Control Theory and Multiproduct Nonlinear Pricing

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Abstract

I derive results from Armstrong's (1996) paper on mechanism design with several dimensions of private information using a result from distributed parameter control theory. I indicate how Armstrong's analysis might be extended to a wider class of problems.
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1. Introduction

In an important paper, Armstrong (1996) analyses a nonlinear pricing problem in which consumer types have more than one dimension. The standard approach to this problem with scalar consumer types is to use the necessary conditions for an optimum to derive optimal consumer demand functions, and then afterwards to demonstrate that these demand functions satisfy an implementability condition. In the scalar case, Armstrong simplifies the firm's objective function by integrating it by parts, using a first order condition to simplify the calculation. He reduces the multidimensional case to a similar problem by integrating the objective function along rays from the origin.

A standard textbook method for solving the scalar case uses optimal control theory. In this paper I demonstrate that Armstrong's method in higher dimensions of "integrating along a ray" is equivalent to a method in distributed-parameter control theory which is due to Derzko, Sethi and Thompson (1984). Several state equations are feasible in this framework: Armstrong's path of integration corresponds just one of these.

2. The Model

Armstrong considers a firm which has a monopoly over n goods. Consumers who pay t to consume a bundle $x \in \Re^n$ of goods receive utility $u(\alpha, x) - t$, where the consumer's type $\alpha \in \Re^m$ is her private knowledge. Armstrong assumes that u is homogenous of degree 1 in α with $u(0, x) = u(\alpha, x) = 0$. Types have density function $f(\alpha)$ with support $A \subseteq \Re^m_+$.

The cost to the firm of serving x to any customer is $c(x)$ for a smooth $c(.)$ with $c(0) = 0$. The firm charges a tariff $t(x)$ for the bundle of goods x, with $t(0) \leq 0$. Consumer surplus is then

$$s(\alpha) = \max_{x \geq 0} u(\alpha, x) - t(x). \tag{CS}$$

If $s(.)$ is differentiable at α then we have the following envelope condition:

$$\nabla s(\alpha) = u_\alpha(\alpha, x(\alpha)). \tag{EC}$$

A surplus function $s(.)$ and demand function $x(.)$ are *implementable* if they may be induced from (CS) and (EC) by means of some tariff $t(.)$ such that $t(0) = 0$. Armstrong notes that x is defined and s is differentiable almost everywhere in the Lebesgue sense.

The firm aims to maximize the profit function π below subject to (CS) and (EC) and the requirement that s and x are implementable.

$$\pi = \int_A [u(\alpha, x(\alpha)) - s(\alpha) - c(x(\alpha))] f(\alpha) d\alpha. \tag{1}$$

Armstrong proves (section 3) that in contrast to the scalar case, a positive measure of consumers will be excluded when $m > 1$. He then turns his attention to the solution of the firm's profit maximization problem. When $m = 1$, the utility function is $\alpha u(x)$, the envelope condition is $u(x(\alpha)) = s'(\alpha)$, and $A = [\alpha_*, \alpha^*]$. Integrating $s(\alpha) f(\alpha)$ by parts over $[\alpha_*, \alpha^*]$ allows one to express the firm's pointwise optimization problem as the selection of $x^*(\alpha)$ to maximize $\left(\alpha - \frac{1-F(\alpha)}{f(\alpha)}\right) u(x) - c(x)$. Armstrong provides sufficient conditions for x^* to be implementable.

In the case of vector α ($m > 1$), Armstrong sets $A = \Re^m_+$. Since $s(0) = 0$,

$$s(\alpha) = \int_0^1 \frac{d}{dr} s(r\alpha) dr = \int_0^1 \alpha' u_\alpha(r\alpha, x(r\alpha)) = \int_0^1 \frac{1}{r} u(r\alpha, x(r\alpha)) dr, \tag{2}$$

where the final equality follows from the homogeneity of u in α, which is essential to this technique. Equation (2) allows one to integrate $s(\alpha) f(\alpha)$ by parts and hence, after some manipulation (pages 62, 63 of Armstrong's paper) to show that candidate demand functions x^* solve

$$\max \left(1 - \frac{g(\alpha)}{f(\alpha)}\right) u(\alpha, x) - c(x), \text{ where } g(\alpha) = \int_1^\infty t^{m-1} f(t\alpha)\, dt. \tag{3}$$

Armstrong provides sufficient conditions for implementability of solutions to (3).

3. An Alternative Approach to Finding Candidate Functions

The scalar problem of maximizing $\int_{\alpha_*}^{\alpha^*} [\alpha u(x(\alpha)) - s(\alpha) - c(x(\alpha))] f(\alpha)\, d\alpha$ subject to $s'(\alpha) = u(x(\alpha))$ and $s(0) = 0$ is a standard textbook control problem with state variable s and control x. The Hamiltonian $H(s, x, \lambda, \alpha)$ is $[\alpha u(x) - s - c(x)] f(\alpha) + \lambda(\alpha) u(x)$ and the adjoint equation $-H_s = \lambda'(\alpha)$ with boundary condition $\lambda(\alpha^*) = 0$ (because the surplus function is zero) immediately yields $\lambda(\alpha) = -(1 - F(\alpha))$, from which the maximizing requirement for the Hamiltonian yields in turn Armstrong's characterization of $x^*(\alpha)$ as the maximand of $\left(\alpha - \frac{1-F(\alpha)}{f(\alpha)}\right) u(x) - c(x)$.

In the case when $m > 1$ the state equation becomes a partial differential equation. The envelope condition (EC) which yields the state equation in the scalar case is in the vector case itself a vector equation: to derive a state equation we can premultiply both sides by α' to yield

$$\alpha' \nabla s(\alpha) = \alpha' u_\alpha(\alpha, x(\alpha)) = u(\alpha, x(\alpha)), \tag{4}$$

where as in equation (2), the last equality follows from the homogeneity of $u(.)$. (See below for a discussion of other utility functions and other state equations.) Maximization of π as defined in equation (1) subject to equation (4) and the initial condition $s(0) = 0$ is an example of a *distributed-parameter optimal control problem.* Derzko et al. (1984) provide necessary conditions for its solution.

For an N vector $J = (j_1, \ldots, j_n)$, define $\partial^J \equiv \partial_1^{j_i} \ldots \partial_n^{j_n}$ and let $|J| \equiv j_1 + \ldots + j_n$. Then if the linear partial differential operator A is given by $Ax(\alpha) = \sum_J a_J(\alpha) \partial^J x(\alpha)$, its *adjoint* A' is defined by $A'\lambda \equiv \sum_J (-1)^{|J|} \partial^J (a_J \lambda)$. Consider the problem of maximizing $\int_A F(s, x, \alpha)\, d\alpha$ subject to the requirement that $A(\alpha) x = u(s, x, \alpha)$ and the boundary condition $s(0) = 0$.[1] Form the Hamiltonian $H(s, x, \lambda, \alpha) \equiv F(s, x, \alpha) + \lambda u(s, x, \alpha)$. Then Derzko *et. al.* show that necessary conditions for $s(\alpha), x(\alpha)$ to constitute an optimum are equation (4), $s = 0$,

$$A'\lambda(\alpha) = \partial_s H(s(\alpha), x(\alpha), \lambda(\alpha), \alpha); \tag{5}$$

$$\lambda(\alpha) = 0 \text{ on } \partial A; \tag{6}$$

$$\forall x. H(s(\alpha), x(\alpha), \lambda(\alpha), \alpha) \geq H(s(\alpha), x, \lambda(\alpha), \alpha). \tag{7}$$

In equation (6) ∂A denotes the boundary of A. Let $\|.\|$ denote the Euclidean norm. For now, I assume that $A = \left\{\alpha \in \Re_+^m : \|\alpha\| \leq R\right\}$ is the intersection of the hypersphere of radius R with the positive orthant $\Re_+^m$: I consider Armstrong's limiting case where $A = \infty$ below.

In the special case of tariff selection, the Hamiltonian becomes

$$H(s, x, \lambda, \alpha) = [u(\alpha, x) - s - c(x)] f(\alpha) + \lambda u(\alpha, x(\alpha)),$$

[1] Derzko *et. al.* (equations 27 and 32) consider more complex boundary conditions than Armstrong: as they are unnecessary in this case, we consider only this special case.

and the differential operator A is defined by $As(\alpha) = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial \alpha_i} s(\alpha)$. The adjoint equation is therefore

$$A'\lambda(\alpha) = -\sum_{i=1}^{m} \frac{\partial}{\partial \alpha_i}(\alpha_i \lambda(\alpha)) = -m\lambda(\alpha) - \sum_{i=1}^{m} \alpha_i \frac{\partial \lambda(\alpha)}{\partial \alpha_i} = H_s = -f(\alpha). \quad (8)$$

I solve (8) using the method of characteristics. Set $\alpha_i = \alpha_i(x)$ so that $\lambda = \lambda(x)$. Then we require $\alpha_i'(x) = \alpha_i$, or $\alpha_i(x) = \hat{\alpha}_i e^x$ for any $\hat{\alpha} \equiv (\hat{\alpha}_1, \ldots, \hat{\alpha}_m) \in A$; and $\lambda'(x) = f(\alpha) - m\lambda(x)$, whence $\frac{d}{dx}(\lambda e^{mx}) = e^{mx} f(\alpha)$. Integrate this equation from 0 to $\ln \frac{R}{\|\hat{\alpha}\|}$, noting that as $\hat{\alpha} \in \partial A$, $\lambda\left(\ln \frac{R}{\|\hat{\alpha}\|}\right) = 0$ to get

$$\lambda(0) = \lambda(\hat{\alpha}) = -\int_0^{\ln \frac{R}{\|\hat{\alpha}\|}} f(\hat{\alpha} e^x) e^{mx} dx = -\int_1^{\frac{R}{\|\hat{\alpha}\|}} f(\hat{\alpha} t) t^{m-1} dt \equiv -g(\hat{\alpha}). \quad (9)$$

Condition (7) implies that $x^*(a)$ solves

$$\max \left(1 - \frac{g(\alpha)}{f(\alpha)}\right) u(\alpha, x) - c(x). \quad (10)$$

Equation (10) is analogous to (3), in the case where A is finite. To derive (3) we can let $R \to \infty$.[2]

4. Additional Candidate Functions

Armstrong's original work relies upon the homogeneity (not necessarily of degree 1) of the utility function u. Provided the control equation is soluble, this is not necessary for the method presented here. For example, suppose that $u(\alpha, x) = \sum_{i=1}^{m} \ln \alpha_i + v(x)$. Then $\alpha' u_\alpha(\alpha, x(\alpha)) = m$. Since the operator A is the same in this case as in the previous section, we have the same adjoint equation and $\lambda(a)$ is given by equation (9). The Hamiltonian is therefore $[u(\alpha, x) - s - c(x)] f(\alpha) + mg(\alpha)$ and the candidate function therefore maximizes $u(\alpha, x) - c(x)$.

Equally, premultiplying (EC) by α' does not yield the unique control equation. Consider the case in which $A = \Re_+^m$. Set $u(\alpha, x) = -\frac{x^2}{m} \sum_{i=1}^{m} \alpha_i$. Premultiply both sides of (EC) by the row vector $(1, 1, \ldots, 1)$ to obtain control equation $\sum_{i=1}^{m} \frac{\partial s}{\partial \alpha_i} = -x^2$. The adjoint equation is therefore $\sum_{i=1}^{m} \frac{\partial \lambda}{\partial \alpha_i} = f(\alpha)$. Solving using the method of characteristics yields $\lambda(\alpha) = -G(\alpha) \equiv -\int_0^\infty f(\alpha_1 + x, \alpha_2 + x, \ldots, \alpha_m + x) dx$, from which it follows that x^* maximizes $u(\alpha, x) + \frac{G(\alpha)}{f(\alpha)} x^2 - c(x)$.

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[2]Strictly speaking, we need to check that $\lim_{x \to \infty} \lambda(x) e^{mx} = 0$. The limiting boundary condition for the infinite case from Derzko *et. al.* is that $\lambda(x)(\xi_1(x) + \xi_2(x)) \to 0$ as $x \to \infty$ for any ξ which is zero at the origin: this is sufficient for our needs.