Variational determination of eigenstrain sources of residual stress

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Summary

If a distribution of inelastic strain (eigenstrain) in an engineering component is specified, together with the component geometry and the degree of constraint, then the residual stress field can be uniquely determined by elastic equilibration. We address, using a variational approach, the inverse problem of eigenstrain determination from a set of measurements of residual stress or residual elastic strain.

Introduction

Residual stresses arise in a variety of manufacturing processes: casting, rolling, quenching, hot forging, cold working, shot-peening or laser shock peening, welding, etc. Detailed modelling of the process of residual stress generation requires the use of sophisticated coupled microstructural and thermo-mechanical numerical models relying on deep understanding of constitutive laws and detailed knowledge of the material parameters. In practice this level of insight is often unattainable, leading to simplified treatments unable to predict adequately the resulting residual stress distributions. This in turns leads to the necessity of using increased safety factors and utilising overly conservative designs, since the residual stresses are known to exert considerable influence over the durability of engineering components and assemblies.

It is possible to attempt correlating the various manufacturing processing conditions directly with the stress distributions that they generate. The problem with this strategy lies in the fact that residual stresses depend not only on the processing conditions, but also on the specimen size and shape, etc. For example, for specimens in having the form of thin strips or sheet, the near surface residual stresses arising in shot peening depend on the specimen thickness, and change with it, no matter how carefully the specimen may be thinned down after peening treatment.

An alternative approach is to focus on the distributions of inelastic strains contained in the sample, which act as the sources of residual stresses. Indeed, in the absence of inelastic (permanent) strain of some origin any sample should be stress free in the absence of external loading. If an eigenstrain distribution $\varepsilon^*_{ij}(\mathbf{x}')$ is given, then the elastic strains and residual stresses arising in an infinitely extended elastic body can be found by the formulae (Mura [1]):

$$e_{kl}(\mathbf{x}) = -\varepsilon^*_{kl}(\mathbf{x}) - \int_{-\infty}^{\infty} C_{pqmn}\varepsilon^*_{mn}(\mathbf{x}')G_{kp,ql}(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}', \qquad \sigma_{ij}(\mathbf{x}) = C_{ijkl}e_{kl}(\mathbf{x}). \quad (1)$$

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Here C_{ijkl} are the elastic stiffness coefficients, and G_{kp} denotes the Green's function for an infinite body. The integral is formally carried out over the entire space, but can in fact be restricted to the region of non-zero eigenstrain: $\Omega\{\mathbf{x}'\colon \varepsilon^*_{ij}(\mathbf{x}') \neq 0\}$. In practical problems involving complex finite geometries the Green's function is usually not known. However, the calculation of elastic strains and residual stresses from given eigenstrains (the *direct* problem) is fairly straightforward, and can be accomplished using e.g. finite element or boundary element methods.

We are interested here in the problem that often arises in residual stress measurement and interpretation. Let there be given a set of measurements (with certain *accuracy*) of strains and stresses collected from a finite number of points (sampling volumes) within a bounded specimen. We would like to solve the *inverse* problem about the determination of unknown eigenstrains $\varepsilon^*_{ij}(\mathbf{x}')$ from this *incomplete* knowledge of elastic strains or residual stresses. The limited accuracy and lack of completeness of measurements suggest that direct inversion of (1) may not be the preferred solution. In fact the method chosen must be sufficiently robust to furnish approximate solutions even in this case.

We use a variational formulation for the inverse problem. Firstly, a functional of the following type (here in terms of strain) is defined:

$$J = \sum_{i=1,\ldots N} w_i \left[e_{kl}(\mathbf{x}_i) + \varepsilon^*_{kl}(\mathbf{x}_i) + \int_{-\infty}^{\infty} C_{pqmn}\varepsilon^*_{mn}(\mathbf{x}')G_{kp,ql}(\mathbf{x}_i - \mathbf{x}')\mathrm{d}\mathbf{x}' \right]^2 . \qquad (2)$$

Here w_i denotes the weight assigned to each of the N collocation points $\mathbf{x}_i$ considered[2]. The first term in brackets, $e_{kl}(\mathbf{x}_i)$, has the meaning of the elastic strain that is observed or measured as a function of position. The remaining two terms depend on the choice of eigenstrain distribution $\varepsilon^*_{kl}(\mathbf{x})$, and have the meaning of the predicted elastic strain $\tilde{e}_{kl}(\mathbf{x}_i)$ at a collocation point $\mathbf{x}_i$.

Thus, an alternative form in which equation (2) can be written is as follows:

$$J = \sum_{i=1,\ldots N} w_i \left[e_{kl}(\mathbf{x}_i) - \tilde{e}_{kl}(\mathbf{x}_i) \right]^2 . \qquad (3)$$

Now in order to find the most likely distribution of eigenstrain $\varepsilon^*_{kl}(\mathbf{x})$ that gives rise to the observed elastic strain distribution, $e_{kl}(\mathbf{x})$, we require the minimum value of the functional J in equation (3). This formulation corresponds to the choice of best agreement between prediction and observation in the least squares sense. In practice the step is best accomplished by expressing the unknown distribution $\varepsilon^*_{kl}(\mathbf{x})$ in the form of a truncated

[2] Summation over a finite number of points in equation (2) can be replaced by an integral expression.

series of functions with unknown coefficients, and determining the values of these by minimising J in equation (2). The choice of the weights w_i remains entirely at our disposal, and can be made judiciously (e.g. taking into account the measurement accuracy at each point $\mathbf{x}_i$) to improve the convergence.

In the following section we describe briefly the procedure for residual strain determination in polycrystalline samples that uses diffraction of high energy synchrotron X-rays [2], and its application to the evaluation of residual strains in a laser shock peened (LSP) sample of Ti-6Al-4V alloy. We then present a simple bending theory analysis allowing the underlying eigenstrain distribution to be deduced. In the final section we introduce the variational formulation of the problem, and present the results of this approach. The paper is concluded with a brief discussion.

Experimental evaluation of residual strains in a laser shock peened sample

High energy synchrotron X-ray diffraction allows the residual elastic strain determination deep inside polycrystalline samples within small gauge volumes defined by the intersection of the incident and diffracted beams [2]. For the present study the residual strain component parallel to a surface of a bar of Ti-6Al-4V alloy subjected to laser shock peening (LSP) treatment was mapped along a line across the bar and perpendicular to the peened surface. At station 16.3 at the SRS (Daresbury, UK) beam energy of about 68keV was selected through the use of a bent Laue monochromator. The strains were deduced from the shift of the 110 diffraction peak of α-titanium. The strain-free value of lattice parameter was estimated from the requirement of static equilibrium. The measured residual elastic strain profile is shown in Figure 1(a) (open circles).

Simple bending analysis

We first present a simple analysis method based on the bending approximation [3] that can easily be carried out in full. Denote the bar width by h, and assume that the relevant eigenstrain component, ε^*, is aligned with the bar axis. The following parameters having the dimension of strain are defined:

$$\Gamma = \frac{1}{h}\int_0^h \varepsilon^*(x)dx\,, \quad \Gamma_1 = \frac{1}{h^2}\int_0^h \varepsilon^*(x)xdx\,. \tag{4}$$

In terms of these parameters the residual elastic strain distribution is given by [3]:

$$e(x) = -\varepsilon^*(x) + 2\Gamma(2-3x/h) - 6\Gamma_1(1-2x/h)\,. \tag{5}$$

Note the similarity between the above expression and equation (1): the elastic strain arising in response to an eigenstrain distribution is given by its opposite together with additional integral terms that depend on the problem geometry (hence Green's function).

In order to find the unknown distribution $\varepsilon^*(x)$ certain assumption must first be introduced. For simplicity we assume that the eigenstrain is localised within the near-surface region, $0 < x < c$, and is given by a segment of a cosine wave, $\varepsilon^*(x) = [1 + \cos(\pi(x - x_0)/c)]\varepsilon_0^*/2$. The problem is reduced to that of funding the best values of the depth parameter *c*, the shift x_0, and the eigenstrain magnitude, ε_0^*.

A prediction of the residual strain profile based on this approach is shown in Figure 1 (small markers), together with the measured strains. Figure 1(b) shows the eigenstrain distribution that gives rise to this solution. The solution clearly captures the key features of the distribution qualitatively correctly. However, the agreement is not uniformly good, possibly due to experimental errors.

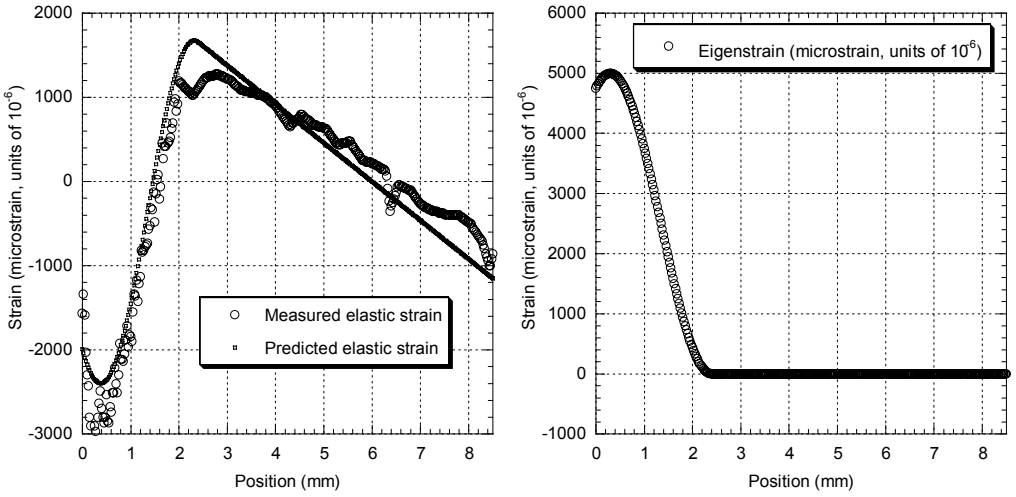

Figure 1. (a) Measured residual elastic strain distribution as a function of position across the bar (open circles) and the prediction based on the simple bending analysis (small circles). (b) The underlying eigenstrain distribution.

Finite element analysis

In this section we present an implementation of the variational method based on finite element analysis. The first step in the method requires us to adopt a functional form for the representation of the eigenstrain distribution, $\varepsilon^*(x)$. We continue to assume, as before, that the only relevant component of eigenstrain is ε_{yy}^* (parallel to the surface *x*=0), and that it is non-zero only within the region $0 < x < c$[3]. We introduce scaled

[3] At this point in the analysis we do not include the choice of c in the variational solution of the problem. Instead, the solution is carried out for different values of c; the best choice is identified by comparison with experimental data.

variables ς and χ, as follows: $\varsigma = (2x - c)/c$, so that $-1 \le \varsigma \le 1$; and $\chi = x/c$, so that $0 \le \chi \le 1$. We use a series of Chebyshev polynomials of the first kind,

$$T_k = \cos(k.\arccos(\varsigma)), \tag{6}$$

to express $\varepsilon^*(x)$, but also impose the requirement that eigenstrain show appropriate decay near the edge of its domain of definition (cf. previous section). We hence use

$$\varepsilon^*_{yy}(x) = G(\chi)\sum_{k=0}^{n} a_k T_k(\varsigma), \tag{7}$$

where the Gaussian distribution $G(\chi) = \exp[-(\xi\chi)^2]$ has been introduced as a 'modulating function'. In the above expression a_k denotes the set of coefficients to be determined by minimization of the functional in equation (3), and n is the number of terms in the series. The elastic strain distribution arising due to each individual term in the series (6) is calculated using finite element analysis, introducing the eigenstrain as thermal strains. Figure 2 shows eigenstrain and corresponding elastic strain distributions for two different components of the series in equation (6) for (a) k= 0 and (b) k = 10.

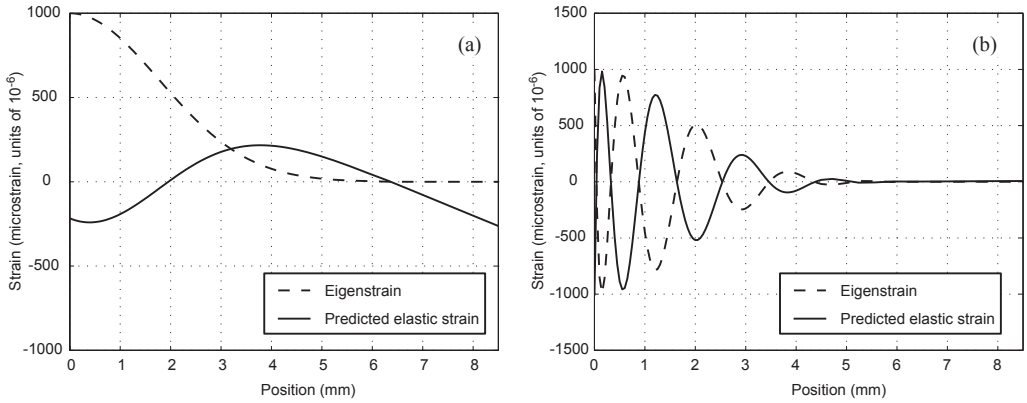

Figure 2. Eigenstrain and corresponding calculated elastic strain for Chebyshev polynomial (T_k) of degree (a) k = 0 and (b) k = 10

The calculation of the coefficients in equation (6) is accomplished by minimising the function in equation (3). A number of collocation points is selected that has to be greater or equal than the number n of unknown coefficients a_k. The problem is then formulated as a linear system for a_k. The final prediction for the strain distribution can be again calculated using FE, or reconstituted directly from individual terms with weights a_k.

Figure 3(a) presents the comparison between experimental residual strain data (open symbols) and the calculated prediction using the method explained above, together with the eigenstrain distribution shown in Figure 3(b). In this example the number of terms in

the series in equation (6) was n=41, and the extent of the source area was to be c=6 mm. Since the weighting in the functional (3) was chosen to favour the near surface peened region, excellent agreement is obtained in this area, but at the expense of some mismatch near the opposite face of the bar.

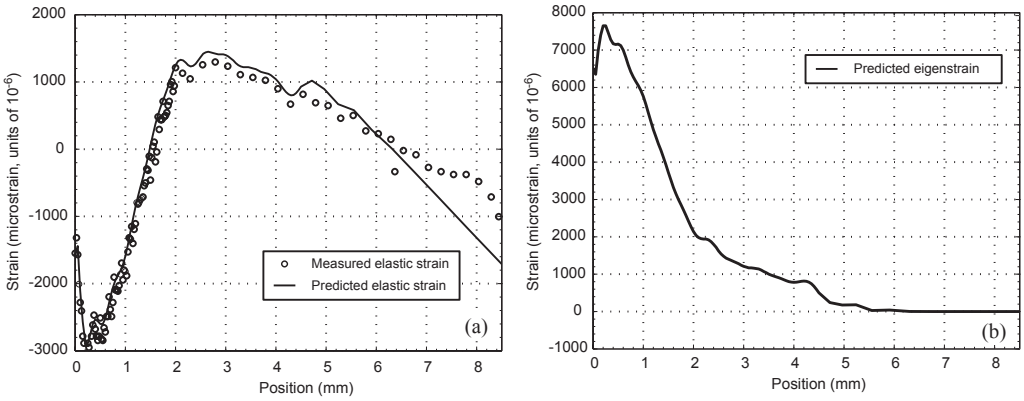

Figure 3. (a) Comparison between the experimentally measured elastic strain distribution and variational FE model prediction, and (b) the eigenstrain distribution.

Discussion

In this paper we presented the framework and several implementations of the variational approach to the determination of eigenstrain distributions, the sources of residual stress. The strength of the proposed procedure lies in the fact that, within the framework and accuracy of the formulation, it allows the complete stress-strain state (i.e. different components) within the component to be deduced on the basis of the eigenstrain distribution.

The approach thus provides a particularly useful tool for experimental residual stress analysis, where typically the complete determination of the three-dimensional strain state at every point is more problematic than the measurement of only the dominant components of strain that govern the overall residual stress state.

References

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