi_1, \ldots, \psi_n \vdash \phi$ is short for $\{\psi_1, \ldots, \psi_n\} \vdash \phi$.

Return to the rule of assumption.

ASSUMPTION RULE

The occurrence of a sentence ϕ with no sentence above it is an assumption. An assumption of ϕ is a proof of ϕ.

This may seem odd.
Suppose I assume, the following:

$$\exists x \exists y (Rxy \vee P)$$

By the rule, this counts as a proof of $\exists x \exists y (Rxy \vee P)$

But it is not an *outright* proof of $\exists x \exists y (Rxy \vee P)$

- This proof does *not* show $\vdash \exists x \exists y (Rxy \vee P)$
- Instead it shows $\exists x \exists y (Rxy \vee P) \vdash \exists x \exists y (Rxy \vee P)$

Rules for $\vee$

The introduction rules are straightforward.

$$\frac{\begin{array}{c}\vdots\\ \phi\end{array}}{\phi \vee \psi}\ \vee\text{Intro1} \qquad \frac{\begin{array}{c}\vdots\\ \psi\end{array}}{\phi \vee \psi}\ \vee\text{Intro2}$$

The elimination rule is a little more complex.

$$\dfrac{\begin{array}{c} \\ \vdots \\ \phi \vee \psi \end{array} \qquad \begin{array}{c} [\phi] \\ \vdots \\ \chi \end{array} \qquad \begin{array}{c} [\psi] \\ \vdots \\ \chi \end{array}}{\chi}\ \vee\text{Elim}$$

Proof by cases in informal reasoning

(1) Either you don't play and you quit or you do something quiet and don't play.
So, (C) You don't play.

Informal proof. Suppose (1)

Case (i): You don't play and you quit. So: you don't play

Case (ii): You do something quiet and don't play. So: you don't play.

Either way then, (C) follows: you don't play.

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

The rules for $\neg$

Here are the rules for $\neg$.

$$\dfrac{\begin{array}{c} [\phi] \\ \vdots \\ \psi \end{array} \qquad \begin{array}{c} [\phi] \\ \vdots \\ \neg\psi \end{array}}{\neg\phi}\ \neg\text{Intro} \qquad\qquad \dfrac{\begin{array}{c} [\neg\phi] \\ \vdots \\ \psi \end{array} \qquad \begin{array}{c} [\neg\phi] \\ \vdots \\ \neg\psi \end{array}}{\phi}\ \neg\text{Elim}$$

The proof technique is known as *reductio ad absurdum*.

Example

$\neg(P \to Q) \vdash \neg Q$

Rules for $\leftrightarrow$

These are reminiscent of the rules for $\rightarrow$

$$\dfrac{\begin{matrix}[\phi] & [\psi] \\ \vdots & \vdots \\ \psi & \phi\end{matrix}}{\phi \leftrightarrow \psi}\leftrightarrow\text{Intro}$$

$$\dfrac{\begin{matrix}\vdots & \vdots \\ \phi \leftrightarrow \psi & \phi\end{matrix}}{\psi}\leftrightarrow\text{Elim1} \qquad \dfrac{\begin{matrix}\vdots & \vdots \\ \phi \leftrightarrow \psi & \psi\end{matrix}}{\phi}\leftrightarrow\text{Elim2}$$

Rules for $\forall$

$$\dfrac{\begin{matrix}\vdots \\ \forall v\,\phi\end{matrix}}{\phi[t/v]}\forall\text{Elim}$$

In this rule:

- ϕ is a formula in which only the variable v occurs freely
- t is a constant
- $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

Substitution

$\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

- Recall: a free occurrence of v is one not bound by $\forall v$ or $\exists v$

Compute the following

- $Px[a/x] =$
- $\forall x Px[a/x] =$
- $\forall y(\exists x Px \vee Qx \rightarrow Py)[a/x] =$

Example

$\forall x\,(Px \rightarrow Qx), Pa \vdash Qa$

Here's the introduction rule for ∀

$$\frac{\begin{array}{c}\vdots\\ \phi[t/v]\end{array}}{\forall v\, \phi}\ \forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

Informal proof. Let an arbitrary thing be given.
Call it 'Jane Doe'.

Clearly, *if* Jane Doe is a pedestrian , then Jane Doe is either a qualified driver or a pedestrian .

So: every pedestrian is either a qualified driver or a pedestrian .

Example

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

Rules for ∃

The introduction rule is straightforward.

$$\frac{\phi[t/v]}{\exists v\, \phi}\ \exists\text{Intro}$$

Example

$Rcc \vdash \exists y\, Rcy$

The elimination rule is as follows.

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\, \phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka.
So, something is a quindarka.

So (C), follows from (1) and (2).

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

Let Γ be a set of $\mathcal{L}_2$-sentences and ϕ a $\mathcal{L}_2$-sentence.

Two notions of consequence

$\Gamma \vdash \phi$ iff there is a proof of ϕ with only sentences in Γ as non-discharged assumptions.

$\Gamma \vDash \phi$ iff there is no $\mathcal{L}_2$-structure in which all sentences in Γ are true and ϕ is false.

Theorem

(a) Soundness: $\Gamma \vdash \phi$ only if $\Gamma \vDash \phi$

(b) Completeness: $\Gamma \vDash \phi$ only if $\Gamma \vdash \phi$

Proof. Elements of Deductive Logic.