CES Preferences: Demands, Gravity and Variety
Notes for Graduate International Trade Lectures

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Plan of Lectures

1. CES Preferences
2. The Gravity Equation
3. Measuring Gains from Variety
4. Supplementary Material
Plan of Lectures

1. CES Preferences
   - The CES Utility Function
   - Preference for Variety
   - Implications of Taste for Diversity
   - Demands

2. The Gravity Equation

3. Measuring Gains from Variety

4. Supplementary Material
The CES Utility Function

\[ u = \left( \sum_{i=1}^{n} x_i^\theta \right)^{1/\theta} \]

- A symmetric CES function:
  - \( x_i \) is the consumption of variety \( i \)
  - \( n \), the number of varieties, is given to consumers;
    - In monopolistically competitive equilibrium, it is endogenous.
  - The index \( \theta \) is a measure of substitutability, and must lie in \([−∞, 1]\)
  - As we will show, it is related to the elasticity of substitution \( \sigma \):
    \[ \sigma \equiv \frac{1}{1−\theta} \iff \theta = \frac{\sigma - 1}{\sigma} \]
    So: \[ \{−∞ < \theta < 1\} \iff \{0 < \sigma < ∞\} \]
  - \( 0 < \sigma \leq 1 \) is fine for consumers; but, as we will see:
    - It is inconsistent with a taste for diversity
    - It is inconsistent with firms’ second-order condition
CES preferences imply a taste for variety:

**Proof**: Assume all varieties have the same price $p$ and so are consumed in equal amounts, so total expenditure is $I = npx$:

$$x_i = x = \frac{l}{np} \Rightarrow u = (nx^\theta)^{1/\theta} = n^{1/\theta} x = n^{\frac{1}{\sigma-1}} l/p$$

- This is the indirect utility function in symmetric equilibria.

- Logarithmically differentiating, with $l$ and $p$ fixed: \[
\hat{u} = \frac{1}{\theta} \hat{n} + \hat{x} = \frac{1}{\theta} \hat{n} - \hat{n} = \frac{1}{\sigma-1} \hat{n}
\]

1. Gain at extensive margin; more than offsets: 
2. Loss at intensive margin

- i.e., utility rises with variety for $\sigma > 1$, and by more the lower is $\sigma$.

QED
Implications of Taste for Diversity

CES price index as a function of number of goods
Demands

- Form the Lagrangian: \( L \equiv u^\theta + \lambda \left( I - \Sigma_i p_i x_i \right) \)
  - \( u^\theta \) easier than \( u \); yields same results: utility is ordinal not cardinal.
  - The term multiplied by the Lagrange multiplier \( \lambda \) is written such that it would be positive if the constraint did not strictly bind; this ensures that \( \lambda \) is never negative.

- Take the first-order conditions and manipulate to obtain:
  - Frisch demand functions: \( x_i = \left( \frac{\lambda p_i}{\theta} \right)^{-\sigma} \)
  - Relative demand functions: \( \frac{x_i}{x_j} = \left( \frac{p_i}{p_j} \right)^{-\sigma} \)
  - Marshallian demand functions: \( x_i = \frac{p_i^{-\sigma}}{\Sigma_j p_j^{1-\sigma}} I = \left( \frac{p_i}{P} \right)^{-\sigma} \frac{I}{P} \)

- To derive \( P \), the true cost of living index:
  - Substitute Marshallian demands into \( u \) to get indirect utility function:
    - \( u^\theta = \Sigma_i x_i^\theta = \frac{\Sigma_i p_i^{-\sigma\theta}}{(\Sigma_j p_j^{1-\sigma})^{\theta}} I^\theta = \left( \Sigma_j p_j^{1-\sigma} \right)^{1-\theta} I^\theta \quad \text{(since } \sigma\theta = \sigma - 1) \)
    - \( \Rightarrow \quad V(p, I) = \frac{I}{P(p)}, \quad e(P, u) = P(p) u, \quad P(p) \equiv \left( \Sigma_j p_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \)
Plan of Lectures

1. CES Preferences

2. The Gravity Equation
   - Introduction
   - Digression: Newton and Gravity
   - Anderson-van Wincoop
   - Anderson-van Wincoop (cont.)
   - Anderson-van Wincoop (cont.)
   - Export and Import Multilateral Resistance

3. Measuring Gains from Variety

4. Supplementary Material
Empirically, trade volumes are well explained by simple “gravity” equations of the kind:
\[ \ln V_{jk} = \ln I_j + \ln I_k - \delta d_{jk} \]
- \( V_{jk} \) is the value of exports from \( j \) to \( k \),
- \( I_j \) is the value of GDP in \( j \),
- \( d_{jk} \) is the distance from \( j \) to \( k \),
- \( \delta \) is a parameter, often estimated to be close to 0.6.

Adding additional variables (e.g., dummies for contiguity, latitude, common language etc.) led to increased empirical success but also increased theoretical embarrassment:
- Why does the equation work so well? How should the coefficients be interpreted? [c.f. Leamer and Levinsohn, 1995]
- Problem of interpretation came to a head with McCallum’s (AER 1995) ”border puzzle”:
  - Controlling for GDP and distance, intra-national trade (trade between different Canadian provinces or between different U.S. states)
  - ... is much greater than international trade (trade between a particular Canadian province and a particular U.S. state).
Note that Newton’s theory of gravity gives inspiration but no real guidance:

- The gravitational force between two bodies \( F_{jk} \) is explained by:
  - Their masses \( m_j \) and \( m_k \);
  - The distance between them \( d_{jk} \);
  - \( G \): Newton’s gravitational constant;
  - \( \ln F_{jk} = \ln G + \ln m_j + \ln m_k - 2 \ln d_{jk} \)

But: the equation is exact (except for large masses when recourse must be had to Einstein’s theory of general relativity).

- In any case, it does not apply to more than two bodies: the general \( n \)-body problem is unsolved.
With CES preferences, a rationalization for the gravity equation of trade flows can be given as follows:

- Assume \( n \) countries, each is endowed with a single good and consumes all \( n \) goods.
- The amount of country \( j \)'s good consumed in country \( k \) is denoted \( x_{jk} \).
- Trade costs are of the “iceberg” kind: \( \tau_{jk} \geq 1 \) units must be shipped from \( j \) in order to deliver one unit to consumers in \( k \).
- The “mill price” or “factory-gate price” of country \( j \)'s good is \( p_j \).
- The price of country \( j \)'s good to consumers in country \( k \) is \( p_{jk} = \tau_{jk} p_j \).
- Hence the value of shipments from \( j \) to \( k \), denoted \( V_{jk} \), is the same:
  - whether valued at \( j \)'s prices: \( \tau_{jk} x_{jk} \) units valued at \( p_j \) each
  - or at \( k \)'s prices: \( x_{jk} \) units valued at \( p_{jk} = \tau_{jk} p_j \) each
- The Marshallian demand function can then be reexpressed in this notation and multiplied by \( p_{jk} \) to give the value of trade:

\[
V_{jk} = \frac{p_{jk}^{1-\sigma}}{P_k^{1-\sigma}} I_k = p_j^{1-\sigma} \frac{\tau_{jk}^{1-\sigma}}{P_k^{1-\sigma}} I_k \quad (1)
\]
Anderson-van Wincoop (cont.)

- This is reminiscent of the gravity equation:
  - Depends $-$'ly on trade costs (for which distance is a plausible proxy);
  - Depends $+$'ly on importer GDP;
- But:
  - Exporter GDP $I_j$ is missing;
  - Also depends on individual goods prices $p_j$: usually unobservable.

These problems can be overcome, and eqtn. given a GE underpinning, by invoking the GDP=Total Sales eqtn. for export country $j$:

$$I_j = \sum_h p_{jh} x_{jh} = p_j^{1-\sigma} \sum_h \frac{\tau_{jh}^{1-\sigma}}{P_h^{1-\sigma}} I_h$$

(2)

- Solve this for prices $p_j^{1-\sigma}$ and substitute into (1) to eliminate $p_j$:

$$V_{jk} = \frac{I_j}{\sum_h \frac{\tau_{jh}^{1-\sigma}}{P_h^{1-\sigma}} I_h} \frac{\tau_{jk}^{1-\sigma}}{P_k^{1-\sigma}} l_k = \frac{1}{\sum_h \frac{\tau_{jh}^{1-\sigma}}{P_h^{1-\sigma}} \theta_h} \left( \frac{\tau_{jk}}{P_k} \right)^{1-\sigma} \frac{l_j l_k}{I_W}$$

(3)

- where $\theta_h \equiv \frac{I_h}{I_W}$ is country $h$’s share in world GDP.
Now, define denominator as a new price index:

$$\Pi_j^{1-\sigma} = \sum_h \theta_h \frac{\tau_{jh}^{1-\sigma}}{P_h^{1-\sigma}}$$

A $\theta$-weighted average of the transport costs relative to local prices $P_h$ faced by country $j$ in all its export markets.

Hence the gravity equation takes a simple and elegant form:

$$V_{jk} = \left( \frac{\tau_{jk}}{\Pi_j P_k} \right)^{1-\sigma} \frac{l_j l_k}{l_W} \quad (4)$$

Thus bilateral trade flows depend:

- Log-linearly on both exporter and importer GDP;
- Negatively on bilateral trade costs;
- But: Only when the latter are measured relative to appropriate averages of the multilateral trade costs faced by the two countries.

AvW: $\Pi_j$ and $P_k$ are export and import “multilateral resistance” terms.

$\frac{l_j l_k}{l_W}$ is “Frictionless Trade”; actual trade is lower.

- Depends only on country size - of both exporter and importer
- Recall that this does not hold in Heckscher-Ohlin
Finally, rewrite $P_k$ in a way that shows clearly that it is dual to $\Pi_j$.

To do this, use (3) once again (with suitable changes in variables) to eliminate prices $p_h$ from the importing country’s price index $P_k$, which can then be rewritten as a $\theta$-weighted average of the transport costs relative to export prices $\Pi_h$ faced by country $k$ on all its imports:

$$P_k^{1-\sigma} = \sum_h p_h^{1-\sigma} = \sum_h p_h^{1-\sigma} \tau_h^{1-\sigma} = \sum_h \frac{I_h}{\tau_h^{1-\sigma}} \tau_h^{1-\sigma} = \sum_h \theta_h \frac{\tau_h^{1-\sigma}}{\Pi_h^{1-\sigma}}$$

Note that $\Pi_j$ and $P_k$ are only defined up to a single normalization:
- A 10% increase in all the $\Pi_j$ implies a 10% fall in all the $P_k$ and no change in any other variables.
- More precisely, system is homogeneous of degree zero in $\Pi_j$ and $P_k^{-1}$.
Plan of Lectures

1. CES Preferences

2. The Gravity Equation

3. Measuring Gains from Variety
   - Konüs and Sato-Vartia
   - Proof of the Sato-Vartia Result
   - Feenstra: Measuring the Gains from New Varieties
   - Applications of Sato-Vartia-Feenstra
   - Addendum

4. Supplementary Material
A different application of CES preferences.

First: Derive the true price index when variety is constant.

Digression: Why do we need to derive it? Isn’t it just $P$?

No: $P$ is unobservable in the realistic case of asymmetric utility:

$$u = \left( \sum_{i=1}^{n} \beta_i x_i^\theta \right)^{\frac{1}{\theta}} \Rightarrow P = \left( \sum_{i=1}^{n} \beta_i^\sigma p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

(5)

A classic index number problem: How to find an empirical index (i.e., based on observables) which equals (or approximates) an unobservable true index?

Solution for CES is the Sato-Vartia Index:

$$\ln P^{SV} \equiv \sum_{i=1}^{n} \omega_i \left( \ln p_i^1 - \ln p_i^0 \right)$$

(6)

where: $\omega_i \equiv \mu_i \mu^{-1}$, $\mu_i \equiv \frac{s_i^1 - s_i^0}{\ln s_i^1 - \ln s_i^0}$, $\mu \equiv \sum_{i=1}^{n} \mu_i$

In words: The SV index is a weighted geometric mean of price relatives, where the weights are the normalised logarithmic means of the budget shares in the two periods.
Proof of the Sato-Vartia Result

- Demand functions \( x_i = \beta_i^\sigma \left( \frac{p_i}{P} \right)^{-\sigma} \) imply budget shares \( s_i = \beta_i^\sigma \left( \frac{p_i}{P} \right)^{1-\sigma} \)
- Take logs: \( \ln s_i = \sigma \ln \beta_i + (1 - \sigma)(\ln p_i - \ln P) \)
- Sum over \( i \), with weights \( \omega_i \) to be determined, and take difference between two periods:

\[
\ln P^1 - \ln P^0 = \sum_{i=1}^{n} \omega_i \left( \ln p_i^1 - \ln p_i^0 \right) + \frac{1}{\sigma - 1} \sum_{i=1}^{n} \omega_i \left( \ln s_i^1 - \ln s_i^0 \right) \tag{7}
\]

- Provided tastes are constant \( (\beta_i^1 = \beta_i^0) \), the \( \beta_i \) vanish!
- For the price index to equal a weighted average of log price changes, the second term on the right-hand side must be zero.
- Hence, the true price index between the two periods equals:

\[
\ln P^1 - \ln P^0 = \sum_{i=1}^{n} \omega_i \left( \ln p_i^1 - \ln p_i^0 \right) \tag{8}
\]

- where: \( \omega_i \equiv \mu_i \mu^{-1} \), \( \mu_i \equiv \frac{s_i^1 - s_i^0}{\ln s_i^1 - \ln s_i^0} \), \( \mu \equiv \sum_{i=1}^{n} \mu_i \)
Suppose the set of goods changes, though not fully: $\mathcal{I} = \mathcal{I}^0 \cap \mathcal{I}^1 \neq \emptyset$

Redefine the budget shares with respect to expenditure on common goods:

$$s^t_i(\mathcal{I}^t) = s^t_i(\mathcal{I}) \lambda^t$$

where:

$$\lambda^t \equiv \frac{\sum_{i \in \mathcal{I}} p^t_i x^t_i}{\sum_{i \in \mathcal{I}^t} p^t_i x^t_i}, \quad t = 0, 1 \quad (9)$$

Take difference in log budget shares between periods as before:

$$\ln s^1_i(\mathcal{I}^1) - \ln s^0_i(\mathcal{I}^0) = (1 - \sigma) \left[ (\ln p^1_i - \ln p^0_i) - (\ln P^1 - \ln P^0) \right] \quad (10)$$

Sum this over $i \in \mathcal{I}$ only, with weights to be determined:

$$\ln P^1 - \ln P^0 = \sum_{i \in \mathcal{I}} \omega_i \left( \ln p^1_i - \ln p^0_i \right) + \frac{1}{\sigma - 1} \sum_{i \in \mathcal{I}} \omega_i \left[ \ln s^1_i(\mathcal{I}^1) - \ln s^0_i(\mathcal{I}^0) \right] \quad (11)$$

Using (9), this gives the SV result with a simple correction factor:

$$\ln P^1 - \ln P^0 = \sum_{i \in \mathcal{I}} \mu_i \mu^{-1} \left( \ln p^1_i - \ln p^0_i \right) + \frac{1}{\sigma - 1} \left( \ln \lambda^1 - \ln \lambda^0 \right) \quad (12)$$

where:

$$\mu_i \equiv \frac{s^1_i(\mathcal{I}) - s^0_i(\mathcal{I})}{\ln s^1_i(\mathcal{I}) - \ln s^0_i(\mathcal{I})}, \quad i \in \mathcal{I}, \quad \mu \equiv \sum_{i \in \mathcal{I}} \mu_i$$
\[
\frac{P^1}{P^0} = \left( \frac{\lambda^1}{\lambda^0} \right)^{\frac{1}{\sigma-1}} \prod_{i \in I} \left( \frac{p^1_i}{p^0_i} \right)^{\omega_i}
\]

(13)

- **Interpretation:** If new varieties are important, \( \lambda^1 \) will tend to be small.
  - So, price index will be *lower*
  - Intuitively, a new good in period 1 has an infinite reservation price in period 0.
  - Similarly if varieties are upgraded, so \( b^1_i > b^0_i \) for some \( i \).

- **Correction factor is less important the higher is \( \sigma \)**
  - Ignoring new varieties matters less if they are close substitutes for existing ones.

  - They estimate total gains from increased import varieties as 2.6% of GDP.
Note that Feenstra (1994) defines the CES price index as

\[ P = \left( \sum_{i=1}^{n} b_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}, \] i.e., \( b_i = \beta_i^\sigma \)
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   - Solving for CES Demands
   - Origins of the CES
Solving for CES Demands

\[ L \equiv u^\theta + \lambda \left( I - \sum_i p_i x_i \right) = \sum_{i=1}^n \beta_i x_i^\theta + \lambda \left( I - \sum_{i=1}^n p_i x_i \right) \]  
(14)

\[ \Rightarrow \frac{\partial L}{\partial x_i} = \beta_i \theta x_i^{\theta - 1} - \lambda p_i = 0 \quad \Rightarrow \quad x_i = \left( \frac{\lambda p_i}{\theta \beta_i} \right)^{-\sigma} \]  
(15)

\[ \Rightarrow x_j x_i = \left( \frac{\beta_i p_j}{\beta_j p_i} \right)^{-\sigma} \]  
(16)

Solve for \( x_j \), multiply by \( p_j \) and sum over \( j \):

\[ \Rightarrow \sum_{i=1}^n p_j x_j = \frac{x_i}{\beta_i p_i^{-\sigma}} \sum_{j=1}^n \beta_j^\sigma p_j^{1-\sigma} = I \quad \Rightarrow \quad x_i = \frac{\beta_i^\sigma p_i^{-\sigma}}{\sum_j \beta_j^\sigma p_j^{1-\sigma}} I \]  
(17)

\[ \Rightarrow x_i = \beta_i^\sigma \left( \frac{p_i}{P} \right)^{-\sigma} \frac{I}{P} \quad \text{where:} \quad P \equiv \left[ \sum_j \beta_j^\sigma p_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \]  
(18)
Origins of the CES

- Mathematical form developed by Hardy, Littlewood and Polya (1934)
- Introduced into economics by Arrow, Chenery, Minhas and Solow (1961)
  - As a form for a two-factor production function
- Applied to monopolistic competition by Dixit-Stiglitz (1977) and Spence (1977)
  - Their innovation: Making $n$ endogenous, so allowing it to be used in monopolistic competition
  - Difference between them: Spence assumed quasi-linear utility, so not applicable to general equilibrium