A Quick Presentation of the 1965 Diamond Model

Key assumptions:

- Each generation lives for 2 periods and has identical preferences over period 1 and period 2 consumption (X_t^1 and X_t^2 respectively, so that the utility function of the generation who are young in period t is $U_t = U(X_t^1, X_t^2)$)
- Capital stock K_t in each period determines the wage of young generation w_t via the marginal product of labour $\frac{dY}{dL}$, where $Y_t = F(K_t, L_t)$ and is determined by saving from the previous young generation, who are now old and receive a return on their savings determined by the interest rate r_t , in turn equal to the marginal product of capital $\frac{dY}{dK}$. Capital is assumed not to depreciate.
- The population L is growing at rate n so that $L_{t+1} = (1+n)L_t$ where L_t is the size of generation who are young in period t . Since the population is changing, the key dynamic variable is the *capital stock per worker* in period t , $\tilde{K}_t = \frac{K_t}{L_t}$.
- The behaviour of the Diamond economy can involve cycles where X_t^1 , X_t^2 and $\tilde{K}_t$ change between generations, but we are usually most interested in steady states where they can stay the same in a sustainable manner for all future generations.
- As we will show below, the Golden Rule (the steady state outcome which maximizes the utility of each generation) requires that the steady state interest rate be equal to n . This is called the *biological interest rate* because it is determined entirely by the fact that the population is growing. The role of the biological interest rate can be seen most clearly from the Samuelson (1958) model, in which each member of each generation simply has a wage of 1 when young, and there is no capital accumulation. In this model, because the new young generation is bigger next period, the total amount available from the savings $S_{t+1} = 1 - X_{t+1}^1$ of each individual member of the next generation (who are young in period t+1) is equal to $L_{t+1}S_{t+1} = (1+n)L_t S_{t+1}$. In a steady state, since X^1 is constant, so must be S . The total amount available from the savings of the next generation is therefore $(1+n)L_t S$, and this must be split between the members of the previous generation, who whom there are L_t . The total amount available for each member of the current young generation when old is therefore $(1+n)S$. Since they are required to save S each when young, they get a rate of return equal to $1+n$.

Key issues:

- As with the simpler Samuelson (1958) model without production, the equilibrium without the use of fiat money is the inefficient autarkic one with no inter-temporal trade and each generation consuming its entire wage in the first period. This is because, unlike in a standard general equilibrium model which results in efficient trade (e.g. via the Edgeworth box), once they are old, each generation has no endowment and therefore no bargaining/market power; they rely entirely on their claim to part of the product of the new young generation to be honoured. Unlike the assumptions for the First Theorem of Welfare Economics, which require trade to occur simultaneously, here money must act as an inter-temporal store of value. This requires a social compact (e.g. a stable price level or inflation indexed assets for the young to buy to sell when they are old, and a commitment to not politically renege on the compact, such as via windfall taxation of pensions). Here we should emphasize that both funded pension schemes based on fiat money and PAYG pension schemes are based upon a social compact, and that the old at any one time are vulnerable to being reneged upon in either system.

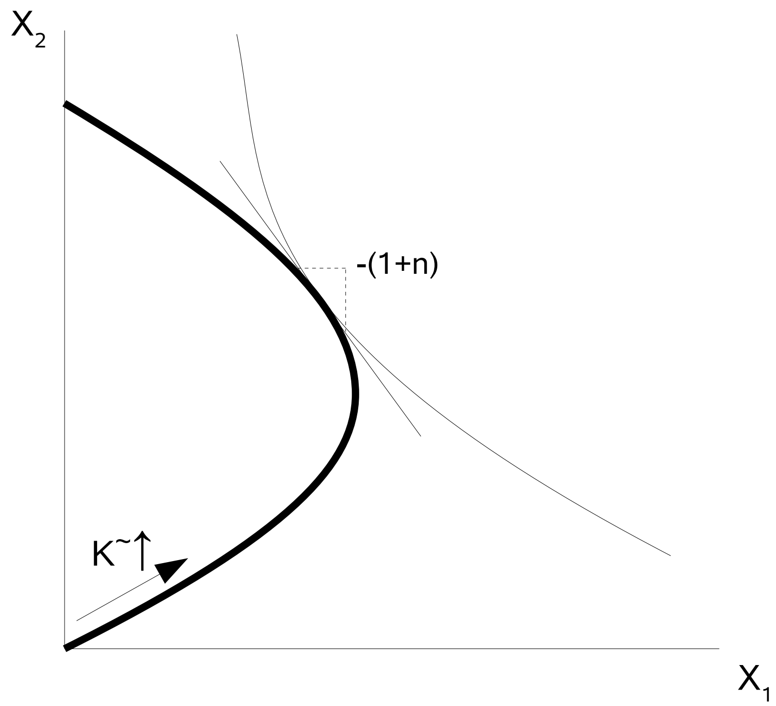
- The Diamond model, unlike the Samuelson model, is sufficiently complex for there to be a difference between PAYG and funded pension schemes (which, remember, are essentially inter-temporal taxes/transfers). This stems from the fact that there is no process built into the model to ensure that the amount saved by the current young generation will achieve the Golden Rule. The reason for this is that each generation is selfish, and does not internalize the externality that is caused by the capital stock they bequeath to the next generation determining their labour productivity and therefore wage. The young generation only takes into account the return they get, determined by the marginal product of capital. Whereas a fully funded government scheme will simply crowd out private saving, leaving the rate of return on the savings of the young, and therefore the steady state, unaffected, a PAYG scheme can alter the link between the amount saved by the young and the amount they get when old. With the optimal pension scheme (which is likely to be a mixture between the two systems), the rate of return on the savings of the young can be brought into line with the biological interest rate, thus fulfilling the Golden Rule.

Derivations:

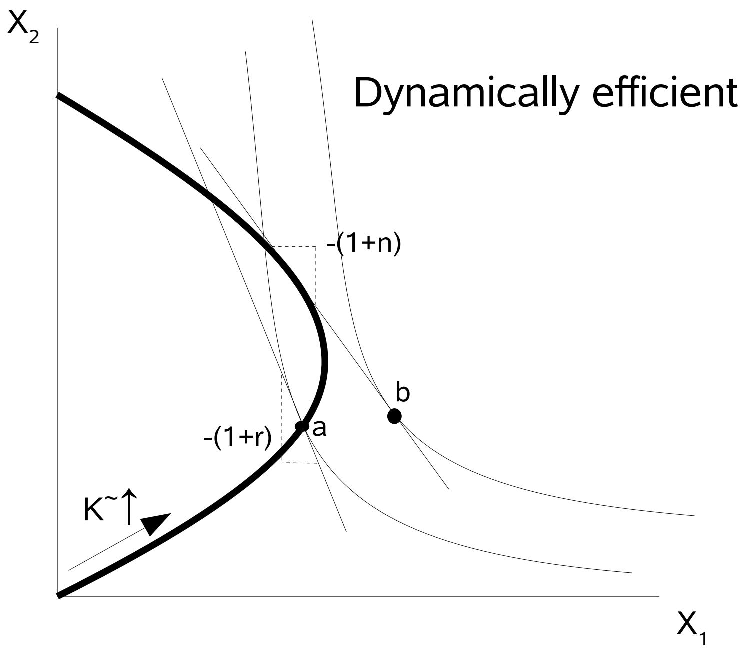
- <u>The per capita production function</u>. Since we are interested in per capita variables, the first thing we need to do is to define output per capita $\tilde{Y}=\frac{Y_t}{L_t}=\frac{F(K_t,L_t)}{L_t}$. If we assume that the production function has constant returns to scale, the $\frac{1}{L_t}$ can be "taken inside" the production function to give $\tilde{Y}=F\left(\frac{K_t}{L_t},1\right)=f(\tilde{K}_t)$. This is the production function for the individual based upon capital stock per individual.
- <u>The wage.</u> Total output is $Y_t=\tilde{Y}_t L_t=L_t f(\tilde{K}_t)$. By differentiating with respect to L_t and applying the product rule and chain rule we get $w_t=MPL=\frac{dY}{dL}=f(\tilde{K}_t)-L_t\left(\frac{K_t}{L_t^2}\right)f'(\tilde{K}_t)$. This simplifies to $w_t=f(\tilde{K}_t)-\tilde{K}_t f'(\tilde{K}_t)$.
- <u>The interest rate.</u> By differentiating total output with respect to K_t and applying the chain rule, we get $r_t=MPK=\frac{dY}{dK}=L_t\left(\frac{1}{L_t}\right)f'(\tilde{K}_t)=f'(\tilde{K}_t)$.
- <u>The Golden Rule.</u> The Golden Rule can most easily be derived from the capital accumulation identity. The capital stock in period $t+1$ must be equal to the amount saved in period t plus the capital stock in period t (since capital does not depreciate). The amount saved in turn is equal to the total product minus the total amount consumed by the young generation in period t and the old generation who were young in period $t-1$. So, $K_{t+1}=K_t+L_t\tilde{Y}_t-L_t X_t^1-L_{t-1}X_{t-1}^2$. Dividing through by $\frac{L_{t+1}}{1+n}$ and using the facts that $L_{t+1}=(1+n)L_t$ and $L_{t-1}=\frac{1}{(1+n)}L_t$, we get $(1+n)\tilde{K}_{t+1}=\tilde{K}_t+\tilde{Y}_t-X_t^1-\frac{1}{1+n}X_{t-1}^2$. Since in a steady state, all these variables are constant, we can drop the time indices and rearrange to give $f(\tilde{K})-n\tilde{K}=X^1+\frac{1}{1+n}X^2$. The LHS of this equation is the consumption possibilities set determined by the output per worker and the need to equip the growing population with more capital in order to keep $\tilde{K}$ constant. The RHS determines the cost of buying consumption in periods 1 and 2 out of this "budget". For the same reason as in the Samuelson model, the opportunity cost of period 2 consumption on terms of period 1 consumption is determined by the biological interest rate. Although many combinations of X_1 and X_2 could fulfil the capital accumulation identity, only when $r=n$ will the opportunity cost faced by individuals in the steady state reflect the true social opportunity

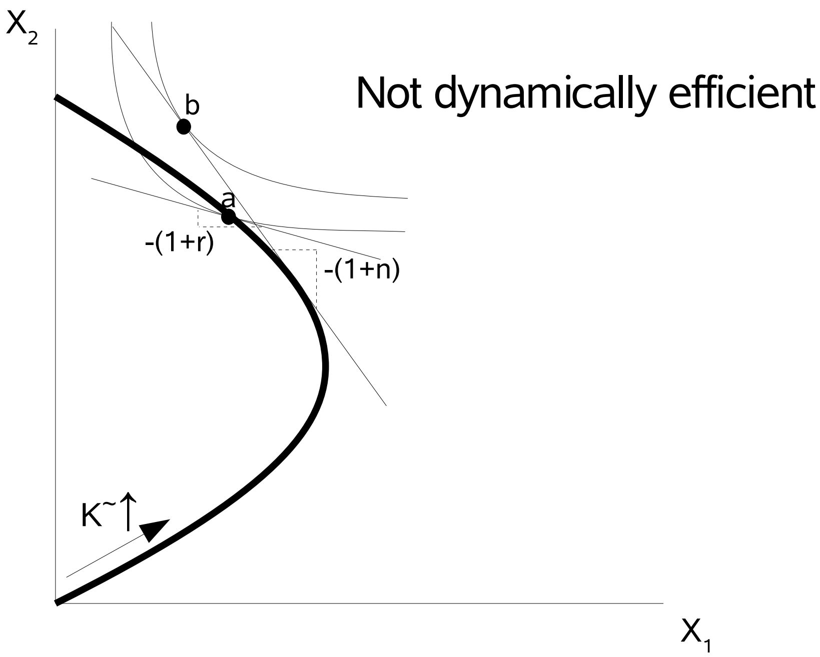
cost determined by the biological interest rate. Hence only the capital stock $\tilde{K}^*$ leading to an allocation where society's entire "budget" is used up and where the budget constraint has a slope $-(1+n)$ will be efficient in terms of maximizing the utility of the representative individual in the steady state.

- The result can be shown graphically using the consumption possibilities frontier. Each point on the CPF is generated by choosing a particular steady steady value of $\tilde{K}$, which then pins down values of X^1 and X^2 via its determination of the steady state wage and interest rate. Initially, as $\tilde{K}$ increases both X^1 and X^2 increase because each generation has a higher wage due to the capital bequeathed to it from the previous generation and so can afford to save enough to keep the per capita capital stock constant whilst still consuming more today and giving more to the current old generation. Eventually, however, declining MPK causes the cost of equipping new workers with capital to lead to a reduction in consumption possibilities in the first period if more consumption is to be gained in the second period.
- Rearranging the steady state capital accumulation identity gives us $X^2=(1+n)\left(f(\tilde{K})-n\tilde{K}\right)-(1+n)X^1$. The Golden Rule is where the consumption possibilities frontier has a slope of $-(1+n)$. If by chance the indifference curve of the representative individual is tangential at this point, then the Golden Rule will be the steady state equilibrium. This is illustrated below:



- In most cases, however the competitive steady state will not fulfil the Golden Rule. There are two possible situations, illustrated graphically below. The first is the case of *dynamic efficiency* - it is not possible to move from the fiat money equilibrium *a* to the Golden Rule equilibrium *b* without saving more in the current period to build up the capital stock. This requires either that the current young consume less, or that they renege on the current old and give them less. There are thus no Pareto improvements to be made. This of course does not imply that the current dynamic equilibrium maximizes the social welfare function. The second possibility is that of dynamic inefficiency. Here, Pareto improvements can be made by consuming the excess capital stock today to go from *a* to *b*, which is then sustainable as a Golden Rule steady state equilibrium for ever after.





Conclusion:

- Overlapping generations models are interesting because they provide an original efficiency rationale for inter-temporal redistribution. Such concerns do not occur in a standard static model with implicitly infinitely-lived consumers. The Diamond model shows us that a (partially) PAYG pension system may cause gains in dynamic efficiency as well as having an equity justification. OLG models also explicitly underline the fact that fiat money and inter-temporal political transfers (e.g. PAYG pensions) are social compacts needed to get out of the autarkic equilibrium with no inter-temporal trade. Fiat money, however, does not guarantee an efficient dynamic equilibrium in the Diamond model because there remains an externality from each generation to the labour productivity of the next generation via the capital stock bequeathed. Government intervention via inter-temporal transfers could, in theory, help to internalize this externality.