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Approximate Equilibrium in Economies with Indivisible Commodities*

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1. INTRODUCTION

This paper investigates how far the techniques and theorems of the study of competitive equilibrium may be extended to a model with indivisible commodities. Specifically, I prove the existence of what comes near to being an equilibrium in an economy with indivisible commodities, providing that at least one commodity is divisible (Theorem 4.11). The allocation described by the theorem is only "nearly" an equilibrium because it has two weaknesses. Firstly, it is only approximately feasible; and secondly, it is not quite certain that each individual prefers his allocated consumption to every other that he can afford.

The first defect seems to be an inevitable accompaniment of nonconvexity. Nonconvex sets are handled in equilibrium theory by a technique which depends essentially on "averaging." Because economics deals in price planes, and only convex sets have supporting hyperplanes, a nonconvexity appears as an inconvenient gap which has to be bridged. If one selects some points from round the edge of the gap, and then takes their average, the resulting point will be somewhere in the middle; it forms, in fact, an element of the bridge. Exactly that process is performed in models which depend on the replication principle. A similar method has become available for a more general case through a theorem by Shapley and Folkman (reported by Starr [10, Appendix 2]). Roughly, the Shapley–Folkman theorem says that the nonconvexities in an aggregate of nonconvex sets do not grow in size with the number of sets making up the aggregate. Thus, relative to the size of the aggregate economy, the nonconvexities become less important in a larger economy. This is the "averaging"; it depends on large numbers. The significant fact is that the

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theorem of near equilibrium determines an allocation which fails to be feasible only by an amount that is independent of the number of individuals in the economy.

There is no need to go in detail into these general problems of nonconvexity. Starr [10] has provided a study of them, and my method follows his closely. Further, Dierker [3, Section 2] gives a rigorous description of the way the necessary approximation becomes less important in a larger economy. These matters, therefore, will be passed over without much comment. It should be noted, though, in the comparison with Starr's work, that the breach in equilibrium is always in *feasibility*; it is what Arrow and Hahn [1, p. 177] call a "social-approximate" equilibrium. An "individual-approximate" equilibrium, where feasibility is maintained but preferences are not quite maximal, is in many ways more interesting; I have not, however, been able to construct one in the indivisible case.

Indivisibility entails nonconvexity, but it has also some special problems of its own; on these I shall place the emphasis of this paper. One of them causes the second flaw in the theorem of near equilibrium. There may be anomalous points within an individual's budget set which he strictly prefers to the consumption the theorem requires him to adopt. The existence of such points is unlikely but definitely possible. Their importance is discussed in Section 5, but we shall have to go more deeply into the model to see how they may appear.

Notation

Following are the meanings of some of the symbols to be used (I do not include those universally understood, nor those defined later). cl Z is the closure of Z. fr Z is the frontier of Z. int Z is the interior of Z. conv Z is the convex hull of Z. "$\notin$" means "is not a member of." "&" means "and." $N(z, \epsilon)$ is the open neighborhood of z with radius ϵ. $\dot{-}$ indicates set-theoretic subtraction: $Y \dot{-} Z$ is $\{y \in Y \mid y \notin Z\}$. Subscripts on vectors always refer to components: z_r is the r-th component of z. Lower case Greek letters stand for scalars. For vectors: "$y > z$" means "$\forall r \in \{1,\dots n\}$: $y_r > z_r$"; "$y \geqslant z$" means "$y \neq z$ & [$\forall r \in \{1,\dots n\}$: $y_r \geqq z_r$]"; "$y \geqq z$" means "$\forall r \in \{1,\dots n\}$: $y_r \geqq z_r$."

As a guide for readers, I repeat here the meanings of some of the particular terms which *are* defined later. $A^i(p)$—see Definition 4.4. $B^i(p) = \{x \mid p \cdot x \leqq p \cdot w^i\}$. $D^i(p)$—see Definition 4.4. $e = (1, 0, 0\dots 0)$. edge $Z = \{z \in Z \mid \forall\lambda > 0: z - \lambda e \notin Z\}$. $F = \{x \in E^n \mid [\forall r \in \{1,\dots n_d\}: x_r$ is real] & [$\forall r \in \{n_d + 1,\dots n\}: x_r$ is integral]$\}$. G^i—"bounding cube" defined after Lemma 4.3. g^i—lower bound on X^i. $I = \{1,\dots m\}$. m—number of consumers. n—number of commodities. n_d—number of divisible commodities. $P^i(\bar{x}) = \{x \in X^i \mid x \succcurlyeq^i \bar{x}\}$. $Q^i(\bar{x}) = \{x \in X^i \mid \bar{x} \succcurlyeq^i x\}$. rest $Z =$

$(Z \div \text{edge } Z)$. $S = \{p \mid \sum_{r=1}^{n} p_r = 1 \;\&\; [\forall r \in \{1,\dots n\}: p_r \geqq 0]\}$. w^i—endowment of i. $w^t = \sum_{i\in I} w^i$. X^i—consumption set. κ^i—see Assumption 2.8.

2. The Model and its Assumptions

The action takes place in a commodity space E^n, representing the n commodities of the economy. Some of these commodities can exist only in integral quantities, so there is a subset F of E^n which contains only those points which can actually exist; these will be called "proper points." $F = \{x \in E^n \mid [\forall r \in \{1,\dots n_d\}: x_r \text{ is real}] \;\&\; [\forall r \in \{n_d + 1,\dots n\}: x_r \text{ is integral}]\}$. n_d is the number of divisible commodities. Production is not included in the model because I believe that the extra complication involved in distributing profits would throw no light on the topic of indivisibilities (for a discussion of efficiency in production in a model with indivisible commodities, see Frank [5]). There are m consumers, indexed for convenience by $i \in \{1,\dots m\}$. Write the index set $\{1,\dots m\}$ as I. For each $i \in I$ there is a consumption set X^i, and X^i is always a subset of F. On each consumption set there is a relation $\succcurlyeq^i$, which is assumed to be a preorder (that is, it is transitive and reflexive). Define the "not-worse-than set" $P^i(\bar{x}) = \{x \in X^i \mid x \succcurlyeq^i \bar{x}\}$ and the "not-better-than set" $Q^i(\bar{x}) = \{x \in X^i \mid \bar{x} \succcurlyeq^i x\}$. As a matter of notation "$x \succ^i \bar{x}$" means "$x \succcurlyeq^i \bar{x}$ & [not $\bar{x} \succcurlyeq^i x$]" and "$x \sim^i \bar{x}$" means "$x \succcurlyeq^i \bar{x}$ & $\bar{x} \succcurlyeq^i x$". The most basic properties assumed on the pairs $(X^i, \succcurlyeq^i)$ are given in the following jumbo assumption.

2.1. Assumption. "Basic properties." $\forall i \in I$:

(a) $[X^i \subset F$ &

(b) ("closedness") X^i *is closed* &

(c) ("boundedness below") $[\exists g^i\, \forall x \in X^i: x \geqq g^i]$ &

(d) ("unlimited consumption") $[\forall x \in X^i\, \forall \bar{x} \in F: [\bar{x} \geqq x \Rightarrow \bar{x} \in X^i]]$ &

(e) $\succcurlyeq^i$ *is a preorder* &

(f) ("completeness") $[\forall x \in X^i\, \forall \bar{x} \in X^i: [x \succcurlyeq^i \bar{x} \text{ or } \bar{x} \succcurlyeq^i x]]$ &

(g) ("monotonicity") $[\forall x \in X^i\, \forall \bar{x} \in X^i: [\bar{x} \geqq x \Rightarrow \bar{x} \succcurlyeq^i x]]$ &

(h) ("continuity") $[\forall \bar{x} \in X^i: [P^i(\bar{x})$ *is closed* & $Q^i(\bar{x})$ *is closed*$]]]$.

All of these are very traditional assumptions, whose meaning (apart from (a)) is not significantly altered in the indivisible context. There are, further, initial endowments w^i. Write $w^t = \sum_{i\in I} w^i$.

The most important special assumption of the model is that there is at least one divisible commodity.

2.2. ASSUMPTION. "One divisible commodity." $n_d \geqq 1$.

This is absolutely crucial to the entire method. Moreover, it is my claim that the way the proof fails in the absence of the assumption reflects the way a real economy would also fail (or work very badly) without a divisible commodity. One of the essential characteristics of money is that it is divisible within anyone's perception, and thus permits easy trade. An economy with nothing smaller than a ten-dollar bill would not turn over smoothly. I shall take an opportunity later of discussing in greater depth this fundamental assumption. It will become obvious very quickly how important technically is its function, especially in smoothing the demand correspondence. Only the divisible commodity makes any sort of continuity remotely possible (but see Henry [7]).

Before proceeding with the description of the structure, we need to set up some more notation and to go through a little mathematics. The "convex hull" of a set Z in E^n is defined as $\operatorname{conv} Z = \bigcap \{C \subset E^n \mid C \text{ is convex } \& \ Z \subset C\}$. The unit vector "in the divisible direction" will be called $e = (1, 0, 0,... 0)$; the first commodity at least is guaranteed to be divisible. A set Z in E^n may be divided into two parts: "the edge of Z" defined as $\operatorname{edge} Z = \{z \in Z \mid \forall\lambda > 0\colon x - \lambda e \notin Z\}$, and "the rest of Z", $\operatorname{rest} Z = (Z \div \operatorname{edge} Z)$. If $x \in \operatorname{rest} X^i$ there is a $\lambda > 0$ such that $x - \lambda e \in X^i$. Hence if X^i satisfies the assumption of unlimited consumption, $x - \lambda e \in \operatorname{rest} X^i$ for *all* sufficiently small $\lambda > 0$. Monotonicity implies the corresponding property for $x \in \operatorname{rest} P^i(\bar{x})$. It also follows very easily (using Carathéodory's theorem below) that if $x \in \operatorname{restconv} X^i$, $x - \lambda e \in \operatorname{restconv} X^i$ for λ sufficiently small, and similarly for $x \in \operatorname{restconv} P^i(\bar{x})$. Heavy use is made of the idea of a "spanning set" which derives from the following well-known theorem on convex hulls.

2.3. "Carathéodory's Theorem." *Let Z be a set in E^n. $\forall z \in \operatorname{conv} Z$ $\exists U(z) \subset Z$: $[U(z)$ has at most $(n + 1)$ members & z is a convex combination of the members of $U(z)]$.*

For a proof see, for instance, Eggleston [4, pp. 35–36]. A "spanning set" of z will be the name given to any *finite* set $U(z)$ such that $z \in \operatorname{conv} U(z)$ (it is a trivial fact that $z \in \operatorname{conv} U(z)$ for a finite set $U(z)$ if and only if z is a convex combination of the members of $U(z)$).

We can now press on to state the remaining special assumptions of the model. One of the important features of the divisible commodity is that it can only perform its smoothing function if it has value in some sense. In fact, I have to make *two* separate assumptions that it is desirable, which seems an unfortunate superfluity. The first of them is the least interesting and is needed only near the end of the proof (Lemma 4.9); at some slight cost in convenience I have spurned its use elsewhere.

2.4. ASSUMPTION. "Strict monotonicity in the divisible commodity." $\forall i \in I\, \forall x \in X^i\, \forall \lambda > 0$: $x + \lambda e \succ^i x$.

The other is this strong assumption:

2.5. ASSUMPTION. "Overriding desirability of the divisible commodity." $\forall i \in I\, \forall x \in X^i\, \forall \bar{x} \in X^i\, \exists \lambda$: $\bar{x} + \lambda e \succcurlyeq^i x$.

It says that a sufficient gain in the divisible commodity will make up for any other loss. Its primary appearance is in the proof of Lemma 4.7, where the upper semicontinuity of the demand correspondence is demonstrated. It can be seen from that proof that without Assumption 2.5, continuity would fail when the price of the divisible commodity is zero. Fortunately, it is possible to illustrate in a rough sort of way the discontinuity in demand even at this early stage, without waiting for the exact definition of the correspondence. Figure 1 represents part of an individual's

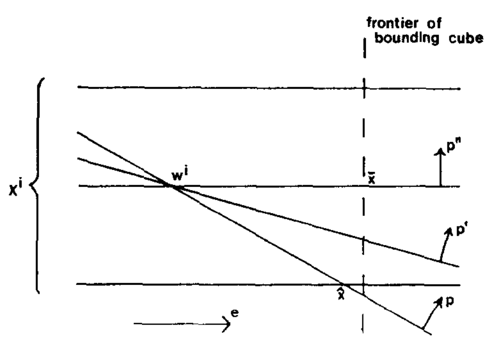

FIG. 1. Discontinuity when $p_1 = 0$.

consumption set; the divisible commodity is measured in the horizontal direction, so the parallel lines are the proper points which constitute the set. Let w^i be the endowment and suppose, in defiance of Assumption 2.5, that w^i is preferred to every point on the line $\{x \mid \exists \lambda: x = \hat{x} + \lambda e\}$, which is the "next" component of the grid which forms X^i. Then, with prices like the p or the p' drawn, we may take it that w^i is the individual's demand, the best point he has available within his budget set. As the price of the divisible commodity gets smaller the budget plane rotates about w^i from p to p' and further to p'', where $p_1'' = 0$. Suddenly it contains all of the line $\{x \mid \exists \lambda: x = w^i + \lambda e\}$, and therefore includes a lot of points strictly preferred to w^i. As usual in proofs of the existence of equilibrium, the demand correspondence is restricted by some bounding cube, so the demand will jump from w^i to, say, $\bar{x}$. Since w^i is definitely excluded from the demand when $p_1 = 0$, this is a breach of upper semicontinuity.

That is only a sketch, of course, but in fact the correspondence defined in Definition 4.4 would behave in exactly that way without Assumption 2.5. The assumption guarantees that the indifference surfaces (strictly their convex hulls) are never parallel to the divisible axis, which in turn guarantees that a tangent price plane can never have $p_1 = 0$. It is at first sight odd that it takes such a strong assumption to do such a little thing. Certainly the example has only shown why there has to be a point at least as good as w^i somewhere on "nextdoor" lines of the grid that makes up X^i. Stretching the picture to three dimensions would make it clear that we cannot restrict attention to the region round w^i. Further, it is not difficult to see that, because of the indivisible structure, if the indifference surfaces are never to be horizontal, then they must "cross" the whole of X^i, just as Assumption 2.5 asks them to do. Exceptions might be made for divisible commodities besides the first, but I have not troubled with that refinement.

The next assumption on the $(X^i, \succcurlyeq^i)$ is one that is not only complicated to state, but also appears to have little contact with intuition.

2.6. ASSUMPTION. $\forall i \in I \quad \forall \bar{x} \in X^i \quad \forall x \in \text{conv } P^i(\bar{x}) \cap \text{restconv } X^i$

$$\exists \{u^1, u^2, \ldots, u^{n'}\} \subset P^i(\bar{x})\, \exists(\alpha^1, \alpha^2, \ldots, \alpha^{n'})\, \exists s' \in \{1, \ldots, n'\}:$$

$$\Big[x = \sum_{s=1}^{n'} \alpha^s u^s \;\&\; \sum_{s=1}^{n'} \alpha^s = 1 \;\&\; [\forall s \in \{1, \ldots, n'\} : 0 \leqq \alpha^s \leqq 1]$$

$$\& \; u^{s'} \in \text{rest } X^i \;\&\; \alpha^{s'} > 0\Big].$$

Most of this assumption is obviously a specification that $\{u^1, u^2, \ldots u^{n'}\}$ spans x. The interpretation is as follows. Any x in conv $P^i(\bar{x})$ is spanned by a finite subset of $P^i(\bar{x})$; that is Carathéodory's theorem. But it is quite possible that all the significant members of any spanning set are on the edge of X^i, even if x itself is in restconv X^i; an example is illustrated in Fig. 2 when $u^1 \succcurlyeq^i u^2 \succcurlyeq^i \bar{x}$. (A "significant" member of a spanning set is one with a strictly positive coefficient α^s; otherwise it may be omitted from the set without making any difference.) Assumption 2.6 is simply the assumption that the possibility is never realized; there is always some spanning set with a significant member in rest X^i. Its use in Lemma 4.6 is fairly transparent, but I am unable to give an explanation of it in more concrete terms.

Debreu [2], in his proof of the existence of equilibrium, assumed, more

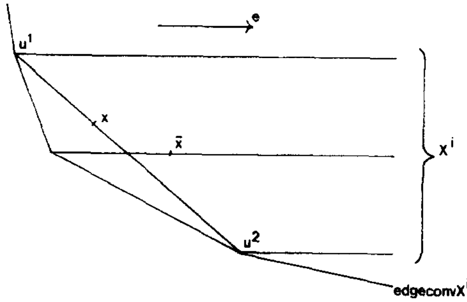

FIG. 2. Counterexample to Assumption 2.6.

or less, that the initial endowment of each consumer is in the interior of his consumption set. McKenzie [8] offered a much weaker assumption to do the same job which, however, complicates the proof a little. My proof is already sufficiently complicated, so I revert to what is undoubtedly a strong assumption after the style of Debreu's. The excuse is that the added difficulty of a weaker one would probably shed no light on indivisibilities.

2.7. ASSUMPTION. "Endowments inside consumption set." $\forall i \in I$: $w^i \in X^i \cap \operatorname{intconv} X^i$.

Finally, it is necessary to bound the size of the nonconvexities in the $P^i(\bar{x})$. This is to be done exactly in the manner of Starr's article [10]. Each point in conv $P^i(\bar{x})$ has spanning sets contained in $P^i(\bar{x})$, and each of the spanning sets has a "radius," defined for a set Z as: $\operatorname{rad} Z = \inf_{a \in E^n} \sup_{z \in Z} | a - z |$. We assume that there is a bound κ^i such that each point has spanning set with a radius not greater than κ^i.

2.8. ASSUMPTION. "Bounded nonconvexities." $\forall i \in I\ \exists \kappa^i\ \forall \bar{x} \in X^i\ \forall x \in \operatorname{conv} P^i(\bar{x})\ \exists U^i(x, \bar{x}) \subset P^i(\bar{x})$: $[U^i(x, \bar{x})$ *spans* x & $\operatorname{rad} U^i(x, \bar{x}) \leqq \kappa^i]$.

I shall write $\kappa^b = \max\{\kappa^i \mid i \in I\}$. For an explanation of the meaning and application of Assumption 2.8, see Starr's paper. It is also explained there that a weaker assumption would suffice, which takes account of the "value" of the nonconvexities rather than of their crude size. To use the weaker assumption presents no further difficulties in depth, but it does further complicate both the notation and the proofs; to save space, therefore, I stick with Assumption 2.8.

3. THE METHOD OF PROOF

At the deepest level the proof contained in Section 4 follows the method of Debreu [2]. The procedure at the heart of Debreu's proof may be sketched as follows. For each individual a "demand correspondence" is set up, from the price simplex

$$S = \left\{ p \,\middle|\, \sum_{r=1}^{n} p_r = 1 \;\&\; [\forall r \in \{1, \dots, n\} : p_r \geqq 0] \right\}$$

into the consumption set X^i. It must have these four properties: (a) the range must be a compact subset of X^i, (b) the image must be convex, (c) the correspondence must be upper semicontinuous, and (d) the set corresponding to a particular price must be the set of the best points available to the individual within his budget set. An aggregate correspondence is next created by vector summation of the individual ones, and, through a particular manipulation involving Walras' law and Kakutani's fixed-point theorem [2, pp. 82–83], an allocation is found which has these two properties: firstly, it is "feasible" in the sense that each individual is allocated a consumption within his consumption set, and the total consumption is not greater than the total endowment; and secondly, there are "supporting" prices such that each individual's allocated consumption is in the set corresponding to those prices. Properties (a)–(c) are the ones which permit the application of the fixed-point theorem. Property (d) is what makes the fixed-point allocation into an *equilibrium*. For, each individual's allocation is in the set corresponding to the supporting prices; if this set contains only the consumptions he might choose at those prices, then the allocation is a proper demand at the prices. This and feasibility are the characteristics of equilibrium.

To see how the method is altered in being transferred to the indivisible case, we can take in turn the four properties of the demand correspondence. A highly artificial correspondence is to be constructed, on the principles to be described, that will satisfy all the requirements except (d), which will be only approximately satisfied. Property (a) gives the least difficulty. We have to choose a bounded set within which to work, instead of the whole unbounded consumption set. The usual method is to use the feasibility constraint, because it is easy to establish that the set of feasible allocations is bounded. Lemma 4.3 is doing the same thing in the present case. Notice that the bounding cube is chosen sufficiently large to include all feasible allocations in its *interior*.

Convexity, property (b), is created entirely artificially, since the model contains no natural convexity at all. Instead of working with the given sets, we take their convex hulls. For instance, the image in the demand correspondence is a subset of conv X^i rather than of X^i, and we use an artificial preference relation created out of the sets conv $P^i(\bar{x})$. In fact, as I shall explain, the technique amounts to constructing an entire convex economy and working in that. The sets conv $P^i(\bar{x})$ are the "synthetic" not-worse-than sets. The result is that the allocation which emerges from the fixed-point theorem is an allocation in the convex construct, so that a further step is needed to get back to the original system. The step is managed by the Shapley–Folkman theorem, and produces an allocation which is only approximately feasible. Divisible nonconvex systems have been tackled in the same way, so for an exposition I refer to Starr [10], and here quote only the useful theorem:

3.1. "Shapley–Folkman Theorem." *For all* $j \in \{1,\dots k\}$, *let* Z^j *be a compact set in* E^n. *Write* $\kappa^b = \max\{\text{rad } Z^j \mid j \in \{1,\dots k\}\}$.

$$\forall z \in \sum_{j=1}^{k} \text{conv } Z^j \; \exists \bar{z} \in \sum_{j=1}^{k} Z^j : | z - \bar{z} | \leqq \kappa^b \sqrt{n}.$$

The proof is in Starr [10, Appendix 2].

It is the assumption that there is one divisible commodity which makes possible property (c), upper semicontinuity. In the constructed convex system, the frontier of the sets conv $P^i(\bar{x})$ form the "synthetic" indifference surfaces. It is clear enough that in the totally indivisible case, the derived synthetic preorder could not exhibit much continuity. Even with the smoothing influence of the divisible commodity, an important vestige of the problem remains. It occurs at the edge of the consumption set and is illustrated in Fig. 3. $\bar{x}$ is in edge X^i; suppose it is indifferent to $\hat{x}$ and $\tilde{x}$.

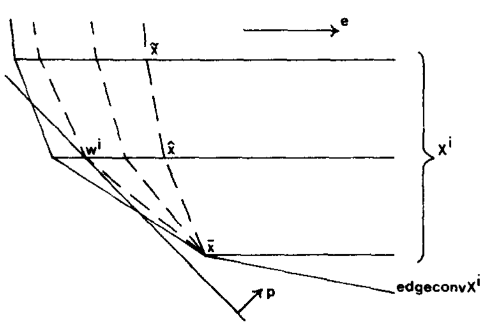

FIG. 3. "The problem of the edge."

The dotted lines represent the frontiers of some sets conv $P^i(x)$; they are thus the synthetic indifference curves in the synthetic consumption set conv X^i. They meet at $\bar{x}$, which is bound to cause trouble because indifference curves cannot possibly meet if the preference relation is continuous. A view of the problem can be found by considering demand at various prices represented by a budget plane. Take w^i as the endowment. Think first of the "genuine" demand, as determined in the given consumption set X^i by the given preorder $\succcurlyeq^i$. With a p like the one drawn, w^i is the best point within the budget set, and thus the proper demand. As the price changes, the budget plane rotates about w^i, and will at a certain moment suddenly come to include $\bar{x}$. $\bar{x}$ is then within the budget set, and since it is strictly preferred to w^i, demand will jump from w^i to $\bar{x}$. This is a breach of upper semicontinuity. Now, in most cases, the correspondence of Definition 4.4 is simply the convex hull of the "genuine" demand correspondence. It would thus be discontinuous in exactly the same way, but for a special trick. At the crucial moment, the constructed correspondence is made artificially to include the *whole* of the line joining $\bar{x}$ and w^i in Fig. 3. It thereby preserves continuity, but in doing so puts the correspondence a little further from property (d). It would have been desirable to preserve the artificial demand correspondence as the convex hull of the genuine demand, for then the application of the Shapley–Folkman theorem would have brought us to an allocation contained in each person's *genuine* demand set. Thus all the requirements of equilibrium would have been satisfied, apart from exact feasibility. But this extra manipulation which has to be performed for the sake of continuity separates us from that achievement. It is what causes the second weakness in the final near equilibrium theorem, and it appears to be ineradicable. The defect will be made as small as possible; Section 5 is a discussion of it. I will call it "the problem of the edge."

The above is a general outline of the way the correspondence is to be set up. It remains to give a more precise explanation of the rather obscure form in which it is actually written (Definition 4.4). The best understanding is to be gained by using Starr's method for constructing a synthetic economy [10]. It happens that the process of synthesis is suppressed in the mathematics of the proof because it turns out to be quicker to cut straight through that stage without bothering with all its details. The correspondence is, nonetheless, almost exactly the one that would arise naturally from Starr's technique, and it is most easily approached from that direction.

Starr defines the synthetic preorder as follows:

3.2. Definition. "Synthetic preörder $S \succcurlyeq^i$." $x \, S \succcurlyeq^i \hat{x} \Leftrightarrow [\forall \bar{x} \in X^i$: $[\hat{x} \in \operatorname{conv} P^i(\bar{x}) \Rightarrow x \in \operatorname{conv} P^i(\bar{x})]]$.

Define the budget set as $B^i(p) = \{x \mid p \cdot x \leqq p \cdot w^i\}$. Then the natural synthetic demand correspondence is

$$p \rightarrow SD^i(p) = \{\hat{x} \in B^i(p) \cap \operatorname{conv} X^i \mid \forall x \in B^i(p) \cap \operatorname{conv} X^i\colon \hat{x} \, S \succcurlyeq^i x\}.$$

In words, this is the set of points in the budget set (and the synthetic consumption set) which are synthetically preferred or indifferent to any other points in the budget set. We may substitute for $S \succcurlyeq^i$ its definition and, with some manipulation, achieve this form for the correspondence: $p \to SD^i(p) = \bigcap_{\bar{x} \in SA^i(p)} (B^i(p) \cap \text{conv } P^i(\bar{x}))$, where

$$SA^i(p) = \{\bar{x} \in X^i \mid B^i(p) \cap \text{conv } P^i(\bar{x}) \text{ is nonempty}\}.$$

In words, the intersection of the budget set with the convex hulls of all those not-worse-than sets whose convex hulls happen to meet the budget set. A very similar formula, employing the intersection of not-worse-than sets, could be used to define the straightforward correspondence in an ordinary convex model.

The correspondence $D^i(p)$ of Definition 4.4 is very like the $SD^i(p)$ just described. There are two differences. The first is that the limitation by the bounding cube G^i has inevitably to be imposed. In doing so I have adopted a minor trick, that int G^i rather than G^i appears in the definition of $A^i(p)$. This shortens the proof, but has otherwise no great significance; the correspondence itself is actually unaltered. The second distinction is the response to the problem of the edge: each conv $P^i(\bar{x})$ of the family defined by $\bar{x} \in A^i(p)$ is required to meet $B^i(p)$ at a point of restconv X^i, not just anywhere in conv X^i. It is to be noted that although the definition requires each set of the intersecting family to meet int G^i and restconv X^i, the intersection of the whole family may include only points in fr G^i and edgeconv X^i.

4. The Theorem of Near Equilibrium

First we note:

4.1. Lemma. *Let Assumption* 2.1 *be satisfied.* $\forall i \in I \, \forall \bar{x} \in X^i$: conv $P^i(\bar{x})$ *is closed.*

The proof is not trivial, but it has been provided by Starr [10, Appendix 3]. In bringing his proof into conformity with the structure of this model, it must be remembered that Starr uses the term "convex hull" for what is here called the closure of the convex hull. An exactly similar proof establishes:

4.2. Lemma. *Let Assumption* 2.1 *be satisfied.* $\forall i \in I$: conv X^i *is closed.*

The next lemma establishes, in a very conventional way from the feasibility constraint, the existence of a suitable "bounding cube" for each consumer.

4.3. LEMMA. *Let Assumption* 2.1 *be satisfied.* $\exists(\bar{g}^1, \bar{g}^2, \ldots \bar{g}^m)$

$$\forall(x^1, x^2, \ldots, x^m) : \left[\left[\sum_{i \in I} x^i \leqq w^t \ \& \ [\forall i \in I : x^i \in \operatorname{conv} X^i]\right] \Rightarrow [\forall i \in I : x^i \leqq \bar{g}^i]\right].$$

Proof. We already have (Assumption 2.1(c)) a lower bound g^i on each X^i. g^i is also a lower bound on conv X^i because $\{x \mid x \geqq g^i\}$ is a convex set containing X^i (then apply the definition of a convex hull). Take $(x^1, x^2, \ldots x^m)$ with $\sum_{i \in I} x^i \leqq w^t$ and $x^i \in \operatorname{conv} X^i$ for all $i \in I$. Then for any $i' \in I$, $x^{i'} \leqq w^t - \sum_{i \neq i'} x^i \leqq w^t - \sum_{i \neq i'} g^i$. So $\bar{g}^{i'} = w^t - \sum_{i \neq i'} g^i$ is an upper bound on $x^{i'}$ as required. ∎

Define the set G^i as $\{x \mid g^i - (1, 1, \ldots 1) \leqq x \leqq \bar{g}^i + (1, 1, \ldots 1)\}$; then it is assured that every feasible allocation has each x^i in the *interior* of G^i. The correspondence to be used is

4.4. DEFINITION. "Correspondence $p \to D^i(p)$." Let
$A^i(p) = \{\bar{x} \in X^i \mid \operatorname{int} G^i \cap B^i(p) \cap \operatorname{conv} P^i(\bar{x}) \cap \operatorname{restconv} X^i$ *is nonempty*$\}$.
$D^i: S \to G^i$ is $p \to D^i(p) = \bigcap_{\bar{x} \in A^i(p)} (G^i \cap B^i(p) \cap \operatorname{conv} P^i(\bar{x}))$.

The object of the next series of lemmas (Lemmas 4.5, 4.6, and 4.7) is to establish the properties of $p \to D^i(p)$ necessary to make applicable the standard procedure which produces a feasible fixed point (Lemma 4.8). Thereafter (Lemma 4.9) the relationship between $D^i(p)$ and a genuine demand correspondence is laid bare in order to make the fixed point into a near equilibrium.

4.5. LEMMA. *Let Assumptions* 2.1, 2.2, *and* 2.5 *be satisfied.* $\forall i \in I \ \ \forall \bar{x} \in X^i \ \ \forall \hat{x} \in \operatorname{conv} X^i$: $[\hat{x} \notin \operatorname{conv} P^i(\bar{x}) \Rightarrow [\exists \bar{\bar{x}} \in X^i: [\bar{\bar{x}} \succ^i \bar{x} \ \ \& \ \hat{x} \notin \operatorname{conv} P^i(\bar{\bar{x}})]]]$.

Proof. The index i will be omitted from the notation in this proof. Take an $\bar{x} \in X$ and an $\hat{x} \in \operatorname{conv} X$ with $\hat{x} \notin \operatorname{conv} P(\bar{x})$. By Lemma 4.1, conv $P(\bar{x})$ is closed, so $\hat{x}$ may be strictly separated from conv $P(\bar{x})$ by a hyperplane. That is, there is a $p' \neq 0$ and a μ' such that $p' \cdot \hat{x} < \mu'$ and for all $x \in \operatorname{conv} P(\bar{x})$ $p' \cdot x > \mu'$. Further, $p' \geqslant 0$, for suppose for some $r \in \{1, \ldots n\}$, $p_r < 0$. Then

$$p' \cdot (\bar{x} + \lambda(0, 0, \ldots 0, 1, 0, \ldots 0)) < \mu'$$

for λ large enough (where the "1" is in the r-th place). Therefore $\bar{x} + \lambda(0, 0, \ldots 0, 1, 0, \ldots 0)$ cannot belong to conv $P(\bar{x})$, which contradicts monotonicity.

Suppose for some $r \in \{1,\dots n\}$, $p_r{}' = 0$. Define $\mu'' = \frac{1}{2}(\mu' + p' \cdot \hat{x})$ and consider a $p_r'' > 0$. Write $p'' = (p_1{}', p_2{}',\dots p_{r-1}', p_r'', p_{r+1}',\dots p_n{}')$. It will be shown that for p_r'' sufficiently small p'' and μ'' define a hyperplane which retains the strict separation properties. g, the lower bound on X, is also a lower bound on conv $P(\bar{x})$ because $\{x \mid x \geqq g\}$ is a convex set containing $P(\bar{x})$. First: if $g_r \geqq 0$ then $p'' \cdot x \geqq p' \cdot x > \mu''$ for all $x \in \text{conv } P(\bar{x})$; and if $g_r < 0$ then, provided $p_r'' < (\mu' - \mu'')/(-g_r)$, $p'' \cdot x = p' \cdot x + p_r'' x_r \geqq p' \cdot x + p_r'' g_r > \mu' - (\mu' - \mu'') = \mu''$ for all $x \in \text{conv } P(\bar{x})$. Second: if $\hat{x}_r \leqq 0$ then $p'' \cdot \hat{x} \leqq p' \cdot \hat{x} < \mu''$; and if $\hat{x}_r > 0$ then, provided $p_r'' < (\mu'' - p' \cdot \hat{x})/\hat{x}_r$, $p'' \cdot \hat{x} = p_r'' \hat{x}_r + p' \cdot \hat{x} < \mu''$.

Repeat the process for any component of p'' which is zero, until eventually we achieve a $p > 0$ and a μ such that $p \cdot \hat{x} < \mu$ and $p \cdot x > \mu$ for all $x \in \text{conv } P(\bar{x})$.

Define T as $\{x \in X \mid p \cdot x \leqq \mu\}$. T is closed because X is closed. T is bounded below by g. T is also bounded above because $\sum_{r=1}^{n} p_r x_r \leqq \mu$ for $x \in T$, so for any $r' \in \{1,\dots n\}$,

$$x_{r'} \leqq (\mu - \textstyle\sum_{r \neq r'} p_r x_r)/p_{r'} \leqq (\mu - \textstyle\sum_{r \neq r'} p_r g_r)/p_{r'}$$

(because $p_r > 0$ for all $r \in \{1,\dots n\}$). So T is compact. T is also nonempty for otherwise $\{x \mid p \cdot x > \mu\}$ is a convex set containing X, which contradicts that $\hat{x} \in \text{conv } X$.

Write $TP(x) = T \cap P(x)$ and consider the family $\{TP(x) \mid x \in T\}$. The family is nonempty because T is nonempty. Each $TP(x)$ in the family is nonempty because x belongs to it. Each $TP(x)$ is closed, being the intersection of two closed sets. The family is totally ordered by inclusion (because it follows directly from the transitivity and completeness of $\succcurlyeq$ that for $\hat{x}$ and $\tilde{x}$ in X either $P(\hat{x}) \subset P(\tilde{x})$ or $P(\tilde{x}) \subset P(\hat{x})$). Therefore any intersection of a finite subfamily of the family is equal to the smallest member of the subfamily and is hence nonempty. Therefore, by the finite intersection property of the compact set T, $\bigcap\{TP(x) \mid x \in T\}$ is nonempty. Take $\tilde{x} \in \bigcap\{TP(x) \mid x \in T\}$, so $\tilde{x} \in P(x)$ for all $x \in T$. Also $\bar{x} \succ \tilde{x}$ because T and $P(\bar{x})$ are disjoint.

By Assumption 2.5 there is a λ such that $\tilde{x} + \lambda e \succcurlyeq \bar{x}$. Take $\bar{\lambda} = \inf\{\lambda \mid \tilde{x} + \lambda e \succcurlyeq \bar{x}\}$, then by the continuity of $\succcurlyeq$, $\tilde{x} + \bar{\lambda} e \sim \bar{x}$. For any $\lambda < \bar{\lambda}$, $\bar{x} \succ \tilde{x} + \lambda e$. Also, since $Q(\tilde{x})$ is closed and $\tilde{x} + \bar{\lambda} e \notin Q(\tilde{x})$, if λ is sufficiently close to $\bar{\lambda}$ then $\tilde{x} + \lambda e \notin Q(\tilde{x})$. Write such a point $\tilde{x} + \lambda e$ (with $\lambda < \bar{\lambda}$ and λ close to $\bar{\lambda}$) as $\bar{\bar{x}}$, so $\bar{x} \succ \bar{\bar{x}} \succ \tilde{x}$.

$P(\bar{\bar{x}})$ and T are disjoint because for all $x \in T$ $\bar{\bar{x}} \succ \tilde{x} \succcurlyeq x$. Therefore the half-space $\{x \mid p \cdot x \geqq \mu\}$ is a convex set which contains $P(\bar{\bar{x}})$ and hence conv $P(\bar{\bar{x}})$. But $\hat{x}$ is outside this half-space, so $\hat{x} \notin \text{conv } P(\bar{\bar{x}})$. ∎

4.6. LEMMA. *Let Assumptions* 2.1, 2.2, *and* 2.6 *be satisfied.*

$$\forall i \in I\ \forall \bar{x} \in X^i\ \forall \bar{\bar{x}} \in X^i\ \forall x \in \text{restconv } X^i:$$
$$[[\bar{x} \succ^i \bar{\bar{x}}\ \&\ x \in \text{conv } P^i(\bar{x})] \Rightarrow x \in \text{restconv } P^i(\bar{\bar{x}})].$$

Proof. The index i is omitted from the notation in this proof. Take an $x \in \text{restconv } X$ with $x \in \text{conv } P(\bar{x})$, and an $\bar{\bar{x}} \in X$ with $\bar{x} \succ \bar{\bar{x}}$. Take a spanning set $U(x) = \{u^1, u^2,\ldots u^{n'}\} \subset P(\bar{x})$ spanning x, with the property of Assumption 2.6. In the notation of that assumption consider $u^{s'}$. $u^{s'} \in P(\bar{x})$ so $u^{s'} \succcurlyeq \bar{x} \succ \bar{\bar{x}}$. Since $Q(\bar{\bar{x}})$ is closed, for a sufficiently small $\lambda' > 0$, $u^{s'} - \lambda' e \notin Q(\bar{\bar{x}})$. Since, by Assumption 2.6, $u^{s'} \in \text{rest } X$, $u^{s'} - \lambda' e \in X$ if λ' is sufficiently small. Therefore $u^{s'} - \lambda' e \in P(\bar{\bar{x}})$. Also, for all $s \neq s'$, $u^s \in P(\bar{x})$ so $u^s \in P(\bar{\bar{x}})$. $x - \alpha^{s'}\lambda' e = \sum_{s \neq s'} \alpha^s u^s + \alpha^{s'}(u^{s'} - \lambda' e)$; that is, $x - \alpha^{s'}\lambda' e$ is spanned by members of $P(\bar{\bar{x}})$ and is hence in $\text{conv } P(\bar{\bar{x}})$. $\alpha^{s'} > 0$ by Assumption 2.6; thus $\alpha^{s'}\lambda' > 0$, so $x \in \text{restconv } P(\bar{\bar{x}})$. ∎

4.7. LEMMA. *Let Assumptions* 2.1, 2.2, 2.5, 2.6, *and* 2.7 *be satisfied.* $\forall i \in I$:

(a) $[[\forall p \in S$: *$D^i(p)$ is convex*] &

(b) $[\forall p \in S$: *$D^i(p)$ is nonempty*] &

(c) $p \to D^i(p)$ *is upper semicontinuous*].

Proof. The index i is omitted from the notation in this proof.

(a) $D(p)$ is the intersection of convex sets.

(b) Take any $p \in S$. $w \in \text{int } G$ by the definition of G and Assumption 2.7. $w \in B(p)$ by the definition of $B(p)$. $w \in \text{restconv } X$ by Assumption 2.7 because $\text{intconv } X \subset \text{restconv } X$. So $w \in \text{int } G \cap B(p) \cap \text{conv } P(w) \cap \text{restconv } X$. Hence $w \in A(p)$, which is therefore nonempty. The sets $G \cap B(p) \cap \text{conv } P(\bar{x})$ for $\bar{x} \in A(p)$ are totally ordered by inclusion (as in the proof of Lemma 4.5), so any finite intersection of them is equal to the smallest member, and by the definition of $A(p)$ that smallest member is nonempty. Each $G \cap B(p) \cap \text{conv } P(\bar{x})$ is closed because G and $B(p)$ are closed by definition and $\text{conv } P(\bar{x})$ is closed by Lemma 4.1. And each $G \cap B(p) \cap \text{conv } P(\bar{x})$ is a subset of G. Therefore, by the finite intersection property of the compact set G, the intersection $D(p)$ of the whole family is nonempty.

(c) Take sequences $\{p^q\} \to \hat{p}$ and $\{x^q\} \to \hat{x}$ with $p^q \in S$ and $x^q \in D(p^q)$ for all $q \in \{1, 2,\ldots\}$. Because S is closed $\hat{p} \in S$ and because G is closed $\hat{x} \in G$. Suppose $\hat{x} \notin B(\hat{p})$, so $\hat{p} \cdot \hat{x} > \hat{p} \cdot w$, then for sufficiently large q, $p^q \cdot x^q > p^q \cdot w$, which contradicts that $x^q \in D(p^q) \subset B(p^q)$. Hence $\hat{x} \in B(\hat{p})$.

Suppose $\hat{x} \notin D(\hat{p})$, then for some $\bar{x} \in A(\hat{p})$ $\hat{x} \notin G \cap B(\hat{p}) \cap \operatorname{conv} P(\bar{x})$. That is, $\hat{x} \notin \operatorname{conv} P(\bar{x})$, since $\hat{x} \in G \cap B(\hat{p})$. By the definition of $A(\hat{p})$, $\operatorname{int} G \cap B(\hat{p}) \cap \operatorname{conv} P(\bar{x}) \cap \operatorname{restconv} X$ is nonempty; let

$$\tilde{x} \in \operatorname{int} G \cap B(\hat{p}) \cap \operatorname{conv} P(\bar{x}) \cap \operatorname{restconv} X.$$

(i) Consider first $\hat{p}$ with $\hat{p}_1 > 0$. By Lemma 4.5 there is an $\bar{\bar{x}}$ with $\bar{x} \succ \bar{\bar{x}}$ and $\hat{x} \notin \operatorname{conv} P(\bar{\bar{x}})$. Since $\operatorname{conv} P(\bar{\bar{x}})$ is closed by Lemma 4.1, there is a neighborhood $N(\hat{x}, \epsilon)$ which is disjoint from $\operatorname{conv} P(\bar{\bar{x}})$. Further, by Lemma 4.6, $\tilde{x} \in \operatorname{restconv} P(\bar{\bar{x}})$, so for λ sufficiently small $\tilde{x} - \lambda e \in \operatorname{conv} P(\bar{\bar{x}})$. Also, $\tilde{x} - \lambda e \in \operatorname{int} G$, and $\tilde{x} - \lambda e \in \operatorname{restconv} X$, for λ sufficiently small. Write such a point $\tilde{x} - \lambda e$ as $\tilde{x}'$. $\hat{p} \cdot \tilde{x}' < \hat{p} \cdot \tilde{x} \leqq \hat{p} \cdot w$ because $\hat{p}_1 > 0$. So for q sufficiently large $p^q \cdot \tilde{x}' \leqq p^q \cdot w$; that is, $\tilde{x}' \in B(p^q)$. Thus $\tilde{x}' \in \operatorname{int} G \cap B(p^q) \cap \operatorname{conv} P(\bar{\bar{x}}) \cap \operatorname{restconv} X$, so $\bar{\bar{x}} \in A(p^q)$. But $x^q \in N(\hat{x}, \epsilon)$ for q sufficiently large, and this means that $x^q \notin \operatorname{conv} P(\bar{\bar{x}})$ which contradicts the definition of $D(p^q)$.

(ii) Consider $\hat{p}$ with $\hat{p}_1 = 0$. There exists a point $v \in X$ with $\hat{p} \cdot v < \hat{p} \cdot w$. For suppose not, then $\{x \mid \hat{p} \cdot x \geqq \hat{p} \cdot w\}$ is a convex set containing X and hence $\operatorname{conv} X$ which does not have w in its interior, and this contradicts Assumption 2.7. By Assumption 2.5, for some λ, $v + \lambda e \succcurlyeq \bar{x}$. Write this point $v + \lambda e$ as $\bar{v}$. Since $p_1 = 0$, $\hat{p} \cdot \bar{v} = \hat{p} \cdot v < \hat{p} \cdot w$. $\bar{v} \in P(\bar{x})$ so $\bar{v} \in \operatorname{conv} P(\bar{x})$. Now write $t(\alpha) = \alpha\bar{v} + (1 - \alpha)\bar{x}$. Since both $\bar{x}$ and $\bar{v}$ belong to $\operatorname{conv} P(\bar{x})$ which is convex, so does $t(\alpha)$ provided $0 \leqq \alpha \leqq 1$. And for $\alpha > 0$ $\hat{p} \cdot t(\alpha) < \hat{p} \cdot w$. For α sufficiently small $t(\alpha) \in \operatorname{int} G$ and $t(\alpha) \in \operatorname{restconv} X$. Choose such an α, then for q sufficiently large $t(\alpha) \in \operatorname{int} G \cap B(p^q) \cap \operatorname{conv} P(\bar{x}) \cap \operatorname{restconv} X$, so $\bar{x} \in A(p^q)$. But because $\operatorname{conv} P(\bar{x})$ is closed (Lemma 4.1) there is a neighborhood $N(\hat{x}, \epsilon)$ which is disjoint from $\operatorname{conv} P(\bar{x})$, and for q sufficiently large, $x^q \in N(\hat{x}, \epsilon)$ which contradicts the definition of $D(p^q)$. ▌

The following lemma has been proved from the results of Lemma 4.7 by Debreu [2, pp. 82–83].

4.8. Lemma. "Fixed point." *Let Assumptions* 2.1, 2.2, 2.5, 2.6, *and* 2.7 *be satisfied.*

$$\exists p^* \in S\, \exists (x^{*1}, x^{*2}, \ldots, x^{*m}) : \left[\sum_{i \in I} x^{*i} \leqq w^t \;\&\; [\forall i \in I : x^{*i} \in D^i(p^*)] \right].$$

Now we come to examine the properties of $p \to D^i(p)$ which allow it to serve in the place of a *demand* correspondence.

4.9. Lemma. *Let Assumptions* 2.1, 2.2, 2.4, *and* 2.5 *be satisfied.* $\forall i \in I\, \forall p^* \in S\, \forall x^{*i} \in \operatorname{int} G^i \cap D^i(p^*)\, \exists \bar{x}^i \in X^i$:

(a) $[x^{*i} \in \operatorname{conv} P^i(\bar{x}^i)$ &

(b) $[\forall x \in B^i(p^*) \cap X^i$: $[x \notin \operatorname{edge} X^i \cap \operatorname{fr} B^i(p^*) \Rightarrow \bar{x}^i \succcurlyeq^i x]]$ &

(c) $\operatorname{fr} B^i(p^*)$ *supports* $\operatorname{conv} P^i(\bar{x}^i)$ at $x^{*i}]$.

Proof. The index i is omitted from the notation in this proof. Take an $x^* \in \operatorname{int} G \cap D(p^*)$. By the definition of $D(p^*)$, $x^* \in \operatorname{conv} P(w)$. Consider points on the line $w + \lambda e$ and take $\bar{\lambda} = \sup\{\lambda \mid x^* \in \operatorname{conv} P(w + \lambda e)\}$. Write $\bar{x} = w + \bar{\lambda} e$; then $\bar{x}$ has the required properties.

(a) Suppose $x^* \notin \operatorname{conv} P(\bar{x})$. By Lemma 4.5 there is an $\bar{\bar{x}} \in X$ with $\bar{x} \succ \bar{\bar{x}}$ and $x^* \notin \operatorname{conv} P(\bar{\bar{x}})$. Since $Q(\bar{\bar{x}})$ is closed, $\bar{x} - \lambda e \succ \bar{\bar{x}}$ for sufficiently small $\lambda > 0$ ($\bar{x} \in \operatorname{rest} X$ because $w \in \operatorname{rest} X$ and $\bar{\lambda} \geqq 0$). Hence $x^* \notin \operatorname{conv} P(\bar{x} - \lambda e)$. But monotonicity implies from the definition of $\bar{x}$ that $x^* \in \operatorname{conv} P(\bar{x} - \lambda e)$ for all $\lambda > 0$. So there is a contradiction.

(b) It follows from (a) that $p_1^* > 0$ (in order for $\operatorname{int} G \cap D(p^*)$ to be nonempty). For suppose $p_1^* = 0$. Take a finite spanning set $U(x^*) \subset P(\bar{x})$. Take a $\lambda > 0$ sufficiently small so that $x^* + \lambda e \in \operatorname{int} G$. By Assumption 2.4, for all $u \in U(x^*)$ $u + \lambda e \succ u \succcurlyeq \bar{x}$. And $x^* + \lambda e$ is spanned by these points $u + \lambda e$. Of the points $u + \lambda e$ for $u \in U(x^*)$, choose the worst, say $u' + \lambda e$, so for all $u \in U(x^*)$ $u + \lambda e \succcurlyeq u' + \lambda e$. Then $x^* + \lambda e \in \operatorname{conv} P(u' + \lambda e)$. Because $p_1^* = 0$, $p^* \cdot (x^* + \lambda e) = p^* \cdot x^* \leqq p^* \cdot w$; also, obviously, $x^* + \lambda e \in \operatorname{restconv} X$. Hence $\operatorname{int} G \cap B(p^*) \cap \operatorname{conv} P(u' + \lambda e) \cap \operatorname{restconv} X$ contains $x^* + \lambda e$; therefore $u' + \lambda e \in A(p^*)$. But $u' + \lambda e \succ \bar{x}$ so there is a $\mu > 0$ such that $u' + \lambda e \succcurlyeq \bar{x} + \mu e$; hence if $x^* \in \operatorname{conv} P(u' + \lambda e)$, then $x^* \in \operatorname{conv} P(\bar{x} + \mu e)$, which contradicts the definition of $\bar{x}$. Therefore $x^* \notin \operatorname{conv} P(u' + \lambda e)$. Thus we have achieved a contradiction of the definition of $D(p^*)$.

Now suppose that there is a point $\hat{x}' \in B(p^*)$ which has $\hat{x}' \succ \bar{x}$ and is either not in edge X or not in $\operatorname{fr} B(p^*)$ (contradicting property (b) of the theorem). If $\hat{x}'$ is not in $\operatorname{fr} B(p^*)$, $p^* \cdot \hat{x}' < p^* \cdot w$, and write $\hat{x} = \hat{x}'$. If $\hat{x}'$ is not in edge X, for sufficiently small $\lambda > 0$, $\hat{x}' - \lambda e \succ \bar{x}$ since $Q(\bar{x})$ is closed. Write this $\hat{x}' - \lambda e$ as $\hat{x}$; then $p^* \cdot \hat{x} < p^* \cdot w$ because $p_1^* > 0$. Write $\delta = (p^* \cdot w - p^* \cdot \hat{x})/p_1^*$, so $p^* \cdot (\hat{x} + \delta e) = p^* \cdot w$.

As in the proof just above that $p_1^* > 0$, take a finite spanning set $U(x^*) \subset P(\bar{x})$ spanning x^*. For any $\lambda > 0$, $x^* + \lambda e$ is spanned by the points $u + \lambda e$ for $u \in U(x^*)$. Out of the set $\{\hat{x}\} \cup \{v \mid \exists u \in U(x^*)\colon v = u + \lambda e\}$ select the worst point $\bar{v}(\lambda)$. Then $x^* + \lambda e \in \operatorname{conv} P(\bar{v}(\lambda))$. As above, if $x^* \in \operatorname{conv} P(\bar{v}(\lambda))$, then $x^* \in \operatorname{conv} P(\bar{x} + \mu e)$ for some $\mu > 0$, which is a contradiction, so $x^* \notin \operatorname{conv} P(\bar{v}(\lambda))$. Further, any point $t(\alpha, \lambda)$, defined as $\alpha(x^* + \lambda e) + (1 - \alpha)\hat{x}$ belongs to $\operatorname{conv} P(\bar{v}(\lambda))$ for $0 \leqq \alpha \leqq 1$ because $\operatorname{conv} P(\bar{v}(\lambda))$ is convex.

Take $\alpha = \delta/(\delta + \lambda)$. Then

$$p^* \cdot t(\alpha, \lambda) = p^* \cdot (\alpha(x^* + \lambda e) + (1 - \alpha)\, \hat{x})$$

$$= \frac{1}{\delta + \lambda} (\delta p^* \cdot x^* + \delta\lambda p_1{}^* + \lambda p^* \cdot \hat{x})$$

$$= \frac{p_1{}^*}{p^* \cdot w - p^* \cdot \hat{x} + \lambda p_1{}^*} \left(\frac{p^* \cdot x^*}{p_1{}^*} (p^* \cdot w - p^* \cdot \hat{x}) + \lambda(p^* \cdot w - p^* \cdot \hat{x} + p^* \cdot \hat{x})\right)$$

$$\leqq \frac{p^* \cdot w(p^* \cdot w - p^* \cdot \hat{x}) + p_1{}^* p^* \cdot w}{p^* \cdot w - p^* \cdot \hat{x} + p_1{}^*} = p^* \cdot w.$$

Thus $t(\alpha, \lambda) \in B(p^*)$.

Now take any $\epsilon > 0$ such that $N(x^*, \epsilon) \subset \operatorname{int} G$. Then if $\lambda < \epsilon\delta/(|x^* - (\hat{x} + \delta e)| - \epsilon)$,

$$|x^* - t(\alpha, \lambda)| = |x^* - (\alpha(x^* + \lambda e) + (1 - \alpha)\, \hat{x})|$$

$$= (1/(\delta + \lambda))|\, \delta x^* + \lambda x^* - \delta x^* - \delta\lambda e - \lambda\hat{x}\, |$$

$$= (\lambda/(\delta + \lambda))|\, x^* - (\hat{x} + \delta e)|$$

$$< \frac{\epsilon\delta\, |x^* - (\hat{x} + \delta e)|(|x^* - (\hat{x} + \delta e)| - \epsilon)}{(|x^* - (\hat{x} + \delta e)| - \epsilon)(\epsilon\delta + \delta(|x^* - (\hat{x} + \delta e)| - \epsilon))} = \epsilon.$$

Thus $t(\alpha, \lambda) \in \operatorname{int} G$. Further, taking any smaller λ' (with $0 \leqq \lambda' < \lambda$) gives $t(\alpha, \lambda') \in \operatorname{conv} X$ so $t(\alpha, \lambda) \in \operatorname{restconv} X$. Hence

$$\operatorname{int} G \cap B(p^*) \cap \operatorname{conv} P(\bar{v}(\lambda)) \cap \operatorname{restconv} X$$

is nonempty because $t(\alpha, \lambda)$ belongs to it. Thus $\bar{v}(\lambda) \in A(p^*)$, and yet $x^* \notin \operatorname{conv} P(\bar{v}(\lambda))$ which contradicts the definition of $D(p^*)$.

(c) Suppose for some $x \in P(\bar{x})$, $p^* \cdot x < p^* \cdot w$. For λ sufficiently small $p^* \cdot (x + \lambda e) < p^* \cdot w$. By the strict monotonicity assumption, $x + \lambda e \succ x \succcurlyeq \bar{x}$. Clearly $x + \lambda e \in \operatorname{rest} X$, which contradicts (b) above. Thus $\{x \mid p^* \cdot x \geqq p^* \cdot w\}$ is a convex set containing $P(\bar{x})$ and hence $\operatorname{conv} P(\bar{x})$. But $x^* \in \operatorname{conv} P(\bar{x})$ and $p^* \cdot x^* \leqq p^* \cdot w$, so the plane $\{x \mid p^* \cdot x = p^* \cdot w\}$ supports $\operatorname{conv} P(\bar{x})$ at x^*. ∎

The center of the proof of (b) is illustrated in Fig. 4, which is meant to be self-explanatory in the notation of the proof. The algebraic complexities appear geometrically as operations on similar triangles. The point of property (c) of Lemma 4.9 is to be seen from the following lemma, which provides an extension to Assumption 2.8.

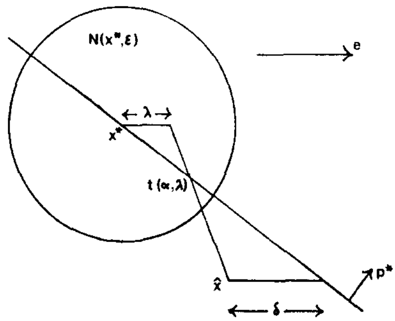

FIG. 4. Proof of Lemma 4.9(b).

4.10. LEMMA. *Let Assumptions* 2.1 *and* 2.8 *be satisfied.* $\forall i \in I\, \forall p^* \neq 0$ $\forall \bar{x}^i \in X^i\, \forall x^{*i} \in \operatorname{conv} P^i(\bar{x}^i)$: $[\{x \mid p^* \cdot x = p^* \cdot x^{*i}\}$ *supports* $\operatorname{conv} P^i(\bar{x}^i) \Rightarrow$ $[\exists^e U^i(x^{*i}, \bar{x}^i) \subset P^i(\bar{x}^i)$: $[^e U^i(x^{*i}, \bar{x}^i)$ *spans* x^{*i} & $\operatorname{rad}{}^e U^i(x^{*i}, \bar{x}^i) \leqq \kappa^i$ & ${}^e U^i(x^{*i}, \bar{x}^i) \subset \{x \mid p^* \cdot x = p^* \cdot x^{*i}\}]]]$.

Proof. The index i is omitted from the notation in this proof. Take an $x^* \in \operatorname{conv} P(\bar{x})$ and a $p^* \neq 0$ such that $\{x \mid p^* \cdot x = p^* \cdot x^*\}$ supports $\operatorname{conv} P(\bar{x})$ at x^* (clearly $x^* \in \operatorname{frconv} P(\bar{x})$). Take the $U(x^*, \bar{x})$ of Assumption 2.8, so $U(x^*, \bar{x}) \subset P(\bar{x})$, $U(x^*, \bar{x})$ spans x^*, and $\operatorname{rad} U(x^*, \bar{x}) \leqq \kappa$. Write $U(x^*, \bar{x}) = \{u^1, u^2, \ldots u^{n'}\}$ and suppose that for some $s' \in \{1, \ldots n'\}$, $p^* \cdot u^{s'} \neq p^* \cdot x^*$. Suppose, in fact, that $p^* \cdot u^{s'} > p^* \cdot x^*$ (the proof is symmetrical for the other case). Then the half-space $\{x \mid p^* \cdot x \geqq p^* \cdot x^*\}$ contains $\operatorname{conv} P(\bar{x})$, so for all $s \in \{1, \ldots n'\}$ $p^* \cdot u^s \geqq p^* \cdot x^*$. There are coefficients $(\alpha^1, \alpha^2, \ldots \alpha^{n'})$ such that $x^* = \sum_{s=1}^{n'} \alpha^s u^s$ and $\sum_{s=1}^{n'} \alpha^s = 1$ and for all $s \in \{1, \ldots n'\}$ $0 \leqq \alpha^s \leqq 1$. Suppose $\alpha^{s'} > 0$; then $p^* \cdot x^* = \sum_{s=1}^{n'} \alpha^s p^* \cdot u^s > \sum_{s=1}^{n'} \alpha^s p^* \cdot x^* = p^* \cdot x^*$ which is absurd. Thus $\alpha^{s'} = 0$. Hence $\sum_{s \neq s'} \alpha^s u^s = x^*$ and $\sum_{s \neq s'} \alpha^s = 1$, so the set $U'(x^*, \bar{x}) = U(x^*, \bar{x}) \div \{u^{s'}\}$ spans x^*. ($U'(x^*, \bar{x})$ is $U(x^*, \bar{x})$ with $u^{s'}$ left out). Clearly $\operatorname{rad} U'(x^*, \bar{x}) \leqq \operatorname{rad} U(x^*, \bar{x}) \leqq \kappa$. If there is any other $s'' \in \{1, \ldots n'\}$ with $p^* \cdot u^{s''} > p^* \cdot x^*$, $u^{s''}$ may be removed similarly from the spanning set, and the process may be repeated. Finally, we achieve a set ${}^e U(x^*, \bar{x}) \subset \{x \mid p^* \cdot x = p^* \cdot x^*\}$. ∎

We can now prove the main theorem.

4.11. THEOREM. "Near equilibrium." *Let Assumptions* 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, *and* 2.8 *be satisfied. Write* $\kappa^b = \max\{\kappa^i \mid i \in I\}$. $\exists p^* \geqslant 0$ $\exists (x^{**1}, x^{**2}, \ldots x^{**m})$:

(a) $[[\forall i \in I: x^{**i} \in X^i]$ &

(b) $[\forall i \in I: p^* \cdot x^{**i} \leqq p^* \cdot w^i]$ &

(c) $[\forall i \in I\ \forall x \in X^i$: $[[p^* \cdot x \leqq p^* \cdot w^i$ & $x \notin \text{edge } X^i \cap \{x \mid p^* \cdot x = p^* \cdot w^i\}] \Rightarrow x^{**i} \succcurlyeq^i x]]$ &

(d) $[\exists x^{*t} \leqq w^t$: $| x^{*t} - \sum_{i \in I} x^{**i} | \leqq \kappa^b \sqrt{n}]]$.

Proof. Take the p^* of Lemma 4.8. $p^* \geqslant 0$ because $p^* \in S$. Take the x^{*i} of Lemma 4.8, and take x^{*t} to be $\sum_{i \in I} x^{*i}$, so $x^{*t} \leqslant w^t$. Each $x^{*i} \in \text{conv } X^i$ and $\sum_{i \in I} x^{*i} \leqq w^t$, so by the definition of G^i and Lemma 4.3, $x^{*i} \in \text{int } G^i$. Thus the prerequisites of Lemma 4.9 are satisfied; take the $\bar{x}^i$ if that lemma. By (c) of Lemma 4.9, fr $B^i(p^*)$ supports conv $P^i(\bar{x}^i)$ at x^{*i}, and fr $B^i(p^*) = \{x \mid p^* \cdot x = p^* \cdot w^i\} = \{x \mid p^* \cdot x = p^* \cdot x^{*i}\}$ (because it contains x^{*i}). Thus the prerequisites of Lemma 4.10 are satisfied, and there is, for each $i \in I$, an ${}^eU^i(x^{*i}, \bar{x}^i) \subset P^i(\bar{x}^i)$ spanning x^{*i}, with radius not greater than κ^i, and contained in the budget plane $\{x \mid p^* \cdot x = p^* \cdot w^i\}$. $x^{*i} \in \text{conv } {}^eU^i(x^{*i}, \bar{x}^i)$ so $x^{*t} \in \sum_{i \in I} \text{conv } {}^eU^i(x^{*i}, \bar{x}^i)$. By the Shapley–Folkman theorem, therefore, there are x^{**i} in the appropriate ${}^eU^i(x^{*i}, \bar{x}^i)$ with $| x^{*t} - \sum_{i \in I} x^{**i} | \leqq \kappa^b \sqrt{n}$. That is property (d) of the theorem. $x^{**i} \in {}^eU^i(x^{*i}, \bar{x}^i) \subset X^i$, so property (a) is satisfied. $x^{**i} \in {}^eU^i(x^{*i}, \bar{x}^i) \subset \{x \mid p^* \cdot x = p^* \cdot w^i\}$, so property (b) is satisfied. $x^{**i} \in {}^eU^i(x^{*i}, \bar{x}^i) \subset P^i(\bar{x})$, so $x^{**i} \succcurlyeq^i \bar{x}^i$, so property (c) is satisfied by property (b) of Lemma 4.9. ∎

5. "The Problem of the Edge"

Theorem 4.11 suffers from having only approximate feasibility, but its unique ailment is in property (c). It is possible, according to the theorem, that some of the consumers may find they have available within their budget sets some points which are strictly preferred to the consumption that the theorem allocates to them. The theorem also says, however, that if such points exist they will have to be both in the budget *plane* and on the edge of the consumption set.

I claim that such anomalous points could exist only by an unlikely coincidence. First of all, in the case with only one divisible commodity, it is even unlikely that the budget plane will meet the edge of the consumption set at all, as can be seen by considering different possible budget planes in Fig. 5. Figure 5, though, shows that there is a further improbable condition necessary to make the anomaly possible. It is true that with several divisible commodities, the budget plane may meet edge X^i more easily, but to cause trouble it must also be tangent at that meeting point to a significant face of the convex hull of a not-worse-than set. For, in the diagram, x, to be preferred to x^{**i}, cannot be below the budget plane. x^{**i} is also in the budget plane. There must, therefore, be a tangency of

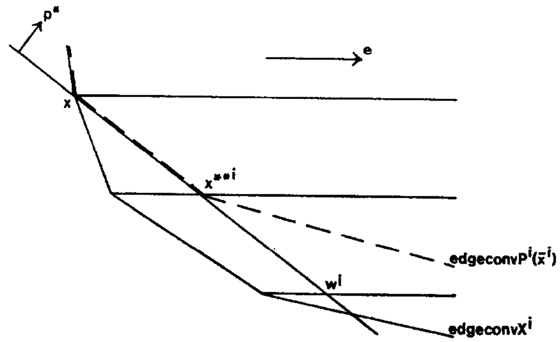

FIG. 5. The anomalous case.

the type illustrated to allow x and x^{**i} to be distinct. And that *is* an unlikely chance.

6. The Literature

The importance of Debreu's work [2] to this study is so obvious that it scarcely needs mention; his approach has determined mine throughout. It is also very clear how closely my method has followed Starr's [10] in dealing with nonconvexities.

Because the synthetic preference relation has not been made explicit in the proof of Theorem 4.11, part of the relationship with Starr's work has been obscured. The chief difficulty has obviously been in the upper semicontinuity of the demand correspondence. In conventional proofs in convex models, that continuity follows naturally from the continuity of the preference preorder; using the technique of synthesis, it could be derived from the continuity of the synthetic preorder. In the indivisible case, that preorder (Definition 3.2) is indeed continuous, under the assumptions of Section 2, except on the edge of the synthetic consumption set conv X^i; most of the elements of the proof may be found in the proofs of Lemmas 4.5, 4.6, and 4.7. Of course, continuity in the divisible but nonconvex case requires fewer assumptions to establish. Nevertheless, it is an intriguing fact that there may possibly be discontinuities in the synthetic preorder at the frontier even of convex consumption sets. Starr devotes little space to the question of continuity, so I have included an appendix which discusses it for a divisible model. The difficulties in that case may help to explain why the indivisible case is even more complicated.

Henry's article [6] is aimed chiefly at demonstrating the intractability of the indivisible model outside two dimensions. He shows convincingly that one should not expect to find an exact equilibrium in a general case, and recommends instead that one look for an approximation, as this paper has done. A more recent paper by Henry [7] has a closer relationship with mine. There, Henry demonstrates the existence of an allocation which is nearly a core. His proof appears to be almost the exact counterpart in core theory to mine in equilibrium theory. The methods are very similar; Henry also uses a convex construction very like Starr's. The core is bigger than the equilibrium, so it should be easier to find, but it is remarkable how *much* more straightforward is Henry's proof than the one in Section 4. It contains none of the irritating complexities and, more important, requires fewer assumptions. Especially, it does not need a divisible commodity. The method employs a theorem of Scarf's [9] that depends on an upper semicontinuous utility *function*. I suspect that the explanation of the greater simplicity of Henry's proof is that the existence of an upper semicontinuous synthetic utility function is equivalent to only one half of the continuity of the synthetic preorder: that the synthetic not-worse-than sets are closed. All the difficulty of the equilibrium proof is in the other half. Anyhow, it is interesting how great is the cost of shrinking a near core to a near equilibrium.

Dierker [3] uses an entirely original technique in his proof of the existence of an approximate equilibrium in an indivisible model. It is impossible to describe the method here, though it is indeed fascinating. It involves setting up a constructed economy in which people are a little insensitive to price changes. After finding a near equilibrium in that, the system is made to tend to the original economy by decreasing the degree of price insensitivity. The use of a limiting process, however, creates a weakness in Dierker's theorem. It is this: the consumption allocated to each individual is not worse than any other consumption which is *strictly* below his budget plane. That is, there may be actually preferred points in the budget plane itself. It is interesting that, with such a radically different approach, Dierker should find a qualification in the result that corresponds quite closely with the weak point in my own Theorem 4.11. But my weakness is not as weak as that. He points out that in his model, which has *no* divisible commodities, there will probably not be any, or many, proper points in the budget plane at all, so the anomaly is unlikely. That is true, but the theorem is thereby irretrievably confined to the totally indivisible case, for if there were one divisible commodity, then the budget plane would contain very many proper points and the theorem would lose its meaning.

Dierker's theorem and the theorem of this paper, therefore, seem to be permanently alienated; they deal with separate and irreconciliable situations. This is a pity, because Dierker's proof has an appeal which is quite absent from the mere complexities of mine. I shall take the opportunity, however, of defending the assumption that at least one commodity is divisible. In introducing it, I claimed that a real economy would not work without a divisible commodity, and that that was one of the functions of money. Therefore, I suggested, it was only to be expected that such a commodity would play an important part in the mathematical study of equilibrium. Yet Dierker has managed without. Does this mean that divisible money might indeed be dispensable?

In Dierker's approximate equilibrium, an individual's allocation is almost always below his budget plane. We can create a picture as follows. The traditional auctioneer of market theory has a warehouse. Everyone brings to it what they want to sell, and takes away goods up to the same value (trade between individuals would turn into a game situation which cannot be allowed). Suppose a farmer wants to buy a cow with sheep when the price is one cow to six and a half sheep. If he wants the cow a lot, he will have to pay seven sheep. This may represent the best point available within his budget set, and correspond to the Dierker equilibrium. Nevertheless, the farmer suffers a definite and conspicuous loss of half a sheep, which he will not be happy about. Anyone whose consumption is left below his budget plane is in the same situation.

The essence of the divisible commodity of the present paper, though, is that it is desirable. If the farmer received change in corn or in valuable money, there will be no loss to him. If it is in money, its value for him must be independent of its use as a medium of exchange; the value has to come from *outside* the system, for otherwise it could be divided out as irrelevant. It might, for example, carry value to a different time period. The farmer might receive a token for half a sheep valid also *next* market day. He might even press the auctioneer to give him such a token. That is a myth about the origin of money which points to its smoothing function over indivisibilities, and which also describes the difference between Dierker's model and mine.

Appendix

The difficulty over continuity in the indivisible case will seem less surprising when one realizes that the divisible model is not straightforward. Even if consumption sets (though not the preference preorders) are *convex* one may encounter discontinuities in the constructed preorder around the frontier of the consumption set. This appendix takes the synthetic preorder as defined by Starr (Definition 3.2) and studies its continuity under the conditions of Assumption 2.1 and

A.1. Assumption. "All commodities divisible." $n_d = n$.

The interest of the matter is that the continuity of the preorder may be used to prove the upper semicontinuity of the synthetic demand correspondence (see Starr [10]). We deal only with the affairs of one individual, so the index i is omitted throughout.

The synthetic preorder $S \succcurlyeq$ is said to be "continuous" if the sets $SP(\hat{x}) = \{x \in \operatorname{conv} X \mid x S \succcurlyeq \hat{x}\}$ and $SQ(\hat{x}) = \{x \in \operatorname{conv} X \mid \hat{x} S \succcurlyeq x\}$ are both always closed. There is no difficulty in proving $SP(\hat{x})$ to be always closed; it follows quickly from Lemma 4.1, and does not even require a single divisible commodity. The closedness of $SQ(\hat{x})$ is harder. It will be shown that it follows from this particular property of the consumption set:

A.2. Definition. "Continuous spannibility." *X is continuously spannible* $\Leftrightarrow [\forall x \in \operatorname{conv} X\ \forall U(x) \subset X\ \forall \delta > 0\ \exists \epsilon > 0$: $[U(x)$ *spans* $x \Rightarrow \operatorname{conv} X \cap N(x, \epsilon) \subset \operatorname{conv} \bigcup_{u \in U(x)} (X \cap N(u, \delta))]]$.

Briefly, to say that a set is continuously spannible means that, given a point and a subset of the given set which spans it, points near the given point are spanned by points near the members of the spanning set. The property lies at the heart of the question of continuity, as Theorem A.5 shows.

A.3. Lemma. *Let Assumptions* 2.1 *and* A.1 *be satisfied.* $\forall \bar{x} \in X$ $\forall \hat{x} \in \operatorname{conv} X$: $[\hat{x} \notin \operatorname{conv} P(\bar{x}) \Rightarrow [\exists \bar{\bar{x}} \in X: [\bar{x} \succ \bar{\bar{x}}\ \&\ \hat{x} \notin \operatorname{conv} P(\bar{\bar{x}})]]]$.

Proof. The proof is identical with the proof of Lemma 4.5, except for the second paragraph from last, which is replaced by:

Write the unit vector (1, 1,... 1) as e^d. For λ large enough $\tilde{x} + \lambda e^d \geqq \bar{x}$, so by monotonicity $\tilde{x} + \lambda e^d \succcurlyeq \bar{x}$. Take $\bar{\lambda} = \inf\{\lambda \mid \tilde{x} + \lambda e^d \succcurlyeq \bar{x}\}$, then by the continuity of $\succcurlyeq$, $\tilde{x} + \bar{\lambda} e^d \sim \bar{x}$. For any $\lambda < \bar{\lambda}$, $\bar{x} \succ \tilde{x} + \lambda e^d$. Also, since $Q(\tilde{x})$ is closed and $\tilde{x} + \bar{\lambda} e^d \notin Q(\tilde{x})$, if λ is sufficiently close to $\bar{\lambda}$, $\tilde{x} + \lambda e^d \notin Q(\tilde{x})$. Write such a point $\tilde{x} + \lambda e^d$ as $\bar{\bar{x}}$, so $\bar{x} \succ \bar{\bar{x}} \succ \tilde{x}$. ∎

A.4. Lemma. *Let Assumption* 2.1 *be satisfied, and let X be continuously spannible.* $\forall \bar{x} \in X$ $\forall \bar{\bar{x}} \in X$ $\forall x \in \operatorname{conv} P(\bar{x})$: $[\bar{x} \succ \bar{\bar{x}} \Rightarrow x$ *belongs to the interior of* $\operatorname{conv} P(\bar{\bar{x}})$ *relative to* $\operatorname{conv} X]$.

Proof. Take a spanning set $U(x) \subset P(\bar{x}) \subset X$, spanning x. By the continuity of $\succcurlyeq$, each $u \in U(x)$ has a $\delta(u)$ such that $X \cap N(u, \delta(u)) \subset P(\bar{\bar{x}})$. Take $\delta = \min\{\delta(u) \mid u \in U(x)\}$ and take the ϵ of Definition A.2. Then $\operatorname{conv} X \cap N(x, \epsilon) \subset \operatorname{conv} \bigcup_{u \in U(x)} (X \cap N(u, \delta)) \subset \operatorname{conv} P(\bar{\bar{x}})$. ∎

A.5. Theorem. *Let Assumptions* 2.1 *and* A.1 *be satisfied, and let X be continuously spannible.* $\forall \hat{x} \in \operatorname{conv} X$: *$SQ(\hat{x})$ is closed.*

Proof. Suppose $SQ(\hat{x})$ is not closed for some $\hat{x} \in \operatorname{conv} X$, and let $x \in \operatorname{cl} SQ(\hat{x})$ with $x \notin SQ(\hat{x})$. $x \in \operatorname{conv} X$ because $\operatorname{conv} X$ is closed (Lemma 42.). $x\, S \succ \hat{x}$ so there is an $\bar{x} \in X$ with $x \in \operatorname{conv} P(\bar{x})$ and $\hat{x} \notin \operatorname{conv} P(\bar{x})$ (by Definition 3.2). Take the $\bar{x}$ of Lemma A.3, so by Lemma A.4, x belongs to the interior of $\operatorname{conv} P(\bar{x})$ relative to $\operatorname{conv} X$. That is, there is an ϵ such that $\operatorname{conv} X \cap N(x, \epsilon) \subset \operatorname{conv} P(\bar{x})$. Because $x \in \operatorname{cl} SQ(\hat{x})$, there is an $\tilde{x} \in SQ(\hat{x})$ which is in $\operatorname{conv} X \cap N(x, \epsilon)$ and hence in $\operatorname{conv} P(\bar{x})$. And this contradicts that $\hat{x} \notin \operatorname{conv} P(\bar{x})$, since $\hat{x}\, S \succcurlyeq \tilde{x}$. ∎

Having established the importance of the concept, I give a more intuitive condition which entails continuous spannibility.

A.6. Definition. "Local conicalness." *X is locally conical* $\Leftrightarrow$ $[\forall \bar{x} \in X \quad \exists \gamma > 0 \quad \forall x \in X \cap N(\bar{x}, \gamma) \quad \forall \lambda > 0\colon [\bar{x} + \lambda(x - \bar{x}) \in N(\bar{x}, \gamma) \Rightarrow \bar{x} + \lambda(x - \bar{x}) \in X]]$.

The definition means that around any point in a locally conical set there is always a neighborhood such that the part of the set inside the neighborhood is a cone with vertex the given point. It can be seen that sets consisting of the intersection of a finite number of closed half-spaces are locally conical. In particular, and this is the significance of the concept, the nonnegative orthant is locally conical. Thus the following lemma, with Theorem A.5, confirms the reliability of the method of synthesis, *provided* the consumption set is the nonnegative orthant (cf. Starr [10]).

A.7. Lemma. *X is convex and locally conical $\Rightarrow$ X is continuously spannible.*

Proof. Take any $\bar{x} \in X$ and any spanning set $U(\bar{x}) = \{u^1,\ldots u^{n'}\} \subset X$ spanning $\bar{x}$. Take any $\delta > 0$. Take a $\gamma > 0$ with the property of the definition of local conicalness. Define $\epsilon = \min\{\gamma, \frac{1}{2}\gamma\delta/(\gamma + \max_s |\bar{x} - u^s|)\}$. It will be shown that ϵ has the property demanded by continuous spannibility. Take any $\tilde{x} \in X \cap N(\bar{x}, \epsilon)$. It will be shown that

$$\tilde{x} \in \operatorname{conv} \textstyle\bigcup_s (X \cap N(u^s, \delta)).$$

Consider first the case when $\epsilon = \gamma$, so $\delta \geqq 2(\gamma + \max_s |\bar{x} - u^s|)$. Then

$$|\tilde{x} - u^s| \leqq |\tilde{x} - \bar{x}| + |\bar{x} - u^s| \leqq y + \tfrac{1}{2}\delta - y$$
$$= \tfrac{1}{2}\delta \qquad \text{for all} \quad s \in \{1,\ldots n'\}.$$

Thus, $\tilde{x} \in X \cap N(u^s, \delta) \subset \operatorname{conv} \bigcup_s (X \cap N(u^s, \delta))$.

Consider next the other case, when $\epsilon = \frac{1}{2}\gamma\,\delta/(\gamma + \max_s |\bar{x} - u^s|)$. Write $\tilde{x}' = (\gamma/\epsilon)\tilde{x} + (1 - (\gamma/\epsilon))\bar{x}$, so $\tilde{x}' \in N(\bar{x}, \gamma)$ and hence by local conicalness $\tilde{x}' \in X$. Because X is convex $\beta\tilde{x}' + (1 - \beta) u^s \in X$ for all $s \in \{1,\dots n'\}$, provided $0 \leqq \beta \leqq 1$. So if $0 < \beta < \delta/(|\tilde{x}' - u^s|)$, then $\beta\tilde{x}' + (1 - \beta) u^s \in X \cap N(u^s, \delta)$. In fact, take

$$\beta = \epsilon/\gamma = \tfrac{1}{2}\,\delta/(\gamma + \max_s |\bar{x} - u^s|) \leqq \tfrac{1}{2}\,\delta/(|\tilde{x}' - u^s|),$$

for all $s \in \{1,\dots n'\}$. Since $U(\bar{x})$ spans $\bar{x}$, we have $\bar{x} = \sum_{s=1}^{n'} \alpha^s u^s$ with $\sum_{s=1}^{n'} \alpha^s = 1$ and $0 \leqq \alpha^s \leqq 1$ for all $s \in \{1,\dots n'\}$. Because for all $s \in \{1,\dots n'\}$ $\beta\tilde{x}' + (1 - \beta) u^s \in X \cap N(u^s, \delta)$,

$$\sum_{s=1}^{n'} \alpha^s(\beta\tilde{x}' + (1 - \beta) u^s) \in \operatorname{conv} \bigcup_s (X \cap N(u^s, \delta)).$$

But this point is $\beta\tilde{x}' + (1 - \beta)\bar{x} = (\epsilon/\gamma)\tilde{x}' + (1 - (\epsilon/\gamma))\bar{x} = \tilde{x}$. ∎

Finally, as promised, I give an example of a convex set in which the synthetic preorder is *not* certainly continuous. This will be a set which is not continuously spannible. Continuous spannibility is not a *necessary* condition for the desired continuity, but what is true is that in any set which is not continuously spannible it would be possible to construct a preference relation such that the synthetic preorder is not continuous.

The set $\{(x, y, z) \mid x \geqq z^2(1 + y^2)\}$ is not continuously spannible. Moreover, the segment of its frontier given by $-1 \leqq y \leqq 0$ and $-1 \leqq z \leqq 0$ could be part of the frontier of a consumption set without violating any of the requirements of Assumption 2.1, and that consumption set would not be continuously spannible. Very briefly, the reason is this: the surface $x = z^2(1 + y^2)$ contains the line $x = z = 0$; points on this line may therefore be spanned by various sets of other points on the line. But everywhere else the surface is curved (technically, every other point on the frontier is an extreme point), so that all other frontier points are significantly spanned only by themselves. But they may be very near a point on the line $x = z = 0$, which can be spanned by quite distant points. Hence nearby points are not always spanned by points near members of the spanning set, as continuous spannibility requires.

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