CONTRACTIVITY AND INVARIANCE IN GAMES WITH VECTOR PAYOFFS

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ABSTRACT

We study a distributed allocation process where, repeatedly in time, every player renegotiates past allocations with neighbors and allocates new revenues. The average allocations evolve according to a doubly (over time and space) averaging algorithm. We study conditions under which the average allocations reach consensus to any point within a predefined target set even in the presence of adversarial disturbances. Motivations arise in the context of coalitional games with transferable utilities (TU) where the target set is any set of allocations that make the grand coalitions stable.

***Index Terms*—** Game theory, networks, allocations.

We consider a two-step distributed allocation process where at every time players first renegotiate their past allocations and second generate a new revenue and allocate it. The cumulative (over time) allocations (states) evolve according to discrete-time linear dynamics which involve an averaging (over space) process. The goal is to let all allocations reach consensus to any value in a predefined set even in the presence of an adversarial disturbance. A similar problem has been addressed in [1] for the time-averaged allocations.

Motivations. The problem arises in the context of dynamic coalitional games with Transferable Utilities (TU games) [5]. A coalitional TU game consists in a set of players, who can form coalitions, and a characteristic function that provides a value for each coalition. The predefined set introduced above can be thought of as (but it is not limited to) the core of the game. This is the set of imputations under which no coalition has a value greater than the sum of its members' payoffs. Therefore, no coalition has incentive to leave the grand coalition and receive a larger payoff.

Main contribution. We analyze conditions under which there exist contractive and invariant sets for the cumulative allocations (Theorem 1).

Related literature. Coalitional games with transferable utilities (TU) were first introduced by von Neumann and Morgenstern [8]. Here, a main issue is to study whether the core is an "approachable" set, and which allocation processes can drive the "complaint vector" to that set. Approachability theory was developed by Blackwell in the early '56, [2], and is captured in the well known Blackwell's Theorem. The geometric (approachability) principle that lies behind the Blackwell's Theorem is among the fundamentals in allocation processes in coalitional games [4]. The discrete-time dynamics analyzed in the paper follows the rules of typical consensus dynamics (see, e.g., [6] and references therein). among multiple agents, where an underlying communication graph for the agents and balancing weights have been used with some variations to reach an agreement on common decision variable in [6, 7] for distributed multi-agent optimization.

The paper is organized as follows. In Section, 1, we formulate the problem In Section 2, we illustrate the main results. Finally, in Section 3, we provide concluding remarks and future directions. Though the proofs of the results are an essential contribution of the paper, we find it more convenient for readability to collect them all in an appendix.

Notation. We let x' denote the transpose of a vector x, and $\|x\|$ denote its Euclidean norm. An $n \times n$ matrix A is row-stochastic if the matrix has nonnegative entries a^i_j and $\sum_{j=1}^n a^i_j = 1$ for all $i = 1, \dots, n$. For a matrix A, we use a^i_j or $[A]_{ij}$ to denote its ijth entry. A matrix A is doubly stochastic if both A and its transpose A' are row-stochastic. We use $|S|$ for the cardinality of a given finite set S. We write $P_X[x]$ to denote the projection of a vector x on a set X, and we write $\mathrm{dist}(x, X)$ for the distance from x to X, i.e., $P_X[x] = \arg\min_{y \in X} \|x - y\|$ and $\mathrm{dist}(x, X) = \|x - P_X[x]\|$, respectively.

1. DISTRIBUTED REWARD ALLOCATION

Every player in a set $N = \{1, \dots, n\}$ is characterized by an average allocation vector $x_i(t) \in \mathbb{R}^n$. At every time he renegotiates with *neighbors* all past allocations and generates a new allocation vector $u_i(t)$. Then, the cumulative (over time) allocation $x_i(t)$ to player i evolves as follows:

$$x_i(t+1) = \left[\sum_{j=1}^n a^i_j(t) x_j(t)\right] + u_i(t),\ t = 0, 1, \dots \quad (1)$$

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where $a^i = (a^i_1, \ldots, a^i_n)'$ is a vector of nonnegative weights consistent with the sparsity of the *communication graph* $\mathcal{G}(t) = (N, \mathcal{E}(t))$. A link $(j,i) \in \mathcal{E}(t)$ exists if player j is a neighbor of player i at time t, i.e. if player i renegotiates allocations with player j at time t.

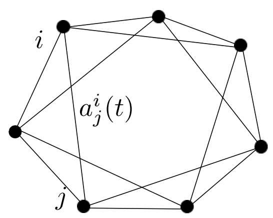

Fig. 1. Communication graph.

For each player $i \in N$, the input $u_i(\cdot)$ is the payoff of a repeated two-player game between player i (Player i_1) and an (external) adversary (Player i_2). Let S_1 and S_2 be the finite set of actions of players i_1 and i_2 respectively and let us denote the set of mixed actions pairs by $\Delta(S_1) \times \Delta(S_2)$ (set of probability distributions on S_1 and S_2). For every pair of mixed strategies $(p(t), q(t)) \in \Delta(S_1) \times \Delta(S_2)$ for player i_1 and i_2 at time t, the expected payoff is

$$\left\{ \begin{array}{l} u_i(t) = \sum_{j \in S_1, k \in S_2} p_j(t)\phi(j,k)q_k(t), \\ \sum_{j \in S_1} p_j(t) = 1 \\ \sum_{k \in S_2} q_k(t) = 1. \end{array} \right. \quad (2)$$

Essentially, in the above game $\phi(j,k)$ is the vector payoff when players i_1 and i_2 play pure strategies $j \in S_1$ and $k \in S_2$ respectively.

Our goal is to study contractivity and invariance of sets for the collective dynamics (1)-(2).

In the sequel, we rewrite dynamics (1)-(2) in the compact form:

$$\left\{ \begin{array}{l} x_i(t+1) = w_i(t) + u_i(t), \\ w_i(t) = \left[\sum_{j=1}^n a^i_j(t) x_j(t) \right]. \\ u_i(t) = \sum_{j \in S_1, k \in S_2} p_j(t)\phi(j,k)q_k(t), \\ (p(t), q(t)) \in \Delta(S_1) \times \Delta(S_2). \end{array} \right. \quad (3)$$

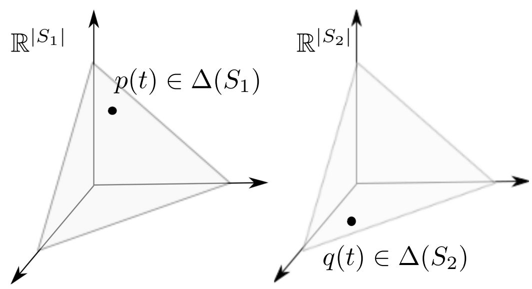

Fig. 2. Spaces of mixed strategies for the two players.

1.1. Motivations

The set X introduced above can be thought of as the core of a coalitional game with Transferable Utilities (TU game).

A coalitional TU game is defined by a pair $< N, \eta >$, where $N = \{1, \ldots, n\}$ is a set of players and $\eta : 2^N \to \mathbb{R}$ a function defined for each coalition $S \subseteq N$ ($S \in 2^N$). The function η determines the value $\eta(S)$ assigned to each coalition $S \subset N$, with $\eta(\emptyset) = 0$. We let η_S be the value $\eta(S)$ of the characteristic function η associated with a nonempty coalition $S \subseteq N$. Given a TU game $< N, \eta >$, let $C(\eta)$ be the core of the game,

$$C(\eta) = \left\{ x \in \mathbb{R}^n \,\Big|\, \sum_{j \in N} [x]_j = \eta_N, \right.$$
$$\left. \sum_{j \in S} [x]_j \geq \eta_S \text{ for all nonempty } S \subset N \right\}.$$

Essentially, the core of the game is the set of all allocations that make the grand coalition stable with respect to all subcoalitions. Condition $\sum_{j \in N} [x]_j = \eta_N$ is also called efficiency condition. Condition $\sum_{j \in S} [x]_j \geq \eta_S$ for all nonempty $S \subset N$ is referred to as "stability with respect to subcoalitions", since it guarantees that the total amount given to the members of a coalition exceeds the value of the coalition itself.

1.2. Main assumptions

Following [6] (see also [5]) we can make the following assumptions on the information structure. We let $A(t)$ be the weight matrix with entries $a^i_j(t)$.

Assumption 1 *Each matrix $A(t)$ is doubly stochastic with positive diagonal. Furthermore, there exists a scalar $\alpha > 0$ such that $a^i_j(t) \geq \alpha$ whenever $a^i_j(t) > 0$.*

At any time, the instantaneous graph $\mathcal{G}(t)$ need not be connected. However, for the proper behavior of the process, the union of the graphs $\mathcal{G}(t)$ over a period of time is assumed to be connected.

Assumption 2 *There exists an integer $Q \geq 1$ such that the graph $\left(N, \bigcup_{\tau=tQ}^{(t+1)Q-1} \mathcal{E}(\tau)\right)$ is strongly connected for every $t \geq 0$.*

For simplicity the one-shot vector-payoff game (S_1, S_2, x_i) is denoted by G_i.

Let $\lambda \in \mathbb{R}^n$. Denote by $\langle \lambda, G_i \rangle$ the zero-sum one-shot game whose set of players and their action sets are as in the game G_i, and the payoff that player 2 pays to player 1 is $\lambda' \phi(j,k)$ for every $(j,k) \in S_1 \times S_2$.

The resulting zero-sum game is described by the matrix

$$\Phi_\lambda = [\lambda' \phi(j,k)]_{j \in S_1, k \in S_2}.$$

As a zero-sum one-shot game, the game $\langle \lambda, G_i \rangle$ has a value, denoted

$$v_\lambda := \min_{p \in \Delta S_1} \max_{q \in \Delta S_2} p' \Phi_\lambda q = \max_{q \in \Delta S_2} \min_{p \in \Delta S_1} p' \Phi_\lambda q.$$

Following [2, 3], we assume next that the value of the projected game is always negative.

Assumption 3 *The payoff of the game $\phi(j,k)$ is bounded and it holds*

$$v_\lambda < 0, \quad \textit{for all } \lambda \in \mathbb{R}^n.$$

The above condition is among the foundations of approachability theory as it guarantees that the average vector payoff of a two-player repeated game converges almost surely to X (see, e.g., [2] and also [3], chapter 7).

2. MAIN RESULT

The main result of this paper establishes contractivity and invariance for the collective dynamics (3). Before stating the theorem, we need to introduce two lemmas. The next lemma establishes that the space averaging step in (1) reduces the total distance (i.e. the sum of distances) of the states from the set X.

Lemma 1 *Let Assumption 1 hold. Then the total distance from X decreases when replacing the states $x_i(t)$ by their space averages $w_i(t)$, i.e.,*

$$\sum_{i=1}^{n} \operatorname{dist}(w_i(t), X) \leq \sum_{i=1}^{n} \operatorname{dist}(x_i(t), X).$$

Proof. Given in the appendix. □

As a preliminary step to the next result, observe that, from the definition of $\operatorname{dist}(\cdot, X)$ and from (1) and (3), it holds

$$\begin{aligned} \operatorname{dist}(x_i(t+1), X)^2 &= \|x_i(t+1) - P_X[x_i(t+1)]\|^2 \\ &\leq \|x_i(t+1) - P_X[w_i(t)]\|^2 \\ &= \|w_i(t) + u_i(t) - P_X[w_i(t)]\|^2 \\ &= \|w_i(t) - P_X[w_i(t)]\|^2 + \|u_i(t)\|^2 \\ &\quad + 2(w_i(t) - P_X[w_i(t)])' u_i(t). \end{aligned} \tag{4}$$

The following lemma states that, under the approachability assumption, the new input $u_i(t)$ reduces the distance of each single average state $w_i(t)$ from X.

Lemma 2 *Let Assumption 3 hold. Then, there exists a lower bound for $\operatorname{dist}(w_i(t), X)$ such that*

$$\operatorname{dist}(x_i(t+1), X) < \operatorname{dist}(w_i(t), X), \quad \forall i = 1, \ldots, n.$$

Proof. Given in the appendix. □

Denote by

$$\begin{aligned} \Phi &= \Big\{ x \in \mathbb{R}^n \,|\, \operatorname{dist}(x_i(t), X) \geq \tfrac{L}{2\phi}, \\ &\qquad \phi \geq -\tfrac{v_\lambda}{\operatorname{dist}(x_i(t), X)}, \text{ for all } \lambda \in \mathbb{R}^n \Big\} \\ \Psi &= \Big\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \,| \\ &\quad \textstyle\sum_{i=1}^n \operatorname{dist}(x_i(t), X)^2 \leq \sum_{i=1}^n \left(\frac{L}{2\phi}\right)^2 + nL \Big\} \end{aligned} \tag{5}$$

We are now ready to state the main result.

Theorem 1 (Contractivity and invariance) *Let Assumptions 1-3 hold and assume that $x_i(t) \in \Phi$ for all $i = 1, \ldots, n$. Then it holds*

$$\sum_{i=1}^{n} \operatorname{dist}(x_i(t+1), X)^2 < \sum_{i=1}^{n} \operatorname{dist}(x_i(t), X)^2. \tag{6}$$

On the other hand, if $x_i(t) \notin \Phi \subset \Psi$ for all $i = 1, \ldots, n$

$$\sum_{i=1}^{n} \operatorname{dist}(x_i(t+1), X)^2 \leq \sum_{i=1}^{n} \left(\frac{L}{2\phi}\right)^2 + nL. \tag{7}$$

Proof. Given in the appendix. □

Essentially the above result proves that there exist both contractive sets and invariant sets for the collective dynamics.

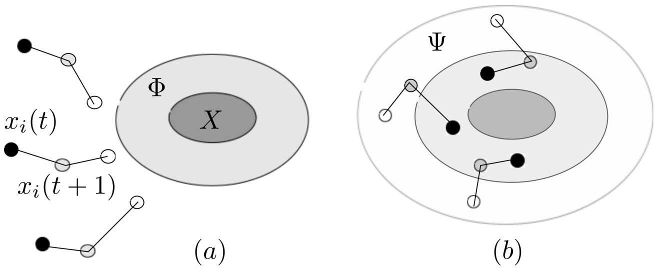

Fig. 3. Theorem 1: contractivity (a) and invariance (b)

3. CONCLUSIONS

We have analyzed convergence conditions of a distributed allocation process arising in the context of TU games. Future directions include the extension of our results to population games with mean-field interactions, and averaging algorithms driven by Brownian motions.

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Appendix

Proof of Lemma 1. By convexity of the distance function $\mathrm{dist}(\cdot, X)$ and from (3) we have

$$\mathrm{dist}(w_i(t), X) \leq \sum_{j=1}^{n} a_j^i(t)\mathrm{dist}(x_j(t), X).$$

Summing over $i = 1, \dots, n$ both sides we obtain

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{dist}(w_i(t), X) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_j^i(t)\mathrm{dist}(x_j(t), X) \\ =\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_j^i(t)\right)\mathrm{dist}(x_j(t), X) &= \sum_{j=1}^{n}\mathrm{dist}(x_j(t), X),\end{aligned}$$

where the last equality follows from the stochasticity of $A(t)$ in Assumption 1. This concludes the proof.

Proof of Lemma 2. Rearranging equation (4) we obtain

$$\begin{aligned}&\|x_i(t+1) - P_X[x_i(t+1)]\|^2 \\ &-\|w_i(t) - P_X[w_i(t)]\|^2 \leq \\ &\|u_i(t)\|^2 + 2(w_i(t) - P_X[w_i(t)])' u_i(t).\end{aligned} \tag{8}$$

Now, from Assumption 3 we also have that for any $w_i(t) \in \mathbb{R}^n$, there exists a mixed strategy $p(t) \in \Delta(S_1)$ for Player i_1 such that, for all mixed strategy $q(t) \in \Delta(S_2)$ of Player i_2, there exists $L > 0$ s.t. $\forall t \geq 0$ $\|u_i(t)\|^2 \leq L$, and $u_i(\cdot)$ satisfies

$$(w_i(t) - P_X(w_i(t)))' u_i(t) \leq -\phi\|w_i(t) - P_X(w_i(t))\| < 0,$$

where $u_i(t) = \sum_{j\in S_1}\sum_{k\in S_2} p_j(t)\phi(j,k)q_k(t)$.

Then there exists a great enough value for $\|w_i(t) - P_X[w_i(t)]\|$ such that from Assumption 3 the left hand side in (8) is negative. From the boundedness of set X and of vectors $u_i(t)$we have

$$\begin{aligned}&\mathrm{dist}(x_i(t+1), X)^2 - \mathrm{dist}(w_i(t), X)^2 \\ &\quad\leq (L - 2\phi\|w_i(t) - P_X[w_i(t)]\|).\end{aligned} \tag{9}$$

Taking $\|w_i(t) - P_X[w_i(t)]\| > \frac{L}{2\phi} > 0$ we have

$$\begin{aligned}&\mathrm{dist}(x_i(t+1), X)^2 - \mathrm{dist}(w_i(t), X)^2 \\ &\quad\leq (L - 2\phi\|w_i(t) - P_X[w_i(t)]\|) < 0\end{aligned} \tag{10}$$

and this concludes the proof.

Proof of Theorem 1. From Lemma 1 and 2 and rearranging (4), we have

$$\begin{aligned}&\sum_{i=1}^{n}\left[\mathrm{dist}(x_i(t+1), X)^2 - \mathrm{dist}(x_i(t), X)^2\right] \\ &\leq \sum_{i=1}^{n}\left[\|u_i(t)\|^2 + 2(w_i(t) - P_X[w_i(t)])' u_i(t)\right] \\ &\leq nL - 2\phi\sum_{i=1}^{n}\|w_i(t) - P_X[w_i(t)]\| \\ &\leq nL - 2\phi\sum_{i=1}^{n}\mathrm{dist}(x_i(t+1, X)) < 0\end{aligned}$$

where the last inequality is due to Assumption 3. Summing over $t = 0, \dots, \tau - 1$, and noting that $\|u_i(t)\|^2$ is bounded (from Assumption 3), we obtain

$$\sum_{i=1}^{n}\left[\mathrm{dist}(x_i(\tau), X)^2 - \mathrm{dist}(x_i(0), X)^2\right] \leq \tau n[L - 2\phi\epsilon]$$

which concludes the first part of the proof (contractivity).

For the second part (controlled invariance) suppose $\mathrm{dist}(x_i(t), X) \leq \frac{L}{2\phi}$. Then we have

$$\sum_{i=1}^{n}\left[\mathrm{dist}(x_i(t+1), X)^2 - \mathrm{dist}(x_i(t), X)^2\right] \leq nL < 0$$

from which we obtain

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{dist}(x_i(t+1), X)^2 &\leq \sum_{i=1}^{n}\mathrm{dist}(x_i(t), X)^2 + nL \\ &\leq \sum_{i=1}^{n}\left(\frac{L}{2\phi}\right)^2 + nL\end{aligned}$$

and this concludes our proof.