Decomposing Linear Representations of Finite Groups

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Abstract

We develop a package using the computer algebra system GAP for computing the decomposition of a representation ρ of a finite group G over $\mathbb{C}$ into irreducibles, as well as the corresponding decomposition of the centraliser ring of $\rho(G)$. Currently, the only open-source programs for decomposing representations are for non-zero characteristic fields. While methods for characteristic zero are known, there are no open-source computer programs that implement these methods, nor are details on how to achieve good performance of such an implementation published. We aim to record such details and demonstrate an application of our program in reducing the size of semidefinite programs.

Contents

1 Introduction

1.1 Motivation and Requirements

In 1971, in his graduate textbook on the linear representations of finite groups [1], Serre specified a method of computing the decomposition of a representation of a finite group G into irreducible subrepresentations. Despite the existence of Serre's text, this algorithm has no open source implementation. Indeed, there is no open source program solving this problem in general at all. This is the problem our project aims to solve, using the computer algebra systems GAP [2] and SageMath [3].

Specifically, we aim to provide a GAP package providing the functionality to compute the following, given a finite dimensional representation of a finite group $\rho : G \to \mathrm{GL}(V)$, where V is a $\mathbb{C}$-vector space.

- A decomposition of V into irreducible subrepresentations
- The corresponding block diagonalisation of ρ, along with the associated basis change matrix
- A basis for the centraliser ring $C_\rho \subseteq \mathrm{End}(V)$

We only deal with the case where the representation ρ is over $\mathbb{C}$ - in particular, this means we are in characteristic zero. In the case where the representation is over a finite field, a user could take advantage of the Meataxe algorithms, introduced by Parker [4] and later improved by various authors. These algorithms have been implemented in GAP and allow computations of decompositions of modules, tests of irreducibility, isomorphisms between modules and so on. None of these methods are applicable in the case of a characteristic zero field, which is what we focus on in this project.

We also aim to extensively test and document our package to ensure correctness and ease of use. After the completion of the project, our goal is for this package to be included with the GAP distribution.

1.2 Outline

In Section 2, we provide some background on representation theory, group theory, and some terminology (some of which is not standardised) that will be used throughout this project.

In Section 3, we describe Serre's algorithms to decompose a representation of a finite group, with some discussion of performance and possible improvements.

In Section 4, we describe an alternate algorithm of our own design, incorporating some results due to Serre and various small performance enhancements.

In Section 5, we describe our testing and benchmarking methodology. We analyse the performance, focusing on the effect of the size of G and the degree of the representation ρ on the running time, as well as measuring the effect of our optimisations.

In Section 6, we describe an application of our program to reducing the dimension of semidefinite programs, using a method due to de Klerk et al. [5]. We reproduce their calculation of a bound on the crossing number of $K_{m,n}$ (a complete bipartite graph) using our method to formulate the program.

In Section 7, we describe how our contributions satisfy the requirements, possible improvements, and further work that could be done on this project.

In the appendix Section A, we implement an algorithm to unitarise representations using methods developed while implementing the main algorithms of this project. We also describe an algorithm due to Dixon [6] for decomposing unitary representations that has some desirable properties. This section is not a part of the requirements set out earlier, but is a possibly useful consequence of our work.

2 Background

In this section, we provide some background information required to understand the results presented in this project. A basic understanding of group theory and linear algebra is assumed, but nothing more.

2.1 Representation Theory

Let G be a finite group. A *representation* of G is a homomorphism ρ from G to the automorphism group of a vector space, $\mathrm{GL}(V)$. We will only consider the case where V is a finite-dimensional vector space over $\mathbb{C}$.

A common abuse of terminology is to refer to V, the vector space, as a representation of G. This is done only when it is clear what the action of G is.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G.

ρ is *isomorphic as a representation* to $\tau : G \to \mathrm{GL}(W)$ if there is a linear isomorphism $\alpha : V \to W$ such that for all $g \in G$, $\alpha \circ \rho(g) = \tau(g) \circ \alpha$. In other words, ρ and τ differ by a change of basis.

A *subrepresentation* of ρ is a representation $\rho|_W : G \to \mathrm{GL}(W)$, where W is a subspace of V which is invariant under the action of G. The action on W is given by $\rho|_W(g)w = \rho(g)|_W w$. This is well defined because W is preserved by $\rho(g)$.

A representation ρ is said to be *irreducible* if it has no subrepresentations other than 0 and V. Note that, for finite group representations over $\mathbb{C}$, this is equivalent to saying that V does not break down into a direct sum of subrepresentations [1]. We will sometimes refer to an "irreducible representation" as an "irreducible".

Let $W_1, \ldots, W_n$ be the complete list of irreducibles of G. If $V = \bigoplus_{i=1}^{m} U_i$ is the decomposition of V into irreducibles, then the *canonical decomposition* of V is $\bigoplus_{k=1}^{n} V_k$ where each V_k is the direct sum of the U_i which are isomorphic as representations to W_k.

The *centraliser* (sometimes called the *commutant*) of a representation ρ is the $\mathbb{C}$-vector space of linear maps A such that $A\rho(g) = \rho(g)A$ for all $g \in G$.

Given ρ, the dual representation ρ^* is the representation defined by:

$$\rho^*(g) = \rho(g^{-1})^T$$

2.2 Orbitals

A more detailed presentation of some of these definitions, with illustrative examples, can be found in P. Cameron's *Permutation Groups* [7].

Frequently, it will be convenient to consider a representation $\rho : G \to \mathrm{GL}(V)$ where all of the images $\rho(g)$ are permutation matrices as a map $\rho : G \to \mathrm{Sym}(X)$, where X is a basis for V. This is a group action of G on X.

An *orbital* of this action is an orbit of G acting on $X \times X$ by: $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

The *orbital (di)graph* associated with an orbital Δ is the directed graph with vertex set X and directed edge set Δ.

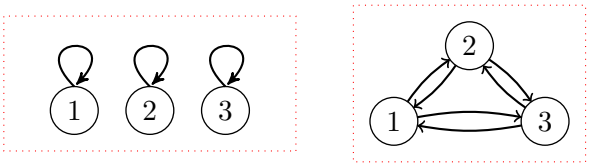

Figure 1: S_3 acting on $\{1, 2, 3\}$ has two orbital digraphs

When we refer to an orbital as a matrix, this refers to the adjacency matrix of the orbital graph.

Computing with the full adjacency matrices is inconvenient due to their space inefficiency. It is more convenient to work with collapsed versions of these matrices, which allow computations to be done much faster.

Let $\alpha \in X$ and fix an ordering of the orbits of G_α (the stabiliser of α): $X_1 = \{\alpha\}, X_2, \ldots, X_r$. Choose representatives $\alpha_i \in X_i$. Let $\Gamma = (X, \Delta)$ be an orbital graph.

A *collapsed adjacency matrix* (in the sense of Praeger and Soicher [8]) for Γ, with respect to the choice of α, representatives, and orderings, is the $r \times r$ integer matrix A defined by:

$$A_{ij} = |\Gamma(\alpha_i) \cap X_j|$$

That is, A_{ij} is the number of neighbours of α_i in the graph Γ which are contained in X_j.

These collapsed matrices are useful because they provide a natural isomorphism from the algebra generated by orbital matrices X to a smaller-dimensional algebra Y, given by the collapsed orbital matrices. This means that whenever we have a problem in X that can be formulated in terms of spectra of matrices, we can pass to Y using the isomorphism and solve the same problem in Y. A detailed explanation of the isomorphism can be found in [5], and an abbreviated explanation in Section 6, where we use this method to reduce the dimension of a semidefinite program.

2.3 Cyclotomic numbers

Cyclotomic numbers are the numbers $z \in \mathbb{C}$ such that $z \in \mathbb{Q}(\zeta_n)$ for some n, where ζ_n is a complex primitive nth root of unity. $\mathbb{Q}(\zeta_n)$ is called a *cyclotomic field*.

Brauer [9] proved that every irreducible complex character of a finite group G can be realised by a representation of G over the cyclotomic field $\mathbb{Q}(\zeta_x)$, where x is the exponent of G (the smallest x such that for all $g \in G$, $g^x = 1_G$). This means that if we restrict our attention to representations with cyclotomic coefficients, we get all representations of G, up to isomorphism.

In GAP (the computer algebra system that we use), we can represent general algebraic numbers, but the algorithms for performing arithmetic with these is not efficient when compared with cyclotomic numbers. Due to Brauer's theorem, we do not need to consider non-cyclotomic algebraic numbers.

Throughout this project, when we take a representation over $\mathbb{C}$, we assume this representation is over the cyclotomic numbers.

2.4 Computer algebra systems used

To implement this project, we used two computer algebra systems: GAP [2] and SageMath [3].

The main part of the project is a GAP package, implemented in the GAP programming language, which is designed to be easy to read for mathematicians and programmers alike. We will refrain from using non-obvious features in code snippets without explanation. A reference for the programming language can be found here: `https://www.gap-system.org/Manuals/doc/ref/chap4.html`.

Some parts of the project are implemented using SageMath, which uses the Python programming language. This was necessary since certain algorithms are only implemented in SageMath, like the computation of a complete list of irreducibles of S_n using integer matrices. SageMath also has a convenient interface to both GAP and a semidefinite program solver, making it an ideal choice to implement Section 6, which explores how our project can be used to speed up computations by finding the optimal block diagonalization for a semidefinite program.

3 Algorithms due to Serre

The algorithms and theorems in this section are taken partially from Serre's text on representation theory [1]. In that text, you will find a more mathematically rigorous treatment of this material, including full proofs of correctness. Our focus is on algorithmic details and performance, which Serre was not overly concerned with.

We first present the basic algorithm as is presented in Serre's text. Then, we describe optimisations that greatly reduce the running time in certain cases.

3.1 Basic algorithm

The algorithm proceeds in two steps. Given a representation $\rho : G \to \mathrm{GL}(V)$, we first compute the canonical decomposition of V, then we decompose each summand into irreducibles.

Let $W_1, \ldots, W_h$ be the complete list of irreducibles of G, $n_1, \ldots, n_h$ their degrees and $\chi_1, \ldots, \chi_h$ their characters.

We use a result due to Serre [1]:

Theorem 3.1. *The projection p_i of V onto V_i associated with the canonical decomposition is given by:*

$$p_i = \frac{n_i}{|G|} \sum_{t \in G} \overline{\chi_i(t)} \rho(t)$$

We can iterate over each irreducible and compute the image of the projection to obtain our canonical decomposition $V = \bigoplus_i V_i$.

The next step is to decompose each V_i into a direct sum of irreducible subrepresentations. The key result we use to compute this decomposition is due to Serre [1]:

Theorem 3.2. *Let n be the degree of the irreducible W_i, with W_i given in matrix form by $r_{\alpha\beta}(s)$. Let $p_{\alpha\beta}$ denote the linear map $V \to V$ given by:*

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho(t)$$

(a) The map $p_{\alpha\alpha}$ is a projection; it is zero on the V_j for $j \neq i$. Its image $V_{i,\alpha}$ is contained in V_i and V_i is the direct sum of the $V_{i,\alpha}$ for $1 \leq \alpha \leq n$. We have $p_i = \sum_\alpha p_{\alpha\alpha}$.

(b) The linear map $p_{\alpha\beta}$ is zero on the V_j for $j \neq i$, as well as on the $V_{i,\gamma}$ for $\gamma \neq \beta$; it defines an isomorphism from $V_{i,\beta}$ onto $V_{i,\alpha}$.

(c) Let x_1 be an element $\neq 0$ of $V_{i,1}$ and let $x_\alpha = p_{\alpha 1}(x_1) \in V_{i,\alpha}$. The x_α are linearly independent and generate a vector subspace $W(x_1)$ stable under G and of dimension

n. For each $s \in G$ we have:

$$\rho(s)(x_\alpha) = \sum_\beta r_{\beta\alpha}(s) x_\beta$$

(in particular, $W(x_1)$ is isomorphic to W_i).

(d) If $(x_1^{(1)}, \ldots, x_1^{(m)})$ is a basis of $V_{i,1}$, the representation V_i is the direct sum of the subrepresentations $(W(x_1^{(1)}), \ldots, W(x_1^{(m)}))$.

We will iterate over the canonical summands V_i, compute a basis for $V_{i,1}$, $(x_1^{(1)}, \ldots, x_1^{(m)})$, and for each j, compute the vector subspace $W(x_1^{(j)})$. Then $V_i = \bigoplus_j W(x_1^{(j)})$ is the irreducible decomposition of this canonical summand.

The full algorithm is as follows:

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function DECOMPOSE REPRESENTATION($\rho : G \to \mathrm{GL}(V)$)
  $n \leftarrow \mathrm{degree}(\rho)$
  $\Delta \leftarrow \{\}$ ▷ This is where we will build the decomposition
  for $\rho_i \in \mathrm{irreducibles}(G)$ **do**
    $\Delta_i \leftarrow \{\}$ ▷ Decomposition of V_i
    $n_i \leftarrow \mathrm{degree}(\rho_i)$
    $\chi_i \leftarrow \mathrm{character}(\rho_i)$
    $p_i \leftarrow \frac{n_i}{|G|} \sum_{t \in G} \overline{\chi_i(t)} \rho(t)$
    $V_i \leftarrow p_i(V)$
    for $1 \le \alpha, \beta \le n_i$ **do**
      $p_{\alpha\beta} \leftarrow= \frac{n_i}{|G|} \sum_{t \in G} (\rho_i(t^{-1}))_{\beta\alpha} \rho(t)$
    end for
    $V_{i,1} \leftarrow p_{11}(V_i)$
    for $x_1^{(j)} \in \mathrm{basis}(V_{i,1})$ **do**
      $W(x_1^{(j)}) \leftarrow \mathrm{span}(\{p_{\alpha 1}(x_1^{(j)}) : 1 \le \alpha \le n_i\})$
      $\Delta_i \leftarrow \Delta_i \cup \{W(x_1^{(j)})\}$
    end for
    $\Delta \leftarrow \Delta \cup \Delta_i$
  end for
  return Δ
end function

This basic version of the algorithm can be called in the GAP package as the function `IrreducibleDecomposition(rho : no_optimisations)`.

3.2 Optimised algorithm

The most obvious optimisation is to prune the set of irreducibles to the W_i that actually appear in the decomposition before performing any expensive per-irreducible computations. Let χ_ρ be the character of ρ and recall that the irreducible characters form an

orthonormal basis for the space of characters. We can restrict our attention to the irreducibles W_i such that $\langle \chi_\rho, \chi_i \rangle > 0$, since these W_i are the only ones that appear with nonzero multiplicity in the decomposition of ρ.

This is a cheap calculation since we can reduce it to a summation over the classes of G. Suppose the conjugacy classes of G are given by $\{t_1^G, \ldots, t_m^G\}$. Then:

$$\langle \chi_\rho, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \chi_i(g)^* = \frac{1}{|G|} \sum_j |t_j^G| \chi_\rho(t_j) \chi_i(t_j)^*$$

This will reduce the amount of work if only a small number of irreducibles actually appear in ρ.

3.2.1 Computing the canonical decomposition

In one case, we can skip the canonical decomposition computation altogether. If we find only one irreducible W_i appears in the decomposition of V, the canonical decomposition is just the whole space V.

Otherwise, we must compute the projections $p_i : V \to V_i$. The main optimisation for this step requires some extra information to be provided by the user.

Suppose we are given a basis $B_1, \ldots, B_d$ for the centraliser ring C of ρ, which is orthonormal with respect to the inner product $\langle A, B \rangle = \text{Trace}(AB^*)$, where B^* is the conjugate transpose of B. We can rewrite the expression for p_i as follows. Let $\{t_1, \ldots, t_m\}$ be a set of representatives for the conjugacy classes of G. Then:

$$\begin{aligned} p_i &= \frac{n_i}{|G|} \sum_{t \in G} \overline{\chi_i(t)} \rho(t) \\ &= \frac{n_i}{|G|} \sum_j \sum_{s \in t_j^G} \overline{\chi_i(s)} \rho(s) \\ &= \frac{n_i}{|G|} \sum_j \overline{\chi_i(t_j)} \sum_{s \in t_j^G} \rho(s) \end{aligned}$$

A matrix M is in the centraliser of ρ exactly when conjugating M by any $\rho(g)$ leaves M unchanged. Say $S = t^G$, the conjugacy class of t in G. Then $\sum_{s \in S} \rho(s)$ is an element of C, since:

$$\rho(g) \left(\sum_{s \in S} \rho(s) \right) \rho(g)^{-1} = \sum_{s \in S} \rho(gsg^{-1}) = \sum_{s \in gSg^{-1}} \rho(s) = \sum_{s \in S} \rho(s)$$

Now we can write $\sum_{s \in S} \rho(s)$ as a vector in the space spanned by the B_k, using the inner product.

We also require that B_k^* is in the centraliser.

$$\begin{aligned}
\left\langle \sum_{s\in S} \rho(s), B_k \right\rangle &= \sum_{s\in S} \langle \rho(s), B_k \rangle \\
&= \sum_{s\in S} \text{Trace}(\rho(s)B_k^*) \\
&= \sum_{s\in S} \text{Trace}(\rho(g_s)\rho(t)\rho(g_s)^{-1}B_k^*) \quad (s = g_s t g_s^{-1} \text{ for some } g_s \in G) \\
&= \sum_{s\in S} \text{Trace}(\rho(g_s)\rho(t)B_k^*\rho(g_s)^{-1}) \qquad\qquad (\text{we require } B_k^* \in C) \\
&= \sum_{s\in S} \text{Trace}(\rho(t)B_k^*) \\
&= |S|\text{Trace}(\rho(t)B_k^*)
\end{aligned} \tag{1}$$

Let $\{t_1, \ldots, t_n\}$ be representatives of the conjugacy classes of G. Now, to sum over the whole group, we need to compute:

$$\sum_{g\in G} \rho(g) = \sum_{i=1}^{n} \sum_{j=1}^{d} |t_i^G| \text{Trace}(\rho(t_i)B_j^*) \tag{2}$$

The number of field operations required to compute the summation (2) depends on both d, the dimension of the centraliser, and n, the number of conjugacy classes of G, but does not depend on $|G|$ at all.

This proof depends on the fact that C is closed under complex conjugation. That is, to use this trick, we require that C is a matrix $*$-algebra.

There are several cases where C has this property. The most obvious is if we have a $\rho : G \to M_n(\mathbb{C})$ that has already been block-diagonalised according to its irreducible decomposition.

Let $\rho = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \rho_i$ where the ρ_i are distinct and irreducible, and m_i is the multiplicity of ρ_i in ρ. Recall that the centraliser of $M_n(\mathbb{C})$ is the set of scalar matrices λI for $\lambda \in \mathbb{C}$.

In our case, each block of our matrix $\rho(g)$ (corresponding to canonical summand V_i) is a block-diagonal matrix $\text{diag}(\rho_i(g), \ldots, \rho_i(g))$. If we now consider the ring of coefficients in our matrix to be $M_{d_i}(\mathbb{C})$, where d_i is the degree of ρ_i, then this is a scalar matrix $\rho_i(g)I_{m_i}$, so commutes with all matrices in $M_{m_i}(S)$ where S is the subring of $M_{d_i}(\mathbb{C})$ consisting of scalar matrices. We know that the ρ_i do not commute with any more matrices due to Schur's lemma, which tells us that the only G-invariant linear endomorphisms of an irreducible are scalar.

To clarify, the centraliser of ρ is spanned as a vector space by matrices with n scalar matrix blocks indexed by i, with block B_i (corresponding to V_i) having size $m_i d_i \times m_i d_i$. Each of these blocks B_i is divided into an $m_i \times m_i$ grid of smaller blocks, each of size $d_i \times d_i$. An orthonormal basis for C is given by a set of $\sum_i m_i^2$ matrices, each with exactly one small block set to the identity matrix and all other blocks set to zero.

For example, if we have a representation $\rho = \rho_1 \oplus 2\rho_2$, where $\deg \rho_1 = 3$ and $\deg \rho_2 = 2$, then a basis for C_ρ is:

$$\left(\begin{array}{c|cc} I_3 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & I_2 & 0 \\ 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & I_2 \\ 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & I_2 & 0 \end{array}\right), \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & I_2 \end{array}\right)$$

It is now clear that given a matrix in the span of these matrices, its conjugate transpose is also in the span.

In the case that we are not given a basis for C, we must calculate one. In general, this may be difficult. In the specific case where ρ is an isomorphism to a permutation group, i.e. the permutation representation of a faithful action of G on some finite set X, we have a basis for C available to use already: the set of orbital matrices (see Section 2 for their definition).

Let the set of orbitals be given by $\{\Delta_i : 1 \leq i \leq r\}$, with Δ_i having matrix A_i. The A_i form an orthonormal basis for C, and can be used with the method of computing the sum described earlier (see equation (1)) since they form a matrix $*$-algebra.

Notice that $\{A_i : 1 \leq i \leq r\}$ span a $*$-algebra. This is because each orbital Δ_i has a paired orbital $\Delta_{i^*} = \{(y, x) : (x, y) \in \Delta_i\}$. It is clear that the associated adjacency matrices are transposes of each other. This means that the algebra generated is closed under $*$.

A permutation representation is a specific case of a more general class of representations that work with this trick: *unitary* representations. A representation ρ is *unitary* if $\rho(g)^* = \rho(g^{-1})$. Then if $A \in C_\rho$, $A^* \in C_\rho$ also:

$$\rho(g)A^* = (\rho(g)A^*)^{**} = (A\rho(g)^*)^* = (A\rho(g^{-1}))^* = (\rho(g^{-1})A)^* = A^*\rho(g^{-1})^* = A^*\rho(g)$$

When implementing the canonical decomposition, we only use this optimisation when ρ is unitary. Although the trick also works if ρ is already block diagonalised, it is a waste of time to check for this: representations are almost never already optimally block diagonal unless they were constructed that way intentionally.

Note that for groups with small numbers of generators that have easily calculated inverses (e.g. S_n), checking whether a representation is unitary is fairly cheap since we only need to check the generators have unitary images. More precisely, for a single generator, the calculation is quadratic in the degree of the representation, since we must transpose the matrix $\rho(g)$, conjugate each cell and check equality with $\rho(g^{-1})$. This cost is small compared to the almost cubic running time of a matrix multiplication.

3.2.2 Unsuitability of the Gram-Schmidt process

In the previous section, we only attempt to find bases for the centraliser ring that are already orthogonal. An alternative approach would be to find a basis that is not necessarily orthogonal and simply apply the Gram-Schmidt orthonormalisation process to obtain an orthonormal basis.

Figure 2: An implementation of the Gram-Schmidt process in GAP

```
OrthonormalBasis@ := function(v, prod)
    local proj, N, u, e, k;

    proj := function(u, v)
        return (prod(u, v) / prod(u, u)) * u;
    end;

    N := Length(v);

    u := [1..N];
    e := [1..N];

    for k in [1..N] do
        u[k] := v[k] - Sum([1..k-1], j -> proj(u[j], v[k]));
        e[k] := (1/Sqrt(prod(u[k], u[k]))) * u[k];
    od;

    return e;
end;
```

This algorithm is simple and does not have an unmanageable complexity in terms of the number of arithmetic operations - only $2nk^2$ operations are required, where k is the number of vectors (in our case, the dimension of the centraliser) and n is the dimension of the vectors (in our case, the square of the degree of the representation). The issue is that the arithmetic operations themselves are do not happen in constant time or space if we are using exact numbers, expressed in terms of cyclotomic numbers.

In GAP, a cyclotomic number $z \in \mathbb{C}$, is represented internally as a list of coefficients $a_i \in \mathbb{Q}$ such that $\sum_{i=0}^{n} \zeta_n^i = z$ for some n. This means that cyclotomic numbers can cause a blowup in the amount of space required to store the values of vectors and consequently, a blowup in the amount of time required to perform arithmetic operations.

For example, suppose we are given the vector $v = (19, 10)$, $\|v\|_2^2 = 461$. If we would like to normalise v, we have to multiply by $\frac{1}{\sqrt{461}}$, but the smallest cyclotomic field of which $\frac{1}{\sqrt{461}}$ is a member is $\mathbb{Q}(\zeta_{461})$, so the scalars required to represent $\frac{v}{\sqrt{461}}$ will be internally represented as lists of 461 coefficients. Arithmetic operations on the normalised vector will thus be much slower than on the original vector.

The Gram-Schmidt orthogonalisation algorithm does not take this into consideration and is unusable in practice when dealing with exact complex numbers.

In general, we try to avoid increasing the degree of the cyclotomic field we are required to work in. This means we allow the usual field operations, but not radicals, which is the issue our example illustrates.

3.2.3 Computing the irreducible decomposition

In this section, we rephrase Serre's formula for the projections $p_{\alpha\beta}$ using tensor products, allowing us to optimise the sums using several observations about summing a representation.

Let ρ_i be the irreducible whose canonical summand we are decomposing, let r give ρ_i in matrix form, with $n = \deg \rho_i$, then:

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1})\rho(t)$$

Serre's method then proceeds by computing various images of vectors under these maps to construct bases for the irreducible subspaces of V (see Section 3.1). The majority of computation time in the basic method is spent computing these $p_{\alpha\beta}$, so this is where we will focus our optimisation efforts.

We can rewrite Serre's formula as:

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{t \in G} (\rho_i^*(t))_{\alpha\beta}\rho(t)$$

where * denotes the dual representation (see Section 2.1 for its definition).

We can regard $p_{\alpha\beta}$ as an $n \times n$ block of a matrix p and notice that p is exactly in the form of a tensor product of ρ_i^* and ρ. Define $\tau = \frac{n}{|G|}(\rho_i^* \otimes \rho)$. Then $\tau(t)_{\alpha\beta} = \frac{n}{|G|}((\rho_i^*(t))_{\alpha\beta}\rho(t)) = \frac{n}{|G|}((\rho_i(t^{-1})_{\beta\alpha}\rho(t))$.

So in fact:

$$p_{\alpha\beta} = \left(\sum_{t \in G} \tau(t) \right)_{\alpha\beta} \tag{3}$$

where we take the $n \times n$ block in the (α, β) position, indexing over blocks.

It is not possible to use the same trick from Section 3.2.1 to speed up the computation of (3). While we may have bases for $C_{\rho_i^*}$ (which is spanned by an identity matrix) and C_ρ (provided by the user), this does not give us a way to compute the basis for $C_{\rho_i^* \otimes \rho}$.

In general, it is *not* the case that a basis for C_f and C_g will give us a basis for $C_{f \otimes g}$. An easy example is when $G = S_3$, which has two degree one irreducibles: ρ_{triv} and ρ_{sign}, and a single degree two irreducible, ρ, the standard representation.

A basis for C_ρ is given by I_2 (see the argument in Section 3.2.1 for why). However, $\rho \otimes \rho = \rho_{triv} \oplus \rho_{sign} \oplus \rho$, so $C_{\rho\otimes\rho}$ has dimension 3. We cannot immediately derive the basis for $C_{\rho\otimes\rho}$ from the basis for C_ρ, since this is the same thing as deriving the block structure of $\rho \otimes \rho$, which might end up being significantly more complicated than that of ρ. In this extreme case, we end up with a direct sum of *every* irreducible by tensoring a single irreducible with itself.

We can, however, take advantage of the fact that the summation (3) has summands given by a group homomorphism.

Suppose we have a finite group G and a monoid homomorphism $f : G \to (R, \cdot)$, where R is a ring with multiplicative monoid $(R, \cdot)$.

We want to compute:

$$\sum_{g\in G} f(g) \tag{4}$$

Given a subgroup $H \leq G$, we can compute the right coset representatives of H: $r_1 = 1, r_2, \ldots, r_{|G:H|}$.

We can now rewrite the summation as:

$$\sum_i \left(\sum_{h\in H} f(h) \right) f(r_i) \tag{5}$$

This requires only $|H| + |G : H|$ image calculations, $|H| + |G : H|$ additions in R, and $|G : H|$ multiplications in R. If H is either trivial or G, this sum is the same as (4). To get an improvement over naively summing, we want to choose a subgroup H with "medium-sized" index, meaning that $|H| + |G : H| \leq |G|$.

Notice that we can now recurse and apply the same method to compute the sum over H. If we were only choosing one H to speed up the sum, we would want this index to be "medium-sized". However, since we can (by recursing) choose an arbitrarily long chain of subgroups and we want to sum as few elements as possible, we want to split the group as many times as possible. So we need to find a chain of subgroups $G_1 \leq \ldots \leq G_n = G$ such that $|G_{i+1} : G_i|$ is minimal (and not 1) at each stage.

If $G = S_n$, it is easy to find a chain with fairly small indices. Consider G to act on $\{1, \ldots, n\}$ and use the chain $\{e\} \leq S_2 \leq \ldots \leq S_n = G$, where we embed $S_i \hookrightarrow S_n$ by considering S_i to be the pointwise stabiliser of $\{i+1, \ldots, n\}$. This gives us an improvement since $|S_{i+1} : S_i| = i + 1$, so there is a significant saving compared to summing over the group directly. This idea generalises.

Let G be a group acting on a set Ω. A *base* for G is a sequence $B = (b_1, \ldots, b_m)$ of points in Ω such that the only $g \in G$ stabilising all b_i is the identity.

The *basic stabiliser chain* for B is a chain of the form:

$$\{1\} = G_m \leq \ldots \leq G_0 = G$$

where G_i is the group of the elements of G stabilising the set $\{b_1, \ldots, b_i\}$.

A *strong generating set* for G is a subset $S \subseteq G$ such that S contains generators for all G_i.

GAP already has an implementation of the Schrier-Sims algorithm [10] [11] to calculate bases and strong generating sets (BSGS), so we need not focus on the algorithmic details of finding a BSGS.

Suppose we have a basic stabiliser chain for G, $G_m \leq \ldots \leq G_0$. First, we compute the sum for G_m, which is trivial. Then, we compute the sum for G_{m-1} using the method described earlier. We continue in this way until we reach G.

Although we are required to have a group with a natural action (i.e. a permutation group) to use the BSGS algorithm to find a chain, the following group summation algorithm is in no way specific to permutation groups. Any chain of subgroups in any type of group (permutation, matrix, finitely presented, and so on) will work: we just do not describe methods of finding chains for these other types of group.

---

function SUM GROUP($f : G \to R$)
  let $\{1_G\} = G_m \leq \ldots \leq G_0 = G$ be a basic stabiliser chain for G
  $\sigma \leftarrow f(1_G)$
  for $i \in \{m-1, \ldots, 0\}$ **do**
    let r_j be a transversal of representatives of $G_i \backslash G$
    $\sigma \leftarrow \sum_j \sigma f(r_j)$
  end for
  return σ
end function

In general, the complexity of this algorithm depends greatly on the basic stabiliser chain, which depends on the structure of the group. To illustrate the speedup, we can consider the case when $G = S_n$ and use the chain $\{e\} \leq S_2 \leq \ldots \leq S_n = G$. We will measure the complexity in terms of ring operations, meaning additions and subtractions in R.

Naively summing over the group requires $\mathrm{O}(n!)$ additions in R.

If we ignore the cost of computing coset representatives (which is reasonable, since the cosets of S_{m-1} in S_m are known), to move from the sum over S_{m-1} to the sum over S_m, we compute:

$$\sum_{i=1}^{m} \left(\sum_{h \in S_{m-1}} f(h) \right) f(r_i)$$

where the r_i are the m coset representatives of S_{m-1} in S_m. Since the inner sum is already known, this takes m multiplications and $m-1$ additions in R.

We do this for $m = 2, 3, \ldots, n$, so the total number of additions is $\sum_{m=2}^{n} m - 1$ which is $\mathrm{O}(n^2)$, and the number of multiplications is similarly $\mathrm{O}(n^2)$. Even considering that R is usually a matrix ring, where multiplication is almost cubic in the number of rows and columns, this is still asymptotically much better than naively summing. In practice, we can sum S_{10} in under a second with the BSGS method, while it takes several minutes with the naive method.

We can use this algorithm to compute the projections $p_{\alpha\beta}$ in Section 3.2.3.

Note that we cannot directly use this algorithm to compute the projections to the canonical summands, since the formula $\sum_{t \in G} \overline{\chi_i(t)} \rho(t)$ does not have the property that the summands are given by a group homomorphism.

However, from (3.2), we have that $p_i = \sum_{\alpha=1}^{n_i} p_{\alpha\alpha}$, so we can proceed by calculating the $p_{\alpha\beta}$ first, then using these to calculate the p_i projections onto the canonical summand and the decomposition of V_i.

3.2.4 Summary

To summarise, we apply the following optimisations:

1. Determine which irreducibles we can ignore using the character inner product.
2. (a) If the matrix of blocks $(p_{\alpha\beta})$ fits into memory, compute it using the group sum trick (5), then use (3.2) to compute the projections p_i.

   (b) If we are given an orthonormal basis for the centraliser and ρ is unitary, compute the p_i using the class sum trick (1).

   (c) If ρ is a permutation representation, compute an orthonormal basis for C_ρ using orbitals, then go back to the previous case to compute p_i.
3. Use the group sum trick to compute $(p_{\alpha\beta})$ if not already done and use these to break V_i into irreducibles.

The effect of these optimisations can be observed in the benchmarks in Section 5.

The optimised version of the algorithm (which attempts to apply each of the optimisations above) is the default when the functions `IrreducibleDecomposition` and `CanonicalDecomposition` are called.

4 Our algorithm

We are trying to find the decomposition of a representation $\rho : G \to M_n(\mathbb{C})$ into irreducibles, but given the irreducible characters, we can immediately construct a representation $\tau : G \to M_n(\mathbb{C})$ that is isomorphic to ρ *and* is block diagonalised according to the irreducible decomposition.

In this section, we present an algorithm to find this τ and use it to compute the irreducible decomposition of V in the original basis.

The GAP functions described in this section must be called with the option `decomp_method := "alternate"` to use the algorithms described here, the default methods are those described in Section 3. For example, to call `IrreducibleDecomposition` in such a way as to use these methods, a user would call `IrreducibleDecomposition(rho : decomp_method := "alternate")`.

4.1 Finding the block diagonalisation

Two representations of a finite G over $\mathbb{C}$ are isomorphic if and only if they have the same character. Let the list of irreducible characters of G by given by $\chi_1, \ldots, \chi_N$. These form an orthonormal basis for the $\mathbb{C}$-vector space of characters, with respect to the inner product $\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)}$.

We can thus determine the irreducible decomposition of ρ (up to isomorphism as representations) by calculating the multiplicities m_i in the expression $\chi_\rho = \sum_{i=1}^{N} m_i \chi_i$: $\langle \chi_\rho, \chi_i \rangle = m_i$.

This means τ, the block diagonalised representation isomorphic to ρ, is given by $\tau = \bigoplus_{i=1}^{N} \bigoplus_{j=1}^{m_i} \rho_i$, where ρ_i is the irreducible corresponding to the character χ_i.

The GAP function `BlockDiagonalRepresentation(rho)` returns this τ, given ρ.

4.2 Finding the intertwining operator

We have constructed a block diagonal τ, but we would now like to know what the irreducible G-invariant spaces are, in our original basis.

In the basis τ is written in, the irreducibles are spanned by certain subsets of $\{e_1, \ldots, e_n\}$. To translate these vectors into the old basis, we need to calculate the linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ with the property that:

$$A^{-1}\tau(g)A = \rho(g) \text{ for all } g \in G \tag{6}$$

A is an *intertwining operator* or isomorphism between the representations.

We can find the intertwining operator by observing that the action of G on matrices given by:

$$g \mapsto (A \mapsto \tau(g)A\rho(g^{-1}))$$

is in fact a linear action, so is a representation of G, call it α.

α is a map from G into the n^2 dimensional (where n is the degree of ρ and τ) $\mathbb{C}$-vector space $M_n(\mathbb{C})$. Using methods from Section 3, we can compute α and find the canonical summand V_{triv} corresponding to the trivial representation $g \mapsto (1)$.

If $A \in V_{triv}$, then since G acts as the identity on this subspace, $\alpha(g)A = A$. This means for all $g \in G$, $A = \tau(g)A\rho(g^{-1})$, which is exactly the property required in (6) for A to be the intertwining operator.

The running time of this method depends on the running time of finding the trivial canonical summand of a representation of G, which is done using Serre's formula for the projection $p : V \to V_{triv}$:

$$p = \sum_{g \in G} \alpha(g) \tag{7}$$

Notice that this summation has summands given by a homomorphism, so we can use the method of building the sum from a BSGS, as in Section 3.2.3.

We then pick a random point B in the domain of p, calculate the image point pB, and rely on the fact that pB will almost always be invertible.

To be precise, the map $B \mapsto \det(pB)$ is polynomial in the entries of B. We know that it is not identically zero since we know an isomorphism between ρ and τ exists, by construction. Nonzero polynomials are zero on hypersurfaces with dimension strictly less than the dimension of the ambient space. In particular, this means the set of B with pB singular has measure zero, since hypersurfaces have measure zero. In the pure mathematical sense, this means that picking B from a uniform distribution on a ball with nonzero radius has probability zero of being singular. In practice, with a computer, the probability is low enough that a single try will almost always[1] be enough to find an invertible pB.

We can directly prove, without relying on Serre's proof, that $A := pB$ satisfies the inter-

[1]Here, I do not mean "almost always" in the precise mathematical sense, I mean that the probability is fairly close to 1

twining operator property. Let $g_0 \in G$. Then:

$$\begin{aligned}
\tau(g_0)A &= \tau(g_0)\sum_{g\in G}\tau(g)B\rho(g^{-1}) \\
&= \sum_{g\in G}\tau(g_0 g)B\rho(g^{-1}) \\
&= \sum_{g\in G}\tau(g)B\rho(g^{-1}g_0) \qquad \text{(by relabelling)} \\
&= \sum_{g\in G}\tau(g)B\rho(g^{-1})\rho(g_0) \\
&= A\rho(g_0)
\end{aligned} \tag{8}$$

While we do know what α is as a linear map, we need to know what it is as a matrix in order to actually compute the sum.

Proposition 4.1. *$\alpha = \tau \otimes \rho^*$, where $\rho^*(g) = \rho(g^{-1})^T$ denotes the dual representation (defined in Section 2.1).*

(We consider $\alpha(g)$ to act on $n \times n$ matrices by reading off each row in sequence, one after the other, to obtain a vector in $\mathbb{C}^{n^2}$)

Proof. Recall that $e_i \otimes e_j = e_i e_j^T = E_{ij}$, and that the E_{ij} form a basis for the space of matrices $M_n(\mathbb{C})$. Then:

$$\begin{aligned}
(\tau \otimes \rho^*)(g)E_{ij} &= \tau(g)e_i \otimes \rho^*(g)e_j \\
&= \tau(g)e_i(\rho^*(g)e_j)^T \\
&= \tau(g)e_i e_j^T \rho(g^{-1}) \\
&= \tau(g)E_{ij}\rho(g^{-1}) \\
&= \alpha(g)E_{ij}
\end{aligned}$$

□

If we directly calculate the images $\alpha(g)$ using the Kronecker product of two $n \times n$ matrices, this will incur $\mathrm{O}(n^4)$ extra space. This is a problem if the degree of the representation is large, since even a single matrix $\alpha(g)$ will not fit in memory.

4.3 Reducing the degree

One method of reducing the degrees we need to compute with is to proceed in two stages. First, split V into canonical summands using the method due to Serre described in Section 3.2.1. Second, apply the method from Section 4.2 on the (hopefully much smaller degree) canonical summands. In the worst case, this will provide no benefit, since it is possible

that there is a single canonical summand - the whole space. Generally, representations do consist of more than one isomorphism class of irreducible, so this optimisation is worthwhile, especially considering that the calculation for each summand can now happen in parallel.

Given a linear map $A : V \to V$, we can restrict it to a subspace W with basis $\{w_j\}_{j=1}^{\dim W}$ by computing the matrix $(A|_W)_{ij} = A(w_j)_i$. We can restrict a representation ρ by computing $\rho(g)|_W$ for each generator g of G.

A key property used in Section 3.2.1 to compute the centraliser basis is that once we have block diagonalised, blocks corresponding to isomorphic irreducibles will have the same matrix coefficients. Restricting to a subspace preserves this property, as long as we are careful to use the correct bases.

4.4 Memory-constrained methods

In some cases, the Kronecker products will still be too large to fit in memory. In this case, we must find a way to represent α without using Kronecker products, then find an element of V_{triv}.

The key space optimisation is to represent a tensor product of matrices as a pair of matrices, without actually calculating the Kronecker product. This takes advantage of the (multiplicative) monoid homomorphism $\phi : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ given by $\phi(A, B) = A \otimes B$. This is a homomorphism since $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

Given $g \in G$, we can represent $\alpha(g)$ as $(\tau(g), \rho^*(g))$. The action of α on $n \times n$ matrices is then given by: $(\tau(g), \rho^*(g))E_{ij} = \tau(g)e_i \otimes \rho^*(g)e_j = \alpha(g)E_{ij}$ (extending to all matrices using linearity). Using the pair representation, we are never required to explicitly compute and store the matrix representing $\alpha(g)$, thus getting rid of the problematic $\mathrm{O}(n^4)$ space usage.

The pair representation does not allow us to sum tensor products, so we cannot use the same method as before. Let $A \in M_n(\mathbb{C})$ and let $G \cdot A$ be its orbit under the action of G given by α. If $v = \sum_{x \in G \cdot A} x$, then $v \in V_{triv}$, since the action of G fixes the set $G \cdot A$ and is linear, so fixes their sum. Define the map $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by:

$$f(A) = \sum_{x \in G \cdot A} x$$

then by a similar argument as for (7), this map almost always gives, in a precise sense, an invertible matrix $f(A)$. This matrix will satisfy the intertwining operator property.

The algorithm used to compute the orbit sum $f(A)$, where the action of G is given by $\rho \otimes \tau$ is as follows:

```
function ORBIT SUM(A, ρ ⊗ τ)
    let {g_1, ..., g_n} be a generating set for G
    σ ← 0
    Δ ← [A]                                          ▷ singleton list
    i ← 1
    while i ≠ |Δ| do
        v ← Δ[i]
        σ ← σ + v
        for 1 ≤ i ≤ n do
            if ρ(g_i) ⊗ τ(g_i) · v ∉ Δ then
                append ρ(g_i) ⊗ τ(g_i) · v to Δ
            end if
        end for
        i ← i + 1
    end while
    return σ
end function
```

A downside to this method is that it is possible that the list used to hold orbit elements could need to grow to a large size, possibly $|G|$. If this orbit cannot fit into memory, this orbit summing method will not work, and we must resort to naively summing over G to calculate an image of the projection to V_{triv}:

$$\sum_{g \in G} \alpha(g) A$$

for some randomly chosen A. This requires a number of matrix additions and multiplications linear in the size of G but only requires us to store a small constant number of $n \times n$ matrices (an accumulator and each summand, one by one), so is not very memory-intensive.

5 Testing and benchmarking

The benchmarks and tests in this section were run on a laptop with an Intel Core i7-4720HQ CPU running at 2.60GHz, with 12 GB of memory. The code was run on Debian GNU/Linux, with the 4.15.0 kernel and GAP version 4.10.1.

All times reported are wall-clock times, measured using GAP's `NanosecondsSinceEpoch` function. Each benchmark was run 3 times and the result averaged to produce each data point.

5.1 Generating test cases

When converting our pseudocode and theorems into runnable code, it is helpful to have a wide range of test cases so that we can be fairly sure our program is correct.

One approach is to think of a wide range of examples and write tests that check the correctness of the algorithm on those specific cases. While this is better than having no tests, it is unlikely that we will be able to come up with a complete set of examples.

The approach we took was to randomly generate test cases and check properties. This is known as *property-based testing* and is inspired largely by QuickCheck[2], a Haskell library for property-based testing.

Our GAP package has a function, `RandomRepresentation` that generates a representation by the following procedure:

- Randomly select a group G from GAP's `SmallGroup` library.
- Compute the list of irreducible representations of G using `IrreducibleRepresentationsDixon`
- Choose a small number of these irreducibles randomly and directly sum them using our function `DirectSumOfRepresentations`.
- Conjugate by a random invertible matrix, generated by `RandomInvertibleMat`.

While computing this random representation, since we constructed it from irreducibles, we know the block structure and centraliser. This knowledge is what we use to check the correctness of the decomposition of the representation after the computation.

For each function in the package, our test suite generates several test cases and checks the correctness of the result.

5.2 Performance comparisons

There are two types of benchmarks that were conducted: benchmarks on a large set of random small groups (generated using `RandomRepresentation`), and benchmarks on some known examples.

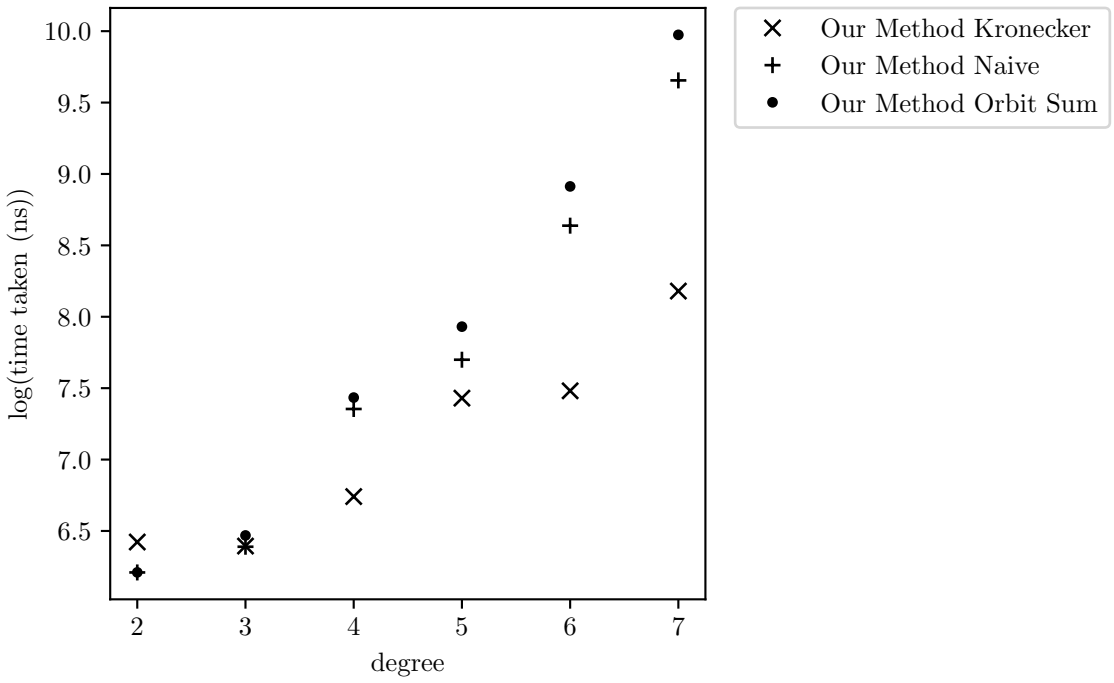
[2] `http://hackage.haskell.org/package/QuickCheck`

Since there are too many combinations of options and flags to test them all, we focus on several cases that demonstrate the effect of our optimisations.

First, we will benchmark the method described in Section 4, comparing the different ways to find intertwining operators (4.2). Recall that to find an isomorphism $\rho \cong \tau$, we need to find a vector in the fixed space V_{triv} of $\alpha = \rho \otimes \tau^*$, there are several ways to do this:

1. Naively summing over G (referred to as the "naive" method)
2. Only summing an orbit of α (referred to as the "orbit sum" method)
3. Computing the projection to V_{triv} using the fast group sum trick (3.2.3) and the Kronecker product of ρ and τ^* (referred to as the "Kronecker" method).

Consider the representation of S_n defined by $\rho_n : S_n \to \mathbb{C}^n$, $\rho(\sigma)_{ij} = 1$ if $j = \sigma(i)$ and 0 otherwise. This is clearly a faithful and unitary representation. It splits into a direct sum of two irreducible representations: the trivial summand spanned by the all-one vector, and the orthogonal complement (by the G-invariant inner product). The nontrivial irreducible is known as the standard representation of S_n.



It is immediately apparent that the orbit sum method is the worst. This is because a random vector will likely have n distinct entries, which means its orbit under S_n will have $n!$ elements: we get every permutation of the entries. Thus summing over the orbit is not an optimisation in this case: our running time is still linear in $|G|$.

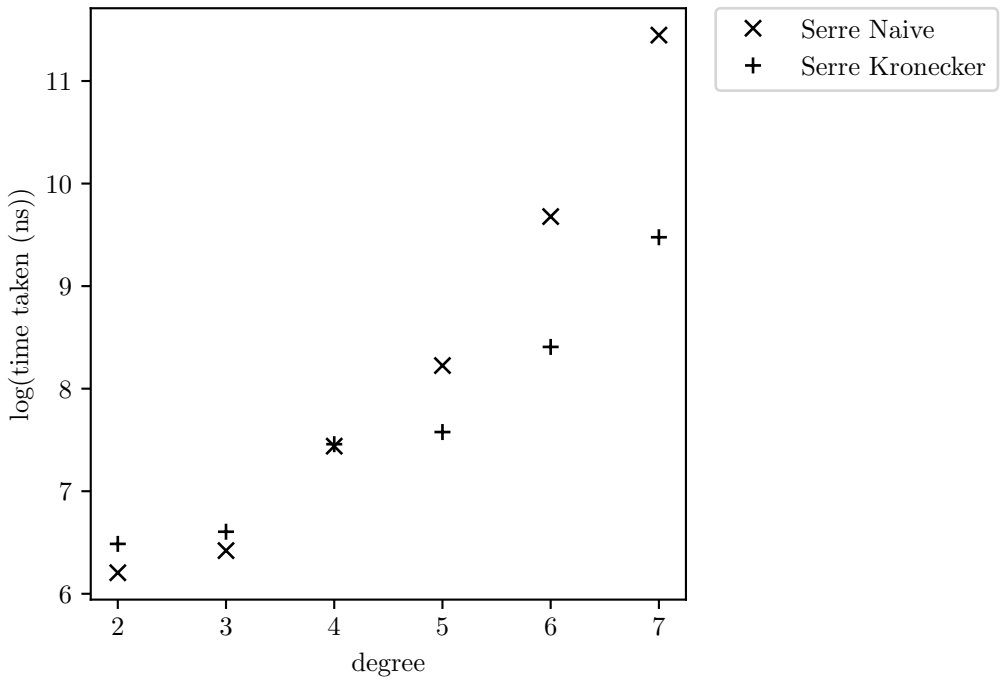
The next worst method is the naive method. This has the same asymptotic behaviour as

the orbit sum method, but with the overhead of keeping track of the orbit eliminated. It is thus faster, but is still linear in $|G|$.

We see a significant improvement when we use the Kronecker method. Note that ρ_n has degree n, and this is very small compared to the size of the group, $n!$. This means that the matrices we are working with are not very large, so the Kronecker products are also not too large.

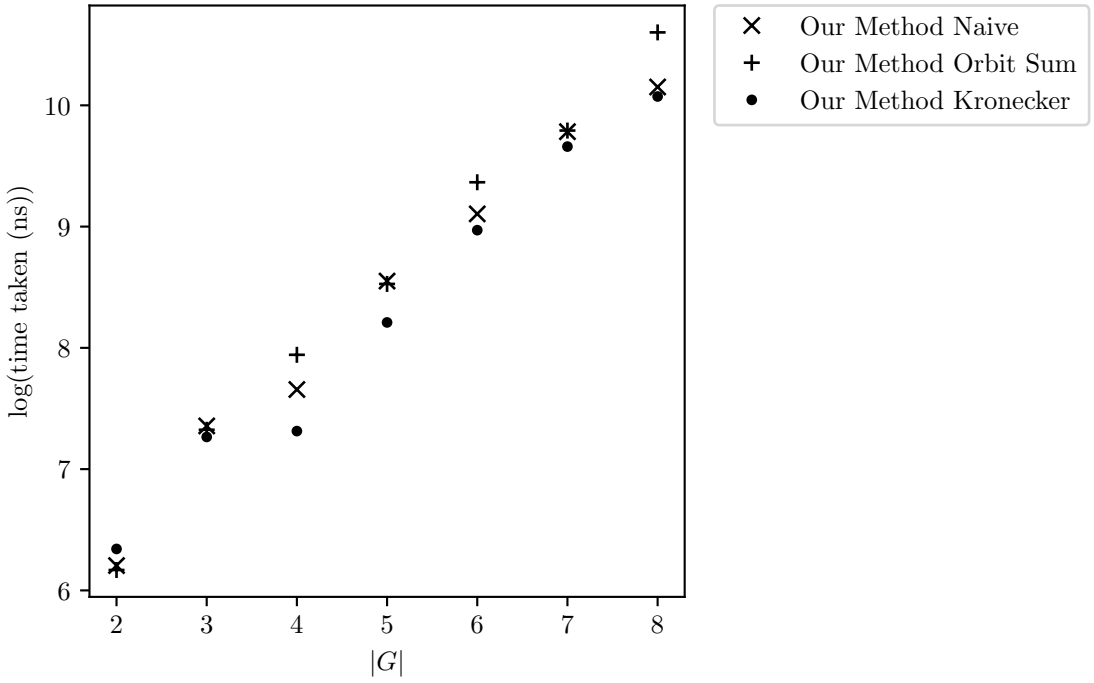
We can observe a similar pattern in two of the methods used to implement Serre's algorithm:

1. Directly computing the projections $p_{\alpha\alpha}$ by summing over G (referred to as the "naive" method)
2. Computing p as the projection to the trivial subspace of $\rho_i^* \otimes \rho$, using the group sum trick from Section 3.2.3 (referred to as the "Kronecker" method).



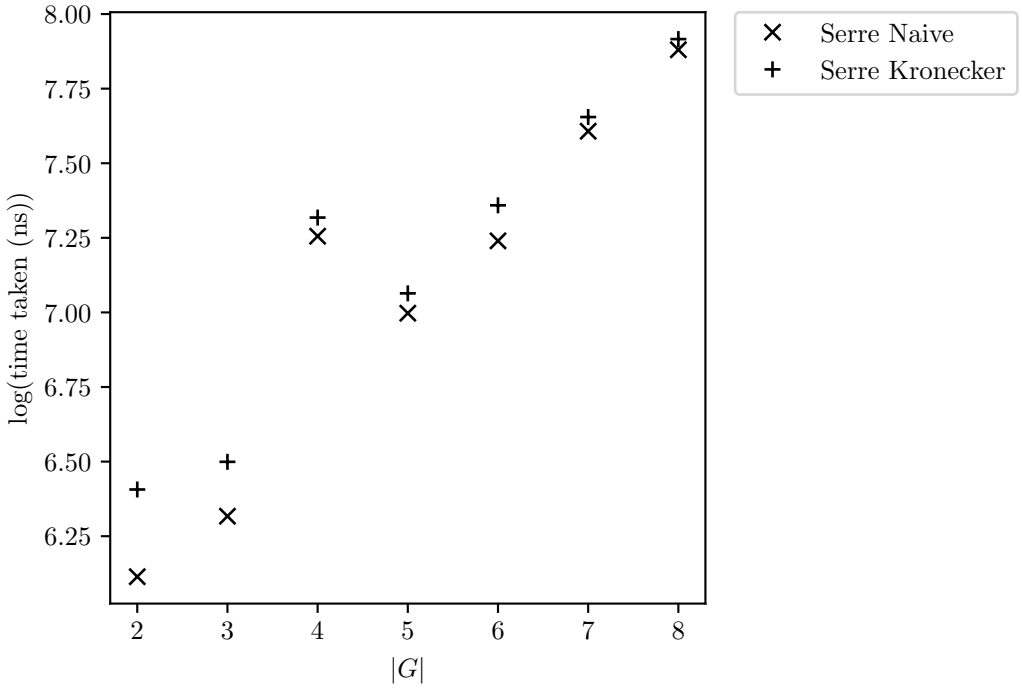
We observe that the Kronecker is superior to the naive method in this case, where the degree of the representation is small and the group is large, as before.

Next, we consider the regular representations of $\mathbb{Z}_n$, the group of integers modulo n. $\mathbb{Z}_n$ is an abelian group, so all irreducibles are of dimension 1, and appear in the regular representation V with multiplicity 1. For the purposes of demonstration, we will consider the tensor product $V \otimes V$, as this has a larger degree, n^2.



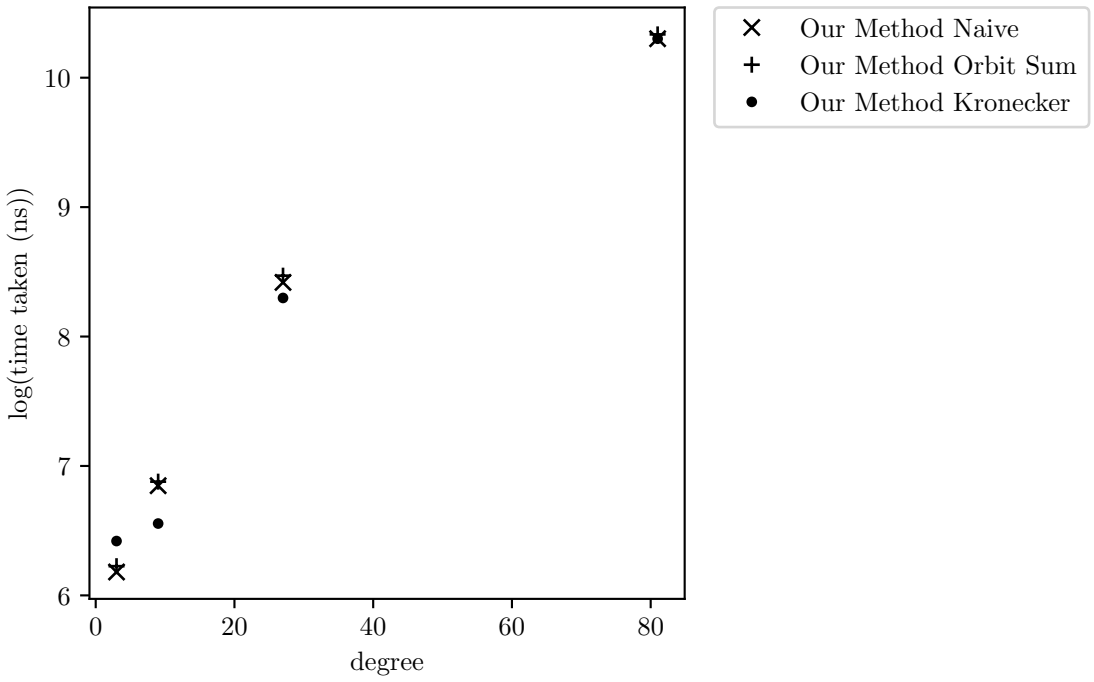
We observe that the Kronecker method is still the most efficient. This is despite the small group size, which improves the running time of the other methods, which sum over the group. Our conclusion is that the Kronecker method is a fairly efficient method. We will compare it to Serre's method later on.

In our implementation of Serre's method, we see the naive method remain competitive. The cyclic group size is small enough that naively summing over the group is faster than the the method using Kronecker products and the group sum trick.



This is demonstrates a special case where the naive method is fastest: when the group is extremely small. In most other cases, the Kronecker method will be better, as we saw when decomposing representations of S_n.

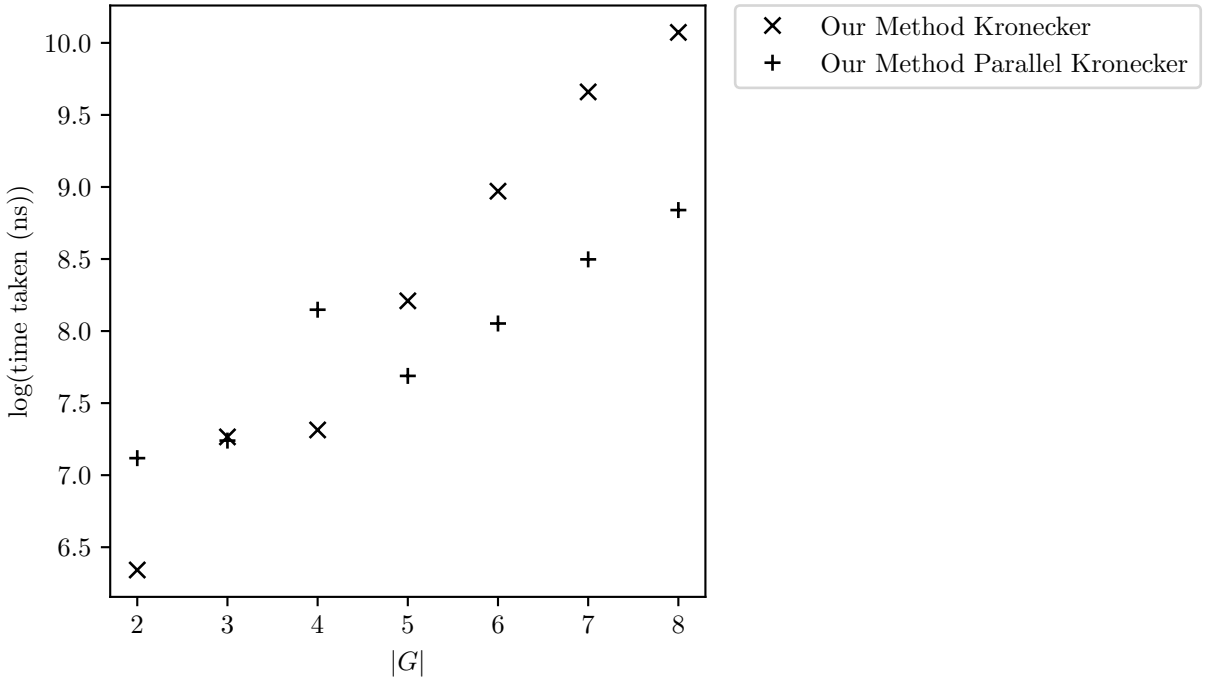
We also want to test the dependency of the running times on the degree of the representations, while keeping the size of the group constant. To demonstrate, we consider the representations $\tau_n : \mathbb{Z}_3 \to \otimes^n V$ where V is the 3 dimensional regular representation of $\mathbb{Z}_3$.

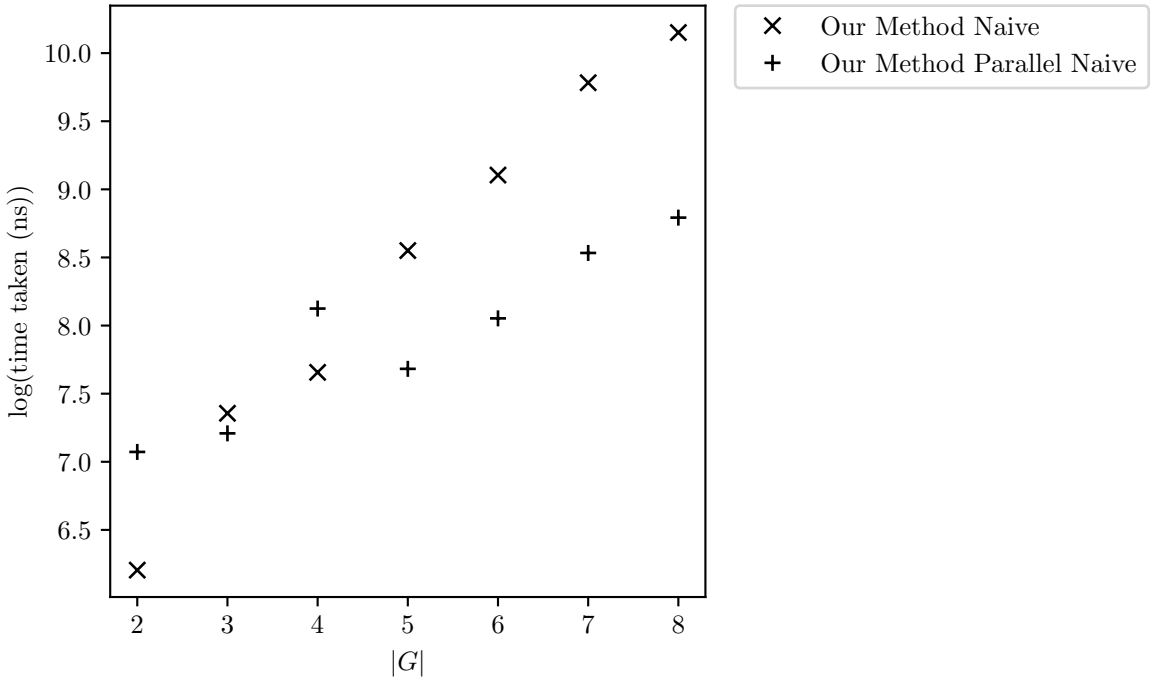


Again, we see that the Kronecker method is the fastest.

Next, we will explore the effect of parallelisation on our method (see Section 4.3 for more details) and Serre's method. We do not expect to see a large improvement in cases where a representation is made up of few irreducibles, so we will benchmark the methods on the tensor $V \otimes V$ of the regular representation V of $\mathbb{Z}_n$ described earlier, since these representations have many irreducibles.

To parallelise our method, we iterate over the list of irreducibles in parallel: for each, we project to the canonical summand using Serre's formula and run the original non-parallel method on the canonical summand.

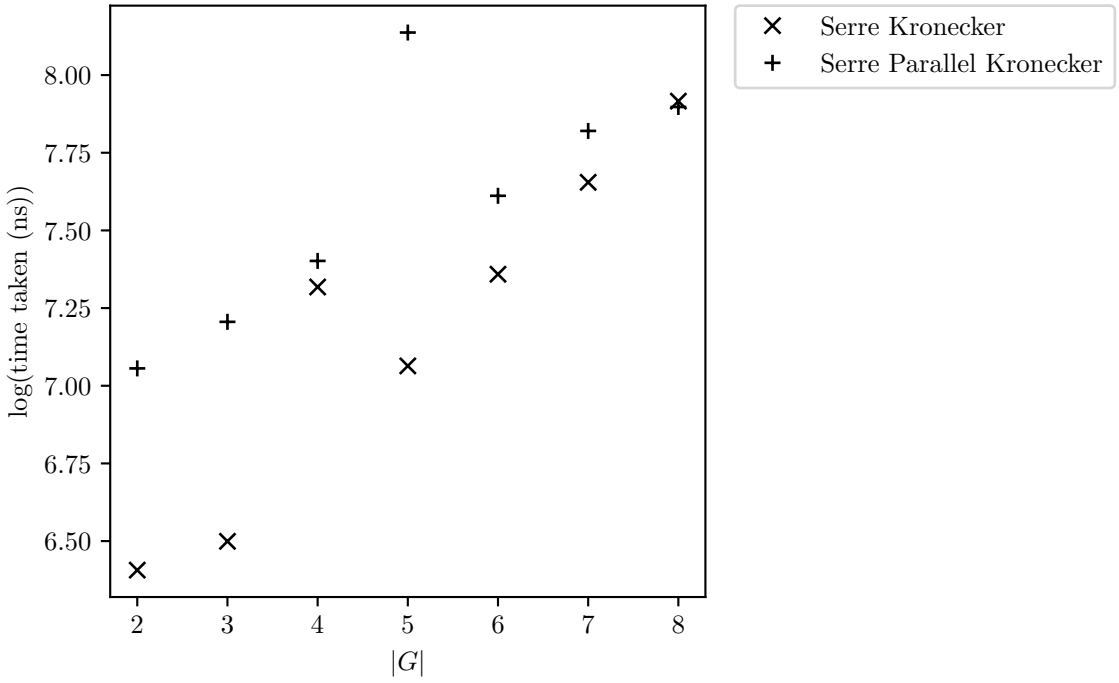


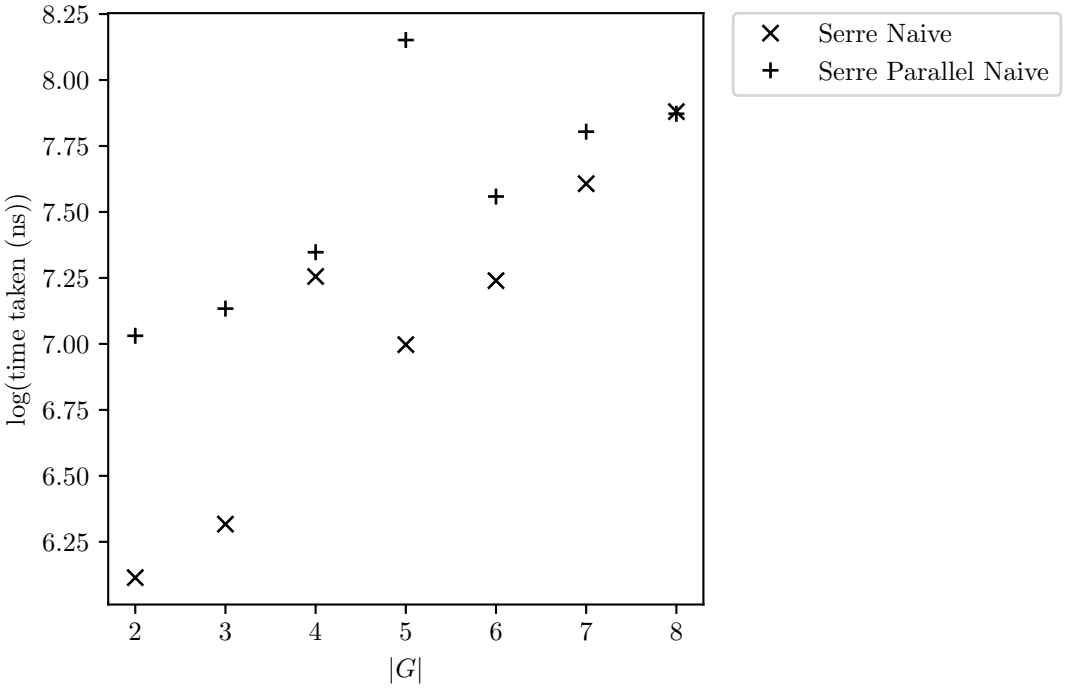


For both methods, we see a large improvement in some cases. The reason for this is that there are n irreducibles of $\mathbb{Z}_n$ over $\mathbb{C}$, one for each nth root of unity. This means there

are n canonical summands, so we could split the work between n processors effectively. The laptop used to run these benchmark only has 4 physical cores, so we do not observe the speedup we would in an ideal scenario, where there is one CPU for each canonical summand.

Similarly to our method, we parallelise Serre's method by splitting into canonical summands and running the original method on each summand.



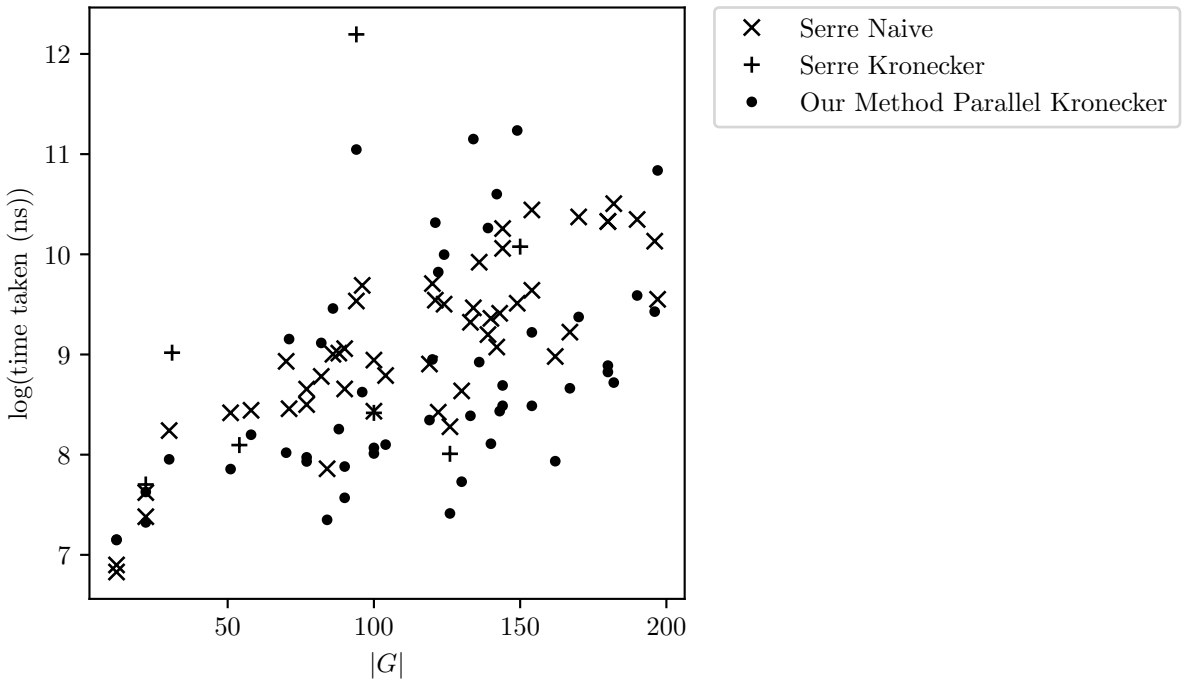


For these examples, the overhead of parallelising is too great and we do not see an improvement in running time for these small examples.

It is work explaining precisely how the parallelisation works, since this reveals why there is no speedup. We use the GAP function `ParListByFork` to process lists in parallel. This function forks child processes, performs the computation, then communicates the results over a pipe to the parent, which builds the result list from each child's result. The time this takes is highly dependent on the performance characteristics of the OS `fork` system call, IPC (interprocess communication) performance, and various GAP serialisation functions. Ideally, we would parallelise using more finely-grained concurrency primitives to avoid the need for any IPC, but none are available in GAP. When profiling, we found that a large amount of running time is spent in functions such as `IO_Pickle` and `IO_Unpickle`, which are serialisation functions, used to transmit results between processes.

Finally, we compare our method to Serre's method. For our method, we will use the paralellised Kronecker product method, since this was the fastest in all cases. For Serre's method, we will benchmark both the naive and Kronecker product methods. This is because the naive method is faster for small groups, while the Kronecker method is faster for large groups.

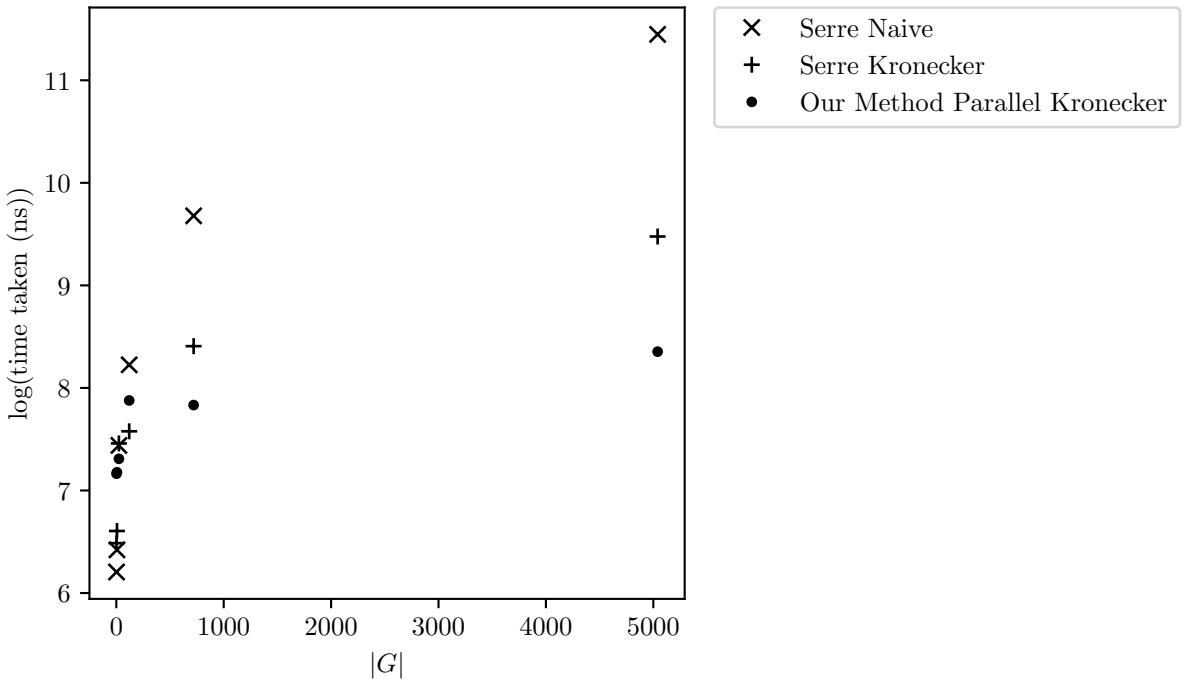
We benchmarked the algorithms for a random selection of 50 test cases, with group sizes from 1 to 200 and the degree being less than 10.



Serre's method with Kronecker products performed extremely poorly in most cases, it was never the best method for any representation tested.

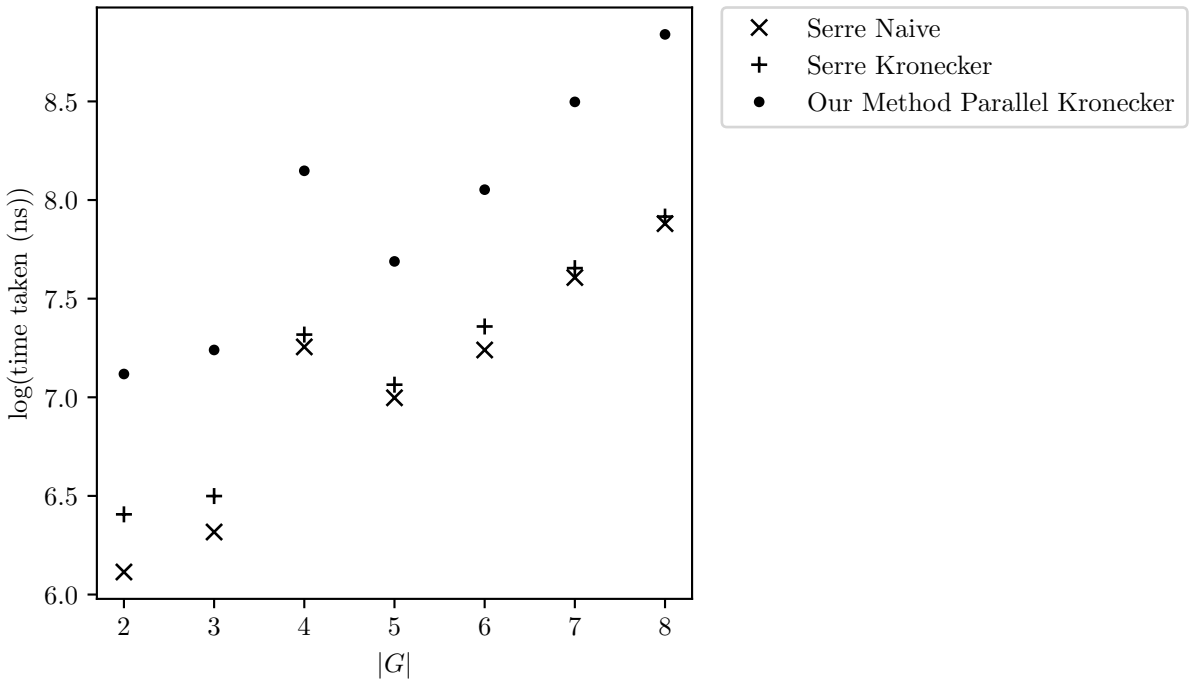
Our method with Kronecker products was the fastest method 35 times, while Serre's method with naive summing was the fastest 15 times.

Next, we benchmarked the three methods on the defining representation of the symmetric group.



This time, we see that Serre's naive method is the worst - this is due to the large size of the symmetric groups. Our method is again the best.

Lastly, for an example where our method is not the best, we benchmarked the tensor product of the regular representation of $\mathbb{Z}_n$.



Here, Serre's naive method is the best: its running time is dominated by the size of the group which is small. Our method has a running time dominated by a polynomial in the degrees of the representations, which grow quadratically ($V \otimes V$ has dimension n^2) with the size of the group.

We can conclude that for small groups, Serre's naive algorithm has good performance. For larger groups, our Kronecker method usually has good performance. In general, it is hard to say anything more concrete than this, since performance depends on too many factors - it is not just the size of the group that affects running time. We have different numbers of conjugacy classes, special cases in GAP algorithms for certain groups (e.g. integers, symmetric groups), different internal representations of certain groups (e.g. polycyclic, presentations) and so on.

6 An application: Bounding the crossing number of $K_{m,n}$

Being able to efficiently decompose representations of finite groups leads to a wide variety of possible applications. In semidefinite programming, it is common in several applications that a program has symmetries which can be expressed as a group action and used to greatly reduce the dimension of the problem. In this section, we present a method due to de Klerk et al. [5] for reducing the dimension of such semidefinite programs. We also apply this method to the problem of computing a lower bound for the crossing number of the complete bipartite graph.

While the reduction in the dimension of the semidefinite program is done by us in the same way as de Klerk et al. [5], the key difference is that we can now optimally block diagonalise the representation of the action and thus the centraliser. This is an improvement over the original method by de Klerk et al. [12], in which no general method of optimally block diagonalising is given.

6.1 Motivation

A *complete bipartite graph* $K_{m,n}$ is a graph that can be partitioned into two sets. One with m vertices, one with n vertices, and each vertex in one set connected with an edge to each vertex in the other set, with no other edges.

In 1954, Zarankiewicz published a proof that the crossing number of $K_{m,n}$, $\operatorname{cr}(K_{m,n}) = \lfloor \frac{1}{4}(m-1)^2 \rfloor \lfloor \frac{1}{4}(n-1)^2 \rfloor$ [13]. His argument contained an error, meaning the proof was only valid for $K_{3,n}$. Other proofs have been published, proving the conjecture for $\min(m,n) \leq 6$ [14] and $m \in \{7,8\}$ and $7 \leq n \leq 10$ [15].

The truth of the conjecture is not known in general, but bounds on $\operatorname{cr}(K_{m,n})$ are known. Let $Z(m,n) = \lfloor \frac{1}{4}(m-1)^2 \rfloor \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

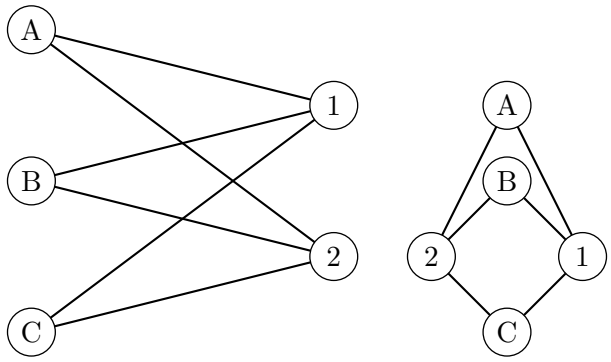


Figure 3: Two drawings of $K_{3,2}$. The second has $Z(3,2) = 0$ crossings.

De Klerk et al. [5] showed that:

$$\lim_{n\to\infty} \frac{\operatorname{cr}(K_{m,n})}{Z(m,n)} \geq 0.8594 \frac{m}{m-1}$$

This is the result that we will reproduce in this section. Our improvement is that, due to a reduction in the sizes of blocks of the matrices we compute with, the computation is more efficient.

6.2 Reducing semidefinite programs

A more detailed presentation of this method, with full proofs of correctness, can be found in the paper by de Klerk, Pasechnik and Schrijver [5].

Suppose we are given a semidefinite program:

$$\min\{\mathrm{tr}(CX) \mid X \text{ positive semidefinite}, X \geq 0, \mathrm{tr}(A_j X) = b_j \text{ for } j = 1, \ldots, m\} \qquad (9)$$

Suppose additionally that we have a finite group G acting on a finite set Z with ρ being the permutation representation of this group action, i.e. $\rho(g)$ is the matrix with rows and columns indexed by Z with $\rho(g)_{xy} = 1$ iff $gx = y$ and 0 otherwise. We also require that the C and A_j matrices in (9) are elements of the centraliser ring of this representation, meaning they commute with all $\rho(g)$.

Let $\{E_1, \ldots, E_d\}$ be the orbital matrices for G acting on Z, and for each i, $B_i = \frac{E_i}{\|E_i\|}$, where the inner product is given by $\langle X, Y \rangle = \mathrm{tr}(XY^*)$. Define L_k such that $B_k B_j = \sum_i (L_k)_{ij} B_i$. This is possible since the centraliser ring (spanned by the E_i) is closed under multiplication.

A key result of de Klerk et al. [5] is that the program (9) is equivalent to the following program:

$$\min \left\{ \sum_{i=1}^{d} \mathrm{tr}(CB_i)x_i \;\middle|\; \sum_{i=1}^{d} x_i L_i \text{ positive semidefinite}, x_i \geq 0 \text{ for } i = 1, \ldots, d, \right.$$
$$\left. \sum_{i=1}^{d} \mathrm{tr}(A_j B_i)x_i = b_j \text{ for } j = 1, \ldots, m \right\}$$

The number of variables x_i is d, the dimension of the centraliser ring: this is in most cases a significant reduction from the number of variables that need to be considered in (9).

Another reduction in complexity comes from the fact that we only condition on positive semidefiniteness of matrices of the size of the L_i, which are $d \times d$ matrices. Compared with (9), which constrains on the positive semidefiniteness of X, a $Z \times Z$ matrix, this can be a large saving. As an example, in this section we will consider a case where $|Z| = 6! = 720$ while $d = 78$.

De Klerk et al. [5] note further that we can always find a symmetric matrix exhibiting the optimal value. Recall that each orbital Δ_i has a paired orbital Δ_{i^*} with all of the pairs swapped. This corresponds to the adjacency matrices being transpose: $E_i = E_{i^*}^T$. A

consequence is that we can perform the restriction to symmetric matrices by adding the constraints $x_i = x_{i^*}$ for each i.

We can apply this to crossing numbers as follows.

Let $G = S_m \times S_2$ act on Z_m, the set of m-cycles in S_m by: $(\sigma, e) \cdot \rho = \sigma\rho\sigma^{-1}$ and $(e, \tau) \cdot \rho = \rho^{\text{sign}(\tau)}$.

Define a matrix C in $\mathbb{R}^{Z_m \times Z_m}$ by $C_{\sigma,\tau}$ = the minimum number of adjacent interchanges required to transform σ to τ^{-1}. An adjacent interchange is a swap of elements that are next to each other when the cycle is written down. For example, (1423) and (4123) differ by an adjacent interchange. Then C is in the centraliser of the G-action. See [12] for more details on the matrix C.

Then define a constant α_m by:

$$\alpha_m := \min\{\text{tr}(CX) \mid X \in \mathbb{R}^{Z_m \times Z_m}, X \text{positive semidefinite}, \text{tr}(JX) = 1\}$$

where J is the all-one matrix.

de Klerk et al. proved [5] that:

$$\text{cr}(K_{m,n}) \geq \frac{m(m-1)}{k(k-1)} \left(\frac{1}{2} n^2 \alpha_k - \frac{1}{2} n \left\lceil \frac{1}{4}(k-1)^2 \right\rceil \right)$$

for all n and $k \leq m$, which implies:

$$\lim_{n\to\infty} \frac{\text{cr}(K_{m,n})}{Z(m,n)} \geq \frac{8\alpha_k m}{k(k-1)(m-1)}$$

for all n and $k \leq m$.

We proceeded by reducing the semidefinite program defining α_m using the method due to de Klerk et al. [5], then block diagonalising the representation using our algorithm (using naive summing) described in Section 4.4. We were forced into using this method since the degrees of the representations are large: the Kronecker product methods quickly ran out of memory.

We provided our algorithm with a complete list of irreducibles of S_n using Young tableaux. This algorithm was not implemented in GAP, so we used the implementation from Sage. The reason for using this algorithm over GAP's generic Dixon algorithm is that computing with Young tableaux is much faster and allows us to produce integer matrices: sidestepping any issues involving cyclotomic fields (see Section 3.2.2 for why this is important).

6.3 Results

These results were computed on a laptop with an Intel Core i7-4720HQ CPU running at 2.60GHz, with 12 GB of memory. The results for α_m for $m \in \{5, 7\}$ match approximately

those obtained in earlier papers [12] [5]. Values for α_m for $m \geq 8$ were not computed here due to the time required, but values for $m = 8, 9$ were computed by de Klerk et al. [5].

The computation took less than a second for $m = 5$, 10 seconds for $m = 6$ and 1.5 hours for $m = 7$. There is much room for improvement on these computation times, but optimising this problem was not the goal of this project - we merely aim to prove that our algorithm works in a real-world scenario.

m	$\approx \alpha_m$
5	1.9472133720059
6	2.9519170848593
7	4.3693933617464

Table 1: Results for $5 \leq m \leq 7$

These were produced by running the script `./run_crossing.sage m` where `m` is the value of m.

To demonstrate the efficiency of our block diagonalisation, we can examine the decomposition of the $m = 7$ representation. $m = 7$ has degree 720 (it acts on the set of 7-cycles, of which there are 6!), and decomposes into blocks as follows:

Block Size	Number of Blocks
1	2
14	8
15	6
20	2
21	6
35	10

Table 2: Block sizes for $m = 7$

Notice that the largest block size is 35, considerably smaller than 720, meaning our method has been effective in reducing the sizes of matrices we are required to compute with.

7 Conclusions and future work

7.1 Conclusions

We have satisfied all of the requirements set out in Section 1. To summarise how we achieved this:

We provide two functions to decompose a representation: `IrreducibleDecomposition-Collected`, which returns a list of lists of irreducibles. The irreducibles are collected into lists according to their isomorphism class, so all spaces appearing in a list are isomorphic as representations. `IrreducibleDecomposition` is the flattened version, a list of irreducibles. In this case, each irreducible is given as a subspace of V, in the form a of a GAP vector space.

To compute the block diagonalisation of ρ, we provide the function `BlockDiagonalRepresentation`, which converts a representation ρ to a block diagonal one τ. τ is given as a GAP homomorphism from the group to a matrix group. The matrix we conjugated by to get τ is given by `BlockDiagonalBasisOfRepresentation` applied to ρ.

To compute a basis for the centraliser ring C_ρ, we provide the function `CentralizerOfRepresentation`, which gives the matrices spanning C_ρ as a vector space, written in the same basis as τ. We also provide `CentralizerBlocksOfRepresentation`, which gives the same matrices, but as lists of blocks rather than full matrices.

Our property-based testing method is described in Section 5.1. We found that this method was superior to manually writing a complete set of examples, since it caught more errors. Since our algorithm applies to representations of any group, it is almost impossible to think of examples covering every possible case: solvable/unsolvable, abelian/nonabelian, cyclic/not cyclic, nilpotent/not nilpotent, and so on. Through the random generation of large numbers of test cases, we discovered edge cases where one of our assumptions broke down. A specific example was encountered when we were testing the centraliser trick (see Section 3.2.1). Originally we only tested on unitary representations and did not realise our assumption that representations were unitary - after implementing randomised testing, this was quickly detected and resolved.

Lastly, we wrote an extensive GAP package manual, documenting all functions, arguments, preconditions, return values and so on. We also made sure to include comments on which algorithms are best in which cases (derived from experiments conducted in Section 5). The source code is also heavily commented, hopefully allowing future contributions without much difficulty.

7.2 Future Work

While we have succeeded in fulfilling the requirements, there are still more improvements and additions that could be made.

We could have written an implementation of Dixon's algorithm for decomposing unitary representations, discussed in the appendix A. This algorithm avoids the need for a complete

list of irreducibles to be provided beforehand, which is a property none of the other algorithms have and warrants further exploration. Once we factor in the computation time for the list of irreducibles, it is possible that we would discover cases where Dixon's algorithm is the best. This task was out of scope for this project, but could form the basis for future contributions to GAP or Sage.

In terms of performance analysis, there is much more that could be done. For example, we could have written tools similar to KCachegrind[3] to produce graphs and analyse the call graphs of our functions in more detail. The analyses we did perform were done with the aid of GAP's profiling tools, which are not as advanced as the tools included with valgrind[4], for example.

We did not profile and analyse memory usage, as this was not a primary concern compared to running time. This is another area of improvement: we did not heavily optimise for memory usage, instead opting to trade memory for a better running time wherever possible. This is most apparent in our usage of Kronecker products, which grow the size of the matrices we are computing with from $n \times n$ to $n^2 \times n^2$. This generally improves running time, but memory usage suffers. We saw this in Section 6, where the Kronecker product methods were too memory intensive when $n = 720$, and could not be used. In some cases, we may have been able to use sparse matrices to improve memory usage, since some representations have matrices which have mostly zeroes as entries. One GAP package that implements this is `Gauss`[5], which implements a sparse matrix data structure that only keeps track of the non-zero entries. We could also use sparse matrix data structures from SageMath[6] or SciPy[7], but this would require implementing parts of our algorithm in SageMath to take advantage of the interoperation between GAP and Python libraries.

7.3 Project Availability

All code associated with the project is currently available at `https://gitlab.com/kaashif/decomp`. The GAP package `RepnDecomp`, containing all functionality implemented as a part of this project, will be submitted for review and will become available at `https://www.gap-system.org/Packages/packages.html` as a deposited package. This means that it will be included in future releases of the GAP distribution.

[3] `http://kcachegrind.sourceforge.net/html/Home.html`
[4] `http://valgrind.org/`
[5] `https://www.gap-system.org/Packages/gauss.html`
[6] `http://doc.sagemath.org/html/en/reference/matrices/sage/matrix/matrix_sparse.html`
[7] `https://docs.scipy.org/doc/scipy/reference/sparse.html`

A Algorithms for unitary representations

During the course of the research for this project, we discovered several interesting algorithms that apply to the special case when a representation is unitary. As we have seen in Section 3.2.1, unitary representations have desirable properties that allow us to perform optimisations that are not possible in general.

In this section, we describe our method to compute a unitary representation isomorphic to a given representation.

Even further, there is an algorithm due to Dixon [6] that allows the decomposition of a unitary representation into irreducibles *without* the need to have a complete list of irreducibles beforehand.

A.1 Unitarising representations

Given a representation $\rho : G \to \mathrm{GL}(V)$, define:

$$S = \sum_{g \in G} \rho(g)\rho(g)^*$$

To clarify: here, $\rho(g)^*$ means the conjugate transpose of $\rho(g)$. This is not the same thing as the dual representation, which we denoted by $\rho^*(g)$, which means $\rho(g^{-1})^T$, the transpose of $\rho(g^{-1})$. The dual representation does *not* appear in this section, * always means "conjugate transpose".

If ρ is already unitary, then S is a scalar matrix. Our strategy is essentially to try to find a change of basis such that S is a scalar matrix, then we will see that this means ρ, after this change of basis, is unitary.

Notice that S is Hermitian, since:

$$S^* = \sum_{g \in G} (\rho(g)\rho(g)^*)^* = \sum_{g \in G} \rho(g)\rho(g)^* = S$$

Notice also that $\rho(g)S\rho(g)^* = S$ by relabelling:

$$\rho(g)S\rho(g)^* = \sum_{t \in G} \rho(g)\rho(t)\rho(t)^*\rho(g)^* = \sum_{g \in G} \rho(gt)\rho(gt)^* = \sum_{g \in G} \rho(g)\rho(g)^* = S$$

We can rephrase the definition of S in terms of a tensor product. We do this with the goal

of using the group sum trick (from Section 3.2.3) used extensively in our other algorithms.

$$\begin{aligned}
S_{ij} &= \sum_{g\in G}(\rho(g)\rho(g)^*)_{ij} \\
&= \sum_{g\in G}\sum_{k}\rho(g)_{ik}(\rho(g)^*)_{kj} \\
&= \sum_{g\in G}\sum_{k}\rho(g)_{ik}(\overline{\rho(g)})_{jk} \\
&= \sum_{k}\sum_{g\in G}(\rho(g)\otimes\overline{\rho(g)})_{ikjk} \\
&= \sum_{k}\left(\sum_{g\in G}(\rho(g)\otimes\overline{\rho(g)})\right)_{ikjk}
\end{aligned}$$

where $(A\otimes B)_{xyij}$ refers to the (i,j) entry in the (x,y) block. A tensor product naturally has this block structure, with each block given by $A_{xy}B$.

The summation $\sum_{g\in G}(\rho(g)\otimes\overline{\rho(g)})$ has summands given by a homomorphism $g\mapsto\rho(g)\otimes\overline{\rho(g)}$, so we can use the fast group sum trick.

Given a Hermitian matrix A, we can decompose A into $A = LDL^*$ where L is lower triangular with all 1 on the diagonal, and D is real, diagonal. We can use the following formulas:

$$\begin{aligned}
D_j &= A_{jj} - \sum_{k=1}^{j-1} L_{jk}L_{jk}^*D_k \\
L_{ij} &= \frac{1}{D_j}\left(A_{ij} - \sum_{k=1}^{j-1} L_{ik}L_{jk}^*D_k\right) \text{ for } i > j
\end{aligned}$$

We can compute this decomposition for S, let $S = LDL^*$. Now:

$$D = L^{-1}LDL^*(L^*)^{-1} = L^{-1}\rho(g)LDL^*\rho(g)^*(L^*)^{-1} = (L^{-1}\rho(g)L)D(L^{-1}\rho(g)L)^*$$

Since D is real, we can take its square root. In fact, D has positive entries, since S is positive definite, which means $\sqrt{D}$ still has real entries. This lets us do the following:

$$\begin{aligned}
I &= \sqrt{D}^{-1}(L^{-1}\rho(g)L)\sqrt{D}\sqrt{D}(L^{-1}\rho(g)L)^*\sqrt{D}^{-1} \\
&= ((L\sqrt{D})^{-1}\rho(g)L\sqrt{D})((L\sqrt{D})^{-1}\rho(g)L\sqrt{D})^* \\
&= \tau(g)\tau(g)^*
\end{aligned}$$

where we define $\tau(g) = (L\sqrt{D})^{-1}\rho(g)L\sqrt{D}$. This shows that τ is unitary.

S is invertible since it is a sum of positive definite Hermitian matrices. This means L is invertible and allows us to define τ as above. A formula for L^{-1} is given by: $L^{-1} = DL^*S^{-1}$.

The main drawback of this method is that it requires computing square roots. As we have seen in Section 3.2.2, square rooting means the minimal cyclotomic extension required to contain all coefficients can be very large. This blowup in degree will cause a corresponding blowup in memory usage and computation time.

A.2 Decomposing unitary representations

We describe an algorithm due to Dixon [6] for decomposing unitary representations into irreducibles. A complete proof of correctness and explanation can be found there, we will focus on describing the algorithm.

In principle, combined with the method for unitarising a representation, this gives a way to find the complete irreducible decomposition of a representation without needing the complete list of irreducibles.

In practice, this algorithm has drawbacks which will become clear. We did not implement this algorithm.

The first step is finding a nonscalar, Hermitian element H of the centraliser ring C_ρ. If E_{rs} denotes the matrix with 1 in the (r, s) position and 0 everywhere else, then define:

$$H_{rs} = \begin{cases} E_{rr} & \text{if } r = s \\ E_{rs} + E_{sr} & \text{if } r > s \\ i(E_{rs} - E_{sr}) & \text{if } r < s \end{cases}$$

For each r, s, compute:

$$H = \frac{1}{|G|} \sum_{g \in G} \rho(g)^* H_{rs} \rho(g)$$

H is Hermitian and commutes with the action of G given by ρ, so is an element of the centraliser. Notice that the H_{rs} give a Hermitian basis for $M_n(\mathbb{C})$.

If H is scalar for all r, s, by Schur's lemma, this means ρ is in fact irreducible, so we can stop here.

Otherwise, for some r, s, H will be nonscalar. H is conjugate symmetric, hence diagonalisable. We can find an orthonormal eigenvector basis for V by diagonalising (using e.g. a Jordan form decomposition), $J = P^{-1}HP$, then orthonormalising the columns of P using the Gram-Schmidt process. P must be unitary for the subrepresentations we will find to be unitary.

The action of G preserves the eigenspaces of H, so the block diagonalisation given by J is in fact a decomposition of V into G-invariant subspaces, giving the unitary subrepresentations.

We then recurse until all representations become irreducible.

The downside to this algorithm is the we require P to be unitary. If it is not, then the subrepresentations are not unitary and we cannot recurse. Computing P requires an orthonormalisation process which, when we require exact coefficients, has problematic performance characteristics when dealing with exact cyclotomic fields (as seen in Section 3.2.2).

Another problem is that computing H requires a summation over G. The reason this algorithm is of interest is that it doesn't require the computation of a complete list of irreducibles for G, which could be intractable if G is large and complex enough. But if G is large, summing over G is also undesirable.

Lastly, in real world cases, we often want to decompose permutation representations $\rho : G \to S_n$. These representations are already unitary, so it may seem as if this algorithm will be a good choice. However, using Young tableaux, a complete list of irreducibles of S_n can be computed very easily (as is done using SageMath in Section 6), while for large n, summing over S_n (as required by the unitary algorithm) will be slow.

For these reasons, this algorithm was not the focus of this project.

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