Stochastic Model Predictive Control

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Stochastic MPC

- Algorithmic considerations
- Implementation issues

Outline

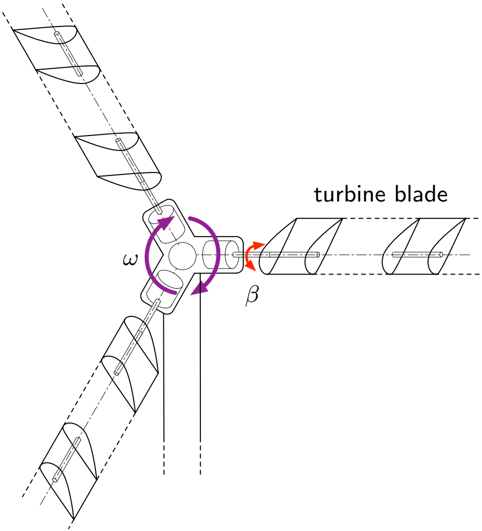
1. Motivation: wind turbine blade fatigue control
2. Stochastic MPC: basic formulations
   - Probabilistic constraints & recursive feasibility
   - Performance costs and stability analyses
3. Implementation
   - Affine model uncertainty: approximate and exact tubes
   - Additive model uncertainty: exact tubes

Wind turbine blade pitch control

- System model
- High-cycle fatigue
- Constraints

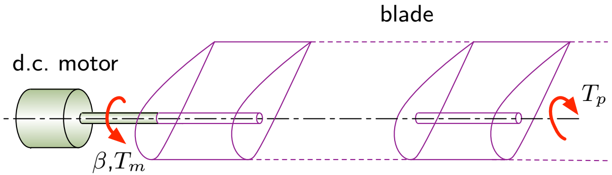
Wind turbine blade pitch control

- Wind speed: v
  assumed random
- Rotational speed: ω
  assumed constant
- Blade pitch angle: β
- Power-optimal
  pitch angle: $\beta^*(v, \omega)$



Control problem: track β^* subject to constraints on fatigue damage

Blade dynamic model



blade pitch angle: β

motor torque: T_m

aerodynamic torque: T_p

friction torque: $c\dfrac{d\beta}{dt}$

Lumped parameter model

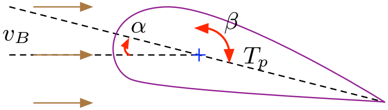
$$\boxed{J\frac{d^2\beta}{dt^2} + c\frac{d\beta}{dt} = T_m - T_p}$$

Aerodynamic torque

$$T_p = T_p(\alpha, v_B)$$

wind speed at blade: $v_B = v_B(\omega, v)$

angle of attack: $\alpha = \alpha(\beta, \omega, v)$



Hence the model

$$J\frac{d^2\beta}{dt^2} + c\frac{d\beta}{dt} = T_m - T_p(\alpha, v_B)$$

contains:

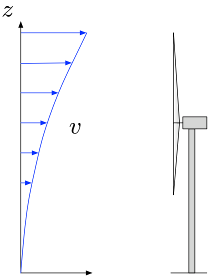
$$\left.\begin{array}{l}\text{multiplicative uncertainty due to } v_B(\omega, v) \\ \text{additive uncertainty due to } \alpha(\beta, \omega, v)\end{array}\right\} \quad \text{statistically dependent}$$

Wind model

Wind speed variation with height z:

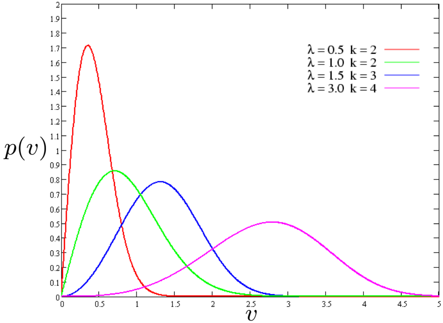
$$v = v_{\mathrm{ref}} \frac{\log(z/z_0)}{\log(z_{\mathrm{ref}}/z_0)}$$

(ground *roughness factor*: $z_0 > 0$)

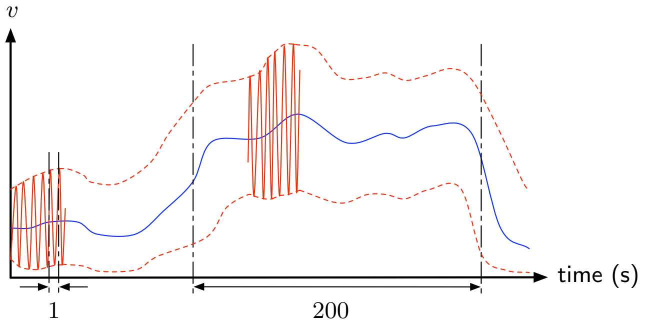


Wind speed pdf $p(v)$

modelled using Weibull
or Gaussian distributions



Wind model



— : v

approx. constant mean and variance over interval ~ 200 s

— : v_B

fluctuates due to blade rotation with period ~ 1 s

Linearized discrete-time system model

- Control input u determines motor torque: $T_m = k_m u$
- System output: net torque $y = T_m - T_p$
- Approximate linear model around a setpoint identified as:

$$\beta_{k+1} = a_{k,1}\beta_k + a_{k,0}\beta_{k-1} + b_{k,1}u_k + b_{k,0}u_{k-1} + \gamma_k$$

using:

NACA 632-215(V) blade data

1 second sampling interval

least squares estimation of $(\bar{\theta}, \Sigma_\theta)$, $\theta = [a_1 \quad a_2 \quad b_1 \quad b_2 \quad \gamma]^T$

Linearized discrete-time system model

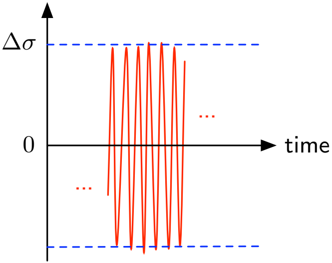
- Model: $x(k+1) = A_k x(k) + B_k u(k) + w_k$,
$$A_k = \begin{bmatrix} 0 & a_{k,2} \\ 1 & a_{k,1} \end{bmatrix}, \quad B_k = \begin{bmatrix} b_{k,2} \\ b_{k,1} \end{bmatrix}, \quad w_k = \begin{bmatrix} 0 \\ \gamma_k \end{bmatrix}$$
with parameters
$$\begin{bmatrix} A_k & w_k \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \end{bmatrix} + \sum_{j=1}^{3} \begin{bmatrix} A^{(j)} & w^{(j)} \end{bmatrix} q_j(k)$$

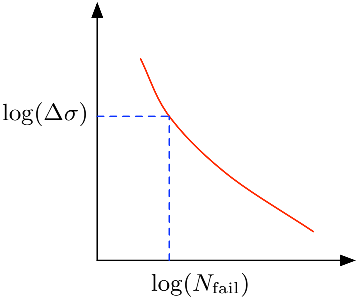
- Output: $y(k) = c_k x(k) + d_k$:
$$\begin{bmatrix} c_k & d_k \end{bmatrix} = \begin{bmatrix} \bar{c} & 0 \end{bmatrix} + \sum_{j=1}^{2} \begin{bmatrix} c^{(j)} & d^{(j)} \end{bmatrix} q_j(k)$$

- Random variable $q(k) \sim \mathfrak{D}$, approximately i.i.d.,
identified empirically as truncated Gaussian

High-cycle fatigue

- S-N curve gives number of cycles to failure (N_{fail}) under cyclical stress loading:





- Extend to: non-zero mean stress (using Goodman's rule)
  combined stress loadings (using Miner's rule)

Constraints

- Blade stress amplitude depends on net torque, y:

$$y = T_m - T_p$$

  hence for a given life-span N_{life}, require:

$$N_{\text{fail}}/N_{\text{life}} \geq \text{rate of violation of bound: } y \leq \bar{y}$$

  for $\bar{y}$ and N_{fail} obtained from S-N curves

- Equivalent probabilistic constraints:

$$\boxed{\Pr\{c_k x(k) + d_k \leq \bar{y}\} \geq p, \qquad p = 1 - \frac{N_{\text{fail}}}{N_{\text{life}}}}$$

- Motor torque saturation: $|T_m| \leq T_{\max}$ implies hard input constraints

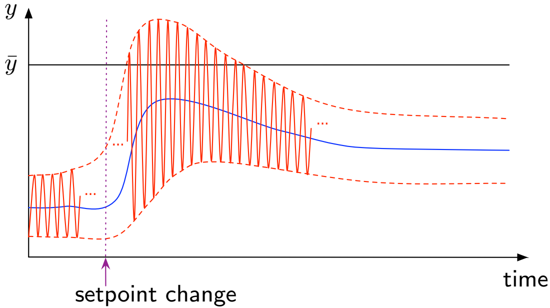
$$\boxed{|u(k)| \leq \overline{u}}$$

Constraints

More accurate probabilistic constraints

... on the average rate of violation of bounds over interval N_s

$$\frac{1}{N_s}\sum_{k=0}^{N_s-1} \Pr\{c_k x(k) + d_k \leq \bar{y}\} \geq p, \qquad p = 1 - \frac{N_{\text{fail}}}{N_{\text{life}}}$$



... on ranges of stress amplitudes, e.g. $\Pr\{c_k x(k) + d_k > \bar{y}_r\} < p_r = N_{\text{fail},r}/N_{\text{life}}$

$$\vdots$$

$$\Pr\{\bar{y}_2 \leq c_k x(k) + d_k > \bar{y}_1\} < p_1 = N_{\text{fail},1}/N_{\text{life}}$$

Problem formulation

- System and constraints:

$$
\begin{aligned}
&x^+ = Ax + Bu + w, && A = A(q),\ B = B(q),\ w = w(q) \\
&\Pr\{cx + d \le \bar{y}\} \ge p, && c = c(q),\ d = d(q) \\
&|u| \le \bar{u}
\end{aligned}
$$

Uncertainty: $q \sim \mathfrak{D}$, where distribution $\mathfrak{D}$ is finitely supported in practice

- MPC law $u(k) = \kappa_{\mathrm{MPC}}(x(k))$ obtained by solving, at $k = 0, 1, \ldots$:

$$
\boxed{
\begin{aligned}
&\min_{\{u(k), u(k+1), \ldots\}} && J\big(x(k), \{u(k), u(k+1), \ldots\}\big) \\
&\text{subject to} && \Pr\big\{c(k+i)x(k+i) + d(k+i) \le \bar{y}\big\} \ge p \\
& && |u(k+i)| \le \bar{u}, \qquad\qquad i = 0, 1, \ldots
\end{aligned}
}
$$

How can we ensure recursive feasibility?

analyse closed loop behaviour?

invoke probabilistic constraints?

parameterize predictions?

Probabilistic constraints & recursive feasibility

Recursive feasibility

Consider the general dynamics: $x^+ = f(x, u, q)$

and probabilistic constraint: $\Pr\{F(x, u, q) \le 1\} \ge p, \;\; q \sim \mathfrak{D}$

Suppose $u(0)$ and $u(1)$ are such that

$$\Pr\{F(x(k), u(k), q(k)) \le 1 \,\big|\, x(0)\} \ge p, \;\; k = 0, 1$$

then:

(i) it is not necessarily true that

$$\Pr\{F(x(k), u(k), q(k)) \le 1 \,\big|\, x(1)\} \ge p, \;\; k = 1$$

e.g. take $x \in \mathbb{R}$, $F(x, u, q) = f(x, u, q)$:

$$\Pr\{F(x(1), u(1), q(1)) \le 1 \,\big|\, x(0)\} \ge p$$

but

$$\Pr\{F(x(1), u(1), q(1)) \le 1 \,\big|\, x_{\max}(1)\} \not\ge p$$

(ii) for some realizations $q(0)$, there may not exist any $u(1)$ satisfying this constraint

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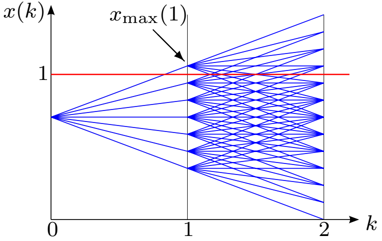

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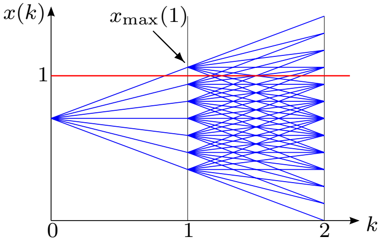

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Recursive feasibility

Define $\mathcal{T}_0 = \{x : \exists u,\ F(x,u,q) \le 1 \text{ w.p. } p\}$,

then $u(1)$ exists s.t. $F(x,u,q) \le 1$ w.p. p iff $x(1) \in \mathcal{T}_0$ w.p. $1 \longleftarrow$ hard constraint

$\exists\{u(0), u(1), \dots\}$ s.t. $\left\{ \begin{array}{l} F(x(k),u(k),q) \le 1 \text{ w.p. } p \\ x(k) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array} \right\}$ $k = 0, 1 \dots$ iff $x(0) \in \mathcal{R}_\infty$

$\mathcal{R}_\infty$ = infinite-time reachability set [Bertsekas 1972]

$\hat{\mathcal{T}}_0 = \{(x,u) : F(x,u,q) \le 1 \text{ w.p. } p\}$, $\qquad \mathcal{R}_0 = \mathcal{T}_0 = \mathrm{Proj}_x(\hat{\mathcal{T}}_0)$

$\hat{\mathcal{R}}_k = \{(x,u) : f(x,u,q) \in \mathcal{R}_{k-1} \text{ w.p. } 1\} \bigcap \hat{\mathcal{T}}_0$, $\qquad \mathcal{R}_k = \mathrm{Proj}_x(\hat{\mathcal{R}}_k)$, $k = 1, 2, \dots$

$\mathcal{R}_\infty$ exists if $\mathcal{T}_0$ and $\mathrm{supp}(\mathfrak{D})$ are compact, and $x^+ = f(x,u,q)$ is stabilizable, with minimal robust control invariant set contained in $\mathrm{int}(\mathcal{T}_0)$.

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$$\exists\{u(0),u(1),\ldots\} \text{ s.t. } \left\{ \begin{array}{l} F(x(k),u(k),q) \leq 1 \text{ w.p. } p \\ x(k) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array} \right\} k = 0,1\ldots \text{ iff } x(0) \in \mathcal{R}_\infty$$

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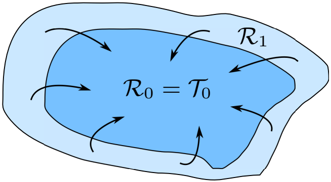

Recursive feasibility

Prototype stochastic MPC optimization:

$$\begin{aligned}\min_{\{u(k),u(k+1),\dots\}} \quad & J\big(x(k),\{u(k),u(k+1),\dots\}\big)\\ \text{subject to} \quad & F(x(k+i),u(k+i),q(k+i)) \leq 1 \text{ w.p. } p\\ & x(k+i)\in\mathcal{T}_0 \text{ w.p. } 1\\ & \qquad\qquad i=0,1,\dots\end{aligned}$$

- Constraint at prediction time $k+i$ is invoked with:
  - worst case $\{q(k),\dots,q(k+i-1)\}$
  - stochastic $q(k+i)$
- Finite horizon implementation requires terminal set, $\mathbb{T}$

$$\text{where } x\in\mathbb{T} \implies \begin{cases} x^+\in\mathbb{T} \text{ w.p. } 1\\ F(x,\kappa_{\mathbb{T}}(x),q)\leq 1 \text{ w.p. } p\end{cases}$$

for some terminal feedback law $u=\kappa_{\mathbb{T}}(x)$

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Terminal set

Let $\mathcal{O}_\infty =$ maximal admissible set for dynamics: $x^+ = f(x, \kappa_{\mathbb{T}}(x), q)$
and constraint: $F(x, \kappa_{\mathbb{T}}(x), q) \leq 1$ w.p. p

where $\quad \mathcal{O}_k = \bigcap_{j=0}^{k} \mathcal{S}_j, \quad k = 0, 1, \ldots$ [cf. Kolmanovsky & Gilbert 1998]

$$\mathcal{S}_0 = \{x : F(x, \kappa_{\mathbb{T}}(x), q) \leq 1 \text{ w.p. } p\}$$

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[cf. Kolmanovsky & Gilbert 1998]

If $\mathcal{O}_k \subseteq \mathcal{S}_{k+1}$, then $\mathcal{O}_k = \mathcal{O}_\infty$

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If $\mathcal{O}_k \subseteq \mathcal{S}_{k+1}$, then $\mathcal{O}_k \bigcap \mathcal{S}_{k+1} = \mathcal{O}_{k+1} = \mathcal{O}_k$, and

(i) $x(0) \in \mathcal{O}_k = \mathcal{O}_{k+1} \Rightarrow \left\{ \begin{array}{c} x(0) \in \mathcal{S}_1 \\ \vdots \\ x(0) \in \mathcal{S}_{k+1} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} x(1) \in \mathcal{S}_0 \text{ w.p. } 1 \\ \vdots \\ x(1) \in \mathcal{S}_k \text{ w.p. } 1 \end{array} \right\} \Rightarrow x(1) \in \mathcal{O}_k$ w.p. 1

i.e. $\mathcal{O}_k$ is positively invariant with probability 1

(ii) $x(0) \in \mathcal{O}_k \Rightarrow x(i) \in \mathcal{O}_k \ \forall i \geq 0 \Rightarrow x(i) \in \mathcal{S}_0$ w.p. 1 $\forall i \geq 0$
$\Rightarrow x(0) \in \mathcal{S}_i \ \forall i \geq 0$

hence $\mathcal{O}_k \subseteq \bigcap_{j=0}^{\infty} \mathcal{S}_j = \mathcal{O}_\infty$

but $\mathcal{O}_\infty \subseteq \mathcal{O}_k$ by definition, so $\mathcal{O}_k = \mathcal{O}_\infty$

Recursive feasibility

Prototype stochastic MPC optimization:

$$\begin{aligned} \min_{\{u(k),u(k+1),\ldots\}} \quad & J\big(x(k),\{u(k),u(k+1),\ldots\}\big) \\ \text{subject to} \quad & \left.\begin{array}{l} F(x(k+i),u(k+i),q(k+i)) \le 1 \text{ w.p. } p \\ x(k+i) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array}\right\} \quad i = 0,\ldots,N-1 \\ & x(k+N) \in \mathbb{T} \text{ w.p. } 1 \end{aligned}$$

- Choose $\mathbb{T}$ as the maximal admissible set $\mathcal{O}_\infty$ (or an invariant inner approximation)

- Recursive feasibility guarantee:

  if $\{u(0),\ldots,u(N-1)\}$ is feasible for given $x(0)$

  then $\{u(1),\ldots,u(N-1),\kappa_{\mathbb{T}}(x(N))\}$ is feasible for $x(1) = f(x(0),u(0),q)$ w.p. 1

- Definition of cost J determines stability properties.

  Consider expected and nominal quadratic costs.

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  Consider expected and nominal quadratic costs.

Recursive feasibility

- Re-parameterize predicted control law: $u = \kappa_{\mathbb{T}}(x) + c$

  then predicted control inputs, $\{u(0), u(1), \ldots\}$, at time 0 are defined by

$$u(k) = \kappa_{\mathbb{T}}(x(k)) + c(k) \ \text{ with } c(k) = 0, \ \forall k \geq N$$
$$\mathbf{c}(0) = \{c(0), \ldots, c(N-1)\}$$

- Define feasible set $\mathcal{F} = \{(x, \mathbf{c}) : \text{constraints are satisfied}\}$, i.e.

$$\mathcal{F} = \left\{ (x(k), \mathbf{c}(k)) : \begin{array}{l} \left. \begin{array}{l} F(x(k+i), u(k+i), q(k+i)) \leq 1 \text{ w.p. } p \\ x(k+i) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array} \right\} i = 0, \ldots, N-1 \\ x(k+N) \in \mathbb{T} \text{ w.p. } 1 \end{array} \right\}$$

  and $\mathcal{F}_x = \mathrm{Proj}_x(\mathcal{F})$

  Then

$$\boxed{\begin{array}{lll} x(0) \in \mathcal{F}_x & \Longrightarrow & (x(1), \tilde{\mathbf{c}}(1)) \in \mathcal{F} \text{ w.p. } 1 \\ & & \text{where } \ x(1) = f(x(0), \kappa_{\mathbb{T}}(x(0)) + c(0), q) \\ & & \qquad\quad \tilde{\mathbf{c}}(1) = \{c(1), \ldots, c(N-1), 0\} \end{array}}$$

Performance cost, stability and convergence

Local Stability

Lyapunov (robust) stability $\rightsquigarrow$ local property of optimal unconstrained MPC law $\kappa_{\mathrm{MPC}}^{\mathrm{uc}}$

Define

$$\mathcal{M}_\infty^{\mathrm{uc}} = \text{minimal robust invariant set under } \kappa_{\mathrm{MPC}}^{\mathrm{uc}}$$

$$\mathcal{F}_x^{\mathrm{uc}} = \left\{ x(0) : \begin{array}{l} (x(k), u(k)) \text{ satisfies constraints } \forall k \geq 0 \\ \text{under } u(k) = \kappa_{\mathrm{MPC}}^{\mathrm{uc}}(x(k)) \end{array} \right\}$$

- ⋆ If $\kappa_{\mathrm{MPC}}^{\mathrm{uc}}$ is asymptotically stabilizing and $\mathcal{M}_\infty^{\mathrm{uc}} \subset \mathrm{int}(\mathcal{F}_x^{\mathrm{uc}})$, then

  $\mathcal{M}_\infty^{\mathrm{uc}}$ is asymptotically stable
  with region of attraction containing $\mathcal{F}_x^{\mathrm{uc}}$

- ⋆ If $\kappa_{\mathbb{T}}$ = optimal unconstrained control law, then

  $$\kappa_{\mathrm{MPC}}^{\mathrm{uc}} = \kappa_{\mathbb{T}} \text{ and } \mathcal{F}_x^{\mathrm{uc}} \supseteq \mathbb{T}$$

  and hence $\mathcal{F}_x^{\mathrm{uc}} \supseteq \mathbb{T} \supset \mathcal{M}_\infty^{\mathrm{uc}}$

Expectation MPC cost

Quadratic expected value cost:

$$J(x(k), \{u(k), u(k+1), \dots\}) = \sum_{i=0}^{\infty} \mathbb{E}\big(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2\big)$$

optimal unconstrained control law: $u(k) = K_{\mathrm{LQ}} x(k)$

- ▷ Finite cost → minimize numerically online
- ▷ Quadratic in dof $\mathbf{c}(k)$:

$$V(x(k), \mathbf{c}(k)) = \mathbf{c}(k)^T P_{cc} \mathbf{c}(k) + 2\, x(k)^T P_{xc} \mathbf{c}(k) + p_k(x(k))$$

P_{cc}, P_{xc} computed offline, $P_{xc} = 0$ if $\kappa_{\mathbb{T}} = K_{\mathrm{LQ}}$

- ▷ Optimal value $V^*(x) = \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$ is lower-bounded in x
- ▷ MPC law: $\kappa_{\mathrm{MPC}}(x) = \kappa_{\mathbb{T}}(x) + c^*$, where $\mathbf{c}^* = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

Expectation MPC cost

Equivalent MPC objective:

$$V(x(k), \mathbf{c}(k)) = \sum_{i=0}^{\infty} \Big[\mathbb{E}\big(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \big) - l_{\mathrm{ss}} \Big]$$

where $l_{\mathrm{ss}} = \lim_{k\to\infty} \mathbb{E}\big(\|x(k)\|_Q^2 + \|u(k)\|_R^2\big)$ under $u(k) = K_{\mathrm{LQ}}x(k)$

- ▷ Finite cost $\rightarrow$ minimize numerically online

- ▷ Quadratic in dof $\mathbf{c}(k)$:

$$V(x(k), \mathbf{c}(k)) = \mathbf{c}(k)^T P_{cc}\mathbf{c}(k) + 2\,x(k)^T P_{xc}\mathbf{c}(k) + p_k(x(k))$$

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Expectation MPC cost

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where $l_{\rm ss} = \lim_{k\to\infty} \mathbb{E}\big(\|x(k)\|_Q^2 + \|u(k)\|_R^2 \big)$ under $u(k) = K_{\rm LQ} x(k)$

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Expectation MPC cost: convergence analysis

Let

$$\mathbf{c}^*(0) = \{c^*(0), \dots, c^*(N-1)\} = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x(0))} V(x(0), \mathbf{c})$$
$$\tilde{\mathbf{c}}(1) = \{c^*(1), \dots, c^*(N-1), 0\}$$

Then, by definition, for $x(0) \in \mathcal{F}_x$:

$$V(x(1), \tilde{c}(1)) = V^*(x(0)) - \left(\|x(0)\|_Q^2 + \|u(0)\|_R^2 - l_{\mathrm{ss}}\right)$$

but $\tilde{\mathbf{c}}(1) \in \mathcal{F}_{\mathbf{c}}(x(1))$ w.p. 1, so the optimal value function satisfies

$$\boxed{\mathbb{E}\big(V^*(x(1))\big) \le V^*(x(0)) - \left(\|x(0)\|_Q^2 + \|u(0)\|_R^2 - l_{\mathrm{ss}}\right)}$$

- summing over r time-steps:

$$\boxed{\frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\left(\|x(k)\|_Q^2 + \|u(k)\|_R^2\right) \le l_{\mathrm{ss}} + \frac{1}{r}\Big[V^*(x(0)) - \mathbb{E}\big(V^*(x(r))\big)\Big]}$$

- hence

$$\lim_{r\to\infty} \frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\left(\|x(k)\|_Q^2 + \|u(k)\|_R^2\right) \le l_{\mathrm{ss}}$$

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- ⋆ hence

$$\lim_{r\to\infty} \frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\big(\|x(k)\|_Q^2 + \|u(k)\|_R^2\big) \leq l_{\text{ss}}$$

Expectation MPC cost: convergence analysis

- If $l_{\mathrm{ss}} = 0$ (no additive disturbance), then

$$\mathbb{E}\big(V^*(x(1))\big) \leq V^*(x(0)) - \big(\|x(0)\|_Q^2 + \|u(0)\|_R^2\big)$$

and $(x(k), u(k)) \to (0, 0)$ as $k \to \infty$

- If $\kappa_{\mathbb{T}} = K_{\mathrm{LQ}}$, then l_{ss} is minimal:

$$l_{\mathrm{ss}} = \min_{\{u(0),u(1),\dots\}} \lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}\big(\|x(k)\|_Q^2 + \|u(k)\|_R^2\big)$$

Hence

(i) the bound $\displaystyle\lim_{r\to\infty} \frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\big(\|x(k)\|_Q^2 + \|u(k)\|_R^2\big) \leq l_{\mathrm{ss}}$ holds with equality

(ii) $\displaystyle\lim_{k\to\infty} \kappa_{\mathrm{MPC}}(x(k)) = K_{\mathrm{LQ}} x(k)$

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Nominal MPC cost

Linear dynamics: $x^+ = Ax + Bu + w, \quad (A, B, w) = (A(q), B(q), w(q)),\ q \sim \mathfrak{D}$

$\mathbb{E}(A, B, w) = (A^0, B^0, 0)$

State decomposition: $x = z + e$

Control decomposition: $u = v + K_{\mathbb{T}} e, \quad v = K_v z + c$

Nominal dynamics: $z^+ = A^0 z + B^0 v$

- Nominal cost:

$$V_0(x(0), \mathbf{c}(0), z(0)) = \sum_{k=0}^{\infty} \left(\|z(k)\|_Q^2 + \|v(k)\|_R^2 \right)$$

- Optimal value: $V_0^*(x) = \min_{(\mathbf{c}, z) \in \mathcal{F}_{\mathbf{c},z}(x)} V_0(x, \mathbf{c}, z)$ is lower-bounded in x

- MPC law: $\kappa_{\mathrm{MPC}}(x) = K_{\mathbb{T}} e + v^*$, where $(\mathbf{c}^*, z^*) = \arg \min_{(\mathbf{c}, z) \in \mathcal{F}_{\mathbf{c},z}(x)} V_0(x, \mathbf{c}, z)$

$$v^* = K_v z^* + c^*$$

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$$v^* = K_v z^* + c^*$$

Nominal MPC cost: convergence analysis

Recursive feasibility:

$$x(0) \in \mathcal{F}_x \quad \Longrightarrow \quad (x(1), \tilde{\mathbf{c}}(1), \tilde{z}(1)) \in \mathcal{F} \text{ w.p. } 1$$

$$\begin{aligned} \text{where } \ x(1) &= Ax(0) + B\kappa_{\mathrm{MPC}}(x(0)) + w \\ \tilde{\mathbf{c}}(1) &= \{c^*(1), \ldots, c^*(N-1), 0\} \\ \tilde{z}(1) &= A^0 z^*(0) + B^0 v^*(0) \end{aligned}$$

Hence $V_0(x(1), \tilde{\mathbf{c}}(1), \tilde{z}(1)) \leq V_0^*(x(0)) - \left(\|z^*(0)\|_Q^2 + \|v^*(0)\|_R^2\right)$, and

$$\boxed{V_0^*(x(1)) \leq V_0^*(x(0)) - \left(\|z^*(0)\|_Q^2 + \|v^*(0)\|_R^2\right)}$$

- sum over r time steps:

$$\sum_{k=0}^{r-1}\left(\|z^*(k)\|_Q^2 + \|v^*(k)\|_R^2\right) \leq V_0^*(x(0))$$

- asymptotically:

$$\lim_{k\to\infty}\left(\|z^*(k)\|_Q^2 + \|v^*(k)\|_R^2\right) = 0 \quad \Longrightarrow \quad \begin{cases} z^*(k) \to 0 \\ v^*(k) \to 0 \\ \kappa_{\mathrm{MPC}}(x(k)) \to K_{\mathrm{T}} x(k) \end{cases} \quad \text{as } k \to \infty$$

Nominal MPC cost: convergence analysis

Recursive feasibility:

$$x(0) \in \mathcal{F}_x \implies (x(1), \tilde{\mathbf{c}}(1), \tilde{z}(1)) \in \mathcal{F} \text{ w.p. } 1$$

$$\begin{aligned} \text{where } x(1) &= Ax(0) + B\kappa_{\text{MPC}}(x(0)) + w \\ \tilde{\mathbf{c}}(1) &= \{c^*(1), \ldots, c^*(N-1), 0\} \\ \tilde{z}(1) &= A^0 z^*(0) + B^0 v^*(0) \end{aligned}$$

Hence $V_0(x(1), \tilde{\mathbf{c}}(1), \tilde{z}(1)) \leq V_0^*(x(0)) - \left(\|z^*(0)\|_Q^2 + \|v^*(0)\|_R^2\right)$, and

$$\boxed{V_0^*(x(1)) \leq V_0^*(x(0)) - \left(\|z^*(0)\|_Q^2 + \|v^*(0)\|_R^2\right)}$$

- ⋆ sum over r time steps:

$$\sum_{k=0}^{r-1} \left(\|z^*(k)\|_Q^2 + \|v^*(k)\|_R^2\right) \leq V_0^*(x(0))$$

- ⋆ asymptotically:

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Nominal MPC cost: convergence analysis

- If $K_{\mathbb{T}}$ is mean-square stabilizing, i.e. $\exists P_{\mathbb{T}} \succ 0$:

$$P_{\mathbb{T}} - \mathbb{E}\big[(A + BK_{\mathbb{T}})^T P_{\mathbb{T}}(A + BK_{\mathbb{T}})\big] = I$$

then the closed loop dynamics under $u = \kappa_{\text{MPC}}(x)$:

$$x^+ = (A + BK_{\mathbb{T}})x + w', \quad w' = B(v^* - K_{\mathbb{T}} z^*) + w$$

has finite l^2-gain $(w' \to x)$, γ, i.e. $\exists \gamma > 0$:

$$\sum_{k=0}^{r-1} \mathbb{E}\big(\|x(k)\|^2\big) \le \|x(0)\|^2_{P_{\mathbb{T}}} + \gamma \sum_{k=0}^{r-1} \mathbb{E}\big(\|Bv^*(k) - K_{\mathbb{T}} z^*(k) + w(k)\|^2\big)$$

- Hence

$$\boxed{\frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\big(\|x(k)\|^2\big) \le \gamma\, \mathbb{E}\big(\|w(0)\|^2\big) + \frac{1}{r}\|x(0)\|^2_{P_{\mathbb{T}}} + \frac{\gamma'}{r} V_0^*(x(0))}$$

Nominal MPC cost: convergence analysis

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Expectation vs Nominal MPC cost

Both expectation and nominal cost ensure: robust asymptotic stability of $\mathcal{M}_\infty^{\mathrm{uc}}$
and finite l^2-gain $(w \to x)$

However

- ▷ nominal can be less representative of predicted performance than expectation cost
  $\uparrow$
  since $e^*(0) = x(0) - z^*(0)$ is not necessarily zero

- ▷ nominal allows different (linear) feedback gains in nominal and disturbed dynamics
  $\uparrow$
  extra flexibility enables e.g. $K_{\mathbb{T}}$ chosen for large $\mathbb{T} = \mathcal{O}_\infty$
  K_v chosen for good l^2 performance: $K_v = K_{\mathrm{LQ}}$

Problem formulation: summary

Prototype stochastic MPC optimization:

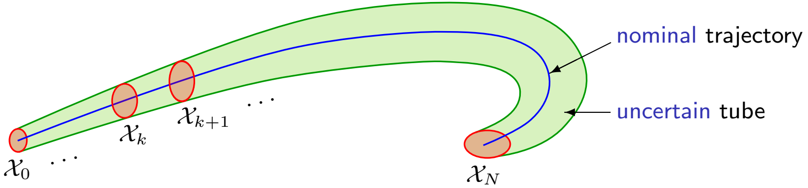
$$\begin{aligned}
\min_{\mathbf{c}} \ \min_{\{u(k),u(k+1),\dots\}} \quad & J\big(x(k),\{u(k),u(k+1),\dots\}\big) \\
\text{subject to} \quad & \left.\begin{array}{l} F(x(k+i),u(k+i),q(k+i)) \le 1 \text{ w.p. } p \\ x(k+i)\in\mathcal{T}_0 \text{ w.p. } 1 \end{array}\right\} \quad i=0,\dots,N-1 \\
& x(k+N)\in\mathbb{T} \text{ w.p. } 1
\end{aligned}$$

- expectation or nominal cost
- constraint at prediction time $k+i$ invoked with:
  - worst-case uncertainty at prediction times $k, k+1, \dots, k+i-1$
  - stochastic uncertainty at prediction time $k+i$

Propagating uncertain predictions

Tube MPC

Split predicted trajectories (input/state) into nominal + uncertain components



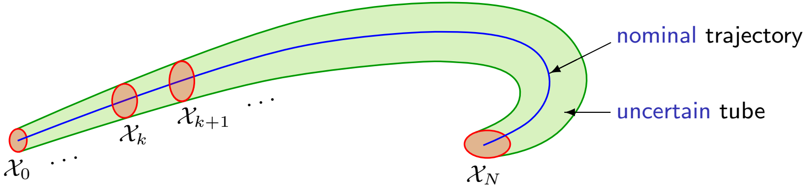
Applications to:

- ⋆ Linear model plus additive uncertainty or uncertain state estimation
  Langson ('04), Mayne ('05)
- ⋆ Nonlinear model plus uncertainty   Blanchini ('90), Lee ('02), Raković ('06)
- ⋆ Linear and nonlinear stochastic systems with additive or multiplicative uncertainty
  Cannon ('09), Cannon ('10)

Propagating uncertain predictions

Tube MPC

Split predicted trajectories (input/state) into nominal + uncertain components



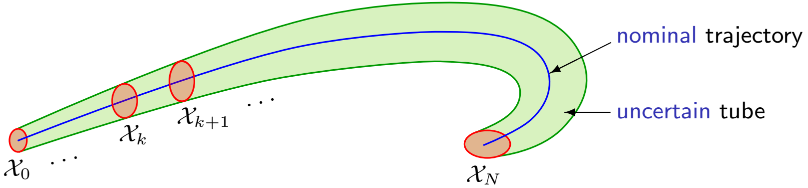
Computational advantages:

- ⋆ Offline computation assuming a fixed disturbance feedback law
  - ▷ linear systems — Gossner ('97), Langson ('04), Mayne ('05)
- ⋆ Outer approximation using one-step-ahead predictions
  - ▷ nonlinear or stochastic systems — Blanchini ('90), Lee ('02), Cannon ('09)

Propagating uncertain predictions

Tube MPC

Split predicted trajectories (input/state) into nominal + uncertain components



Different approaches to computing probability distributions

Model uncertainty	Exact tubes	Approximate tubes
additive	numerical integration	numerical integration
additive and multiplicative	sampling/M-C simulation	parametric bounds

Affine uncertainty: Approximate tubes

Affine uncertainty: Parameter bounds

- Affine in the disturbance dynamics:

$$x_{k+1} = A(q_k)x_k + B(q_k)u_k + d(q_k)$$

$$(A, b, d) = (A^0, B^0, 0) + \sum_{i=1}^{m} (A^{(j)}, B^{(j)}, d^{(j)}) q_{j,k}$$

$$q_k \sim \mathfrak{D} \text{ i.i.d.}, \ \operatorname{supp}(\mathfrak{D}) = \mathcal{Q}$$

- Confidence region: if $q_k \in \hat{\mathcal{Q}}(p)$ w.p. p, where $\hat{\mathcal{Q}}(p) = \mathrm{Co}\{q^{(i)},\ i = 1, \ldots, l\}$, then

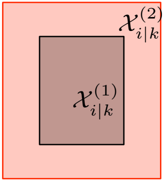
$$\boxed{\begin{aligned} x_{k+1} &\in \mathrm{Co}\{A(q^{(i)})x_k + B(q^{(i)})u_k + d(q^{(i)}),\ i = 1, \ldots, l\} \text{ w.p. } p \\ &= \mathrm{Co}\{\hat{A}^{(i)}x_k + \hat{B}^{(i)}u_k + \hat{d}^{(i)},\ i = 1, \ldots, l\} \end{aligned}}$$

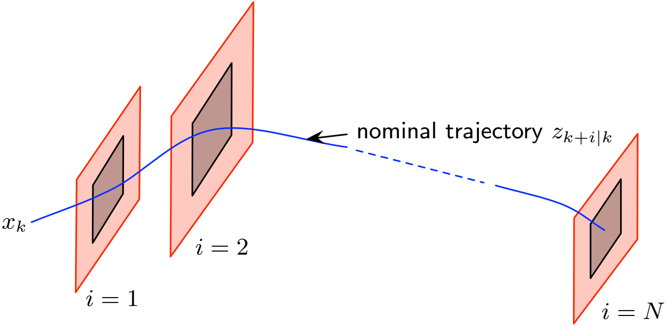
Affine uncertainty: Probabilistic tubes

Tube cross-section at prediction time-step i:

$$\{\mathcal{X}_{i|k}^{(1)}, \dots, \mathcal{X}_{i|k}^{(r)}\}, \text{ e.g. } \mathcal{X}_{i|k}^{(j)} = z_{k+i|k} + \{e : |e| \le \bar{e}_{i|k}^{(j)}\}$$

with $\mathcal{X}_{i|k}^{(1)} \subseteq \cdots \subseteq \mathcal{X}_{i|k}^{(r)}$



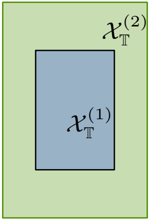


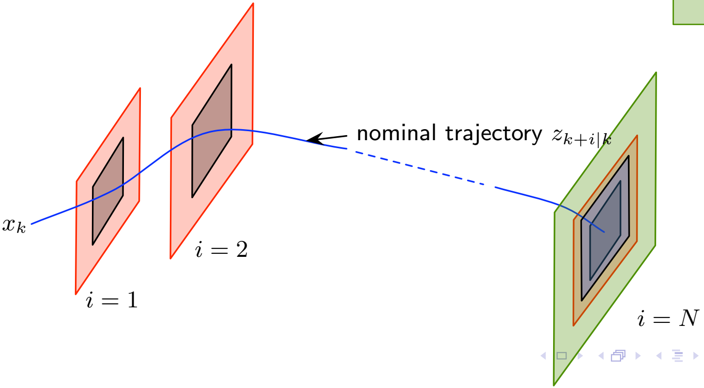
Affine uncertainty: Probabilistic tubes

Terminal sets:

$$\{\mathcal{X}_{\mathbb{T}}^{(1)}, \dots, \mathcal{X}_{\mathbb{T}}^{(r)}\}, \quad \mathcal{X}_{\mathbb{T}}^{(j)} = \{x : |x| \le \bar{x}_{\mathbb{T}}^{(j)}\}$$

with $\mathcal{X}_{\mathbb{T}}^{(1)} \subseteq \cdots \subseteq \mathcal{X}_{\mathbb{T}}^{(r)}$

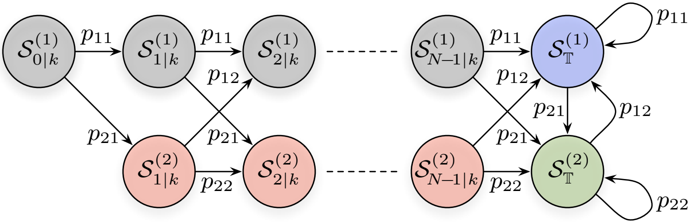




Affine uncertainty: Probabilistic tubes

Let $\mathcal{S}_\star^{(j)} = \begin{cases} \mathcal{X}_\star^{(1)} & j = 1 \\ \mathcal{X}_\star^{(j)} - \mathcal{X}_\star^{(j-1)} & j = 2, \ldots, r \end{cases}$

(i). Define transition probabilities p_{jm}, $j, m = 1, \ldots r$:



then $p_i^{(j)} = \Pr(e_{k+i|k} \in \mathcal{S}_{i|k}^{(j)})$ is given by

$$\begin{bmatrix} p_i^{(1)} \\ p_i^{(2)} \\ \vdots \\ p_i^{(r)} \end{bmatrix} = \Pi^i \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & \ddots & \vdots \\ p_{r1} & \cdots & p_{rr} \end{bmatrix}$$

Affine uncertainty: Probabilistic tubes

Let $\mathcal{S}_\star^{(j)} = \begin{cases} \mathcal{X}_\star^{(1)} & j = 1 \\ \mathcal{X}_\star^{(j)} - \mathcal{X}_\star^{(j-1)} & j = 2, \ldots, r \end{cases}$

(ii). Define p_j as the probability of satisfying soft constraints in $\mathcal{S}_\star^{(j)}$:

$$\Pr\big(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h \mid e_{k+i|k} \in \mathcal{S}_{i|k}^{(j)}\big) < p_j \quad i \leq N-1$$

$$\Pr\big(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h \mid x_{k+i|k} \in \mathcal{S}_{\mathbb{T}}^{(j)}\big) < p_j \quad i \geq N$$

(i) and (ii) imply

$$\boxed{\Pr(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h) < \begin{bmatrix} p_1 & p_2 & \cdots & p_r \end{bmatrix} \Pi^i e_1}$$

for all i

Affine uncertainty: Probabilistic tubes

The constraint

$$\Pr(f^T x_{k+i|k} + g^T u_{k+i|k} > h) < p$$

is satisfied for all i if Π and $p_1, \ldots, p_r$ satisfy

$$\boxed{\begin{bmatrix} p_1 & p_2 & \cdots & p_r \end{bmatrix} \Pi^i e_1 < p, \quad \forall i}$$

- ⋆ Fix Π and $p_1, \ldots, p_r$ offline and optimize $\{\mathcal{X}^{(j)}_{i|k}\}$ online subject to constraints on: (i). transition probabilities
  (ii). probabilities of satisfying soft constraints
- ⋆ Hard constraints are satisfied if feasible for $x \in \mathcal{X}^{(r)}_{i|k}$ and $x \in \mathcal{X}^{(r)}_{\mathbb{T}}$

Affine uncertainty: Probabilistic tubes

Aside: the constraint

$$\frac{1}{N_s}\sum_{i=n+1}^{n+N_s} \Pr(f^T x_{k+i|k} + g^T u_{k+i|k} > h) < p$$

is satisfied for all n if Π and $p_1, \ldots, p_r$ satisfy

$$\boxed{\frac{1}{N_s}\sum_{i=0}^{N_s-1} \begin{bmatrix} p_1 & p_2 & \cdots & p_r \end{bmatrix} \Pi^i e_1 < p}$$

- ⋆ Fix Π and $p_1, \ldots, p_r$ offline and optimize $\{\mathcal{X}_{i|k}^{(j)}\}$ online subject to constraints on: (i). transition probabilities
  (ii). probabilities of satisfying soft constraints
- ⋆ Hard constraints are satisfied if feasible for $x \in \mathcal{X}_{i|k}^{(r)}$ and $x \in \mathcal{X}_{\mathbb{T}}^{(r)}$

Affine uncertainty: Tube constraints

For tractable implementation:

1. apply constraints to $\{\mathcal{X}_{i|k}^{(j)}\}$ instead of $\{\mathcal{S}_{i|k}^{(j)}\}$
2. invoke transition probability constraints via inequality constraints

$$\Downarrow$$

Define $\tilde{p}_{jm} = \sum_{l=1}^{j} p_{lm} \quad j, m = 1, \ldots, r,$ and invoke constraints:

(a). on transition probabilities via

$$\Pr\big(x_{k+i+1|k} \in \mathcal{X}_{i+1|k}^{(j)} \mid x_{k+i|k} \in \mathcal{X}_{i|k}^{(m)}\big) \geq \tilde{p}_{jm} \quad i \leq N-1$$
$$\Pr\big(x_{k+i+1|k} \in \mathcal{X}_{\mathbb{T}}^{(j)} \mid x_{k+i|k} \in \mathcal{X}_{\mathbb{T}}^{(m)}\big) \geq \tilde{p}_{jm} \quad i \geq N$$

(b). on the probability of constraint satisfaction within $\mathcal{X}_{\star}^{(j)}$ via

$$\Pr\big(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h \mid x_{k+i|k} \in \mathcal{X}_{i|k}^{(j)}\big) < p_j \quad i \leq N-1$$
$$\Pr\big(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h \mid x_{k+i|k} \in \mathcal{X}_{\mathbb{T}}^{(j)}\big) < p_j \quad i \geq N$$

Affine uncertainty: Tube constraints

For tractable implementation:

1. apply constraints to $\{\mathcal{X}^{(j)}_{i|k}\}$ instead of $\{\mathcal{S}^{(j)}_{i|k}\}$
2. invoke transition probability constraints via inequality constraints

$$\downarrow$$

Define $\tilde{p}_{jm} = \sum_{l=1}^{j} p_{lm} \quad j, m = 1, \ldots, r,$ and invoke constraints:

(a). on transition probabilities via

$$\Pr\big(x_{k+i+1|k} \in \mathcal{X}^{(j)}_{i+1|k} \mid x_{k+i|k} \in \mathcal{X}^{(m)}_{i|k}\big) \geq \tilde{p}_{jm} \quad i \leq N-1$$
$$\Pr\big(x_{k+i+1|k} \in \mathcal{X}^{(j)}_{\mathbb{T}} \mid x_{k+i|k} \in \mathcal{X}^{(m)}_{\mathbb{T}}\big) \geq \tilde{p}_{jm} \quad i \geq N$$

(b). on the probability of constraint satisfaction within $\mathcal{X}^{(j)}_{\star}$ via

$$\Pr\big(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h \mid x_{k+i|k} \in \mathcal{X}^{(j)}_{i|k}\big) < p_j \quad i \leq N-1$$
$$\Pr\big(f^T x_{k+i+1|k} + g^T u_{k+i+1|k} > h \mid x_{k+i|k} \in \mathcal{X}^{(j)}_{\mathbb{T}}\big) < p_j \quad i \geq N$$

Affine uncertainty: Tube constraints

The constraint

$$\Pr(f^T x_{k+i|k} + g^T u_{k+i|k} > h) < p$$

is satisfied under (a) & (b) for all i if Π and $p_1, \dots, p_r$ satisfy

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \end{bmatrix} \Pi^i e_1 > p, \quad \forall i$$

and, additionally, if p_j and $\tilde{p}_{jm}$ satisfy

$$\boxed{\left.\begin{matrix} p_j < p_{j+1} \\ \tilde{p}_{jm} \geq \tilde{p}_{j\,m+1} \end{matrix}\right\} \text{ for } j = 1, \dots, r-1}$$

- $\star$ $p_j < p_{j+1}$ $\iff$ probability of satisfying soft constraint increases towards centre of tube
- $\star$ $\tilde{p}_{jm} \geq \tilde{p}_{j\,m+1}$ $\longleftarrow$ always holds (since $\mathcal{X}_\star^{(1)} \subseteq \cdots \subseteq \mathcal{X}_\star^{(r)}$)

Affine uncertainty: Tube constraints

Invoke the constraints on: (a). transition probabilities
(b). probabilities of satisfying soft constraints

using polytopic confidence regions for disturbance q, e.g.:

$$x_{k+i+1|k} \in \mathrm{Co}\{\hat{A}^{(j)} x_k + \hat{B}^{(j)} u_k + \hat{d}^{(i)},\ j = 1, \ldots, l\} \text{ w.p. } \tilde{p}_{jm}$$

implies

$$\Pr\big(x_{k+i+1|k} \in \mathcal{X}^{(j)}_{i+1|k} \mid x_{k+i|k} \in \mathcal{X}^{(m)}_{i|k}\big) > \tilde{p}_{jm}$$

whenever

$$\mathrm{Co}\{(\hat{A}^{(j)} + \hat{B}^{(j)} K_{\mathbb{T}}) x^{(m,r)}_{i|k} + \hat{B}^{(j)} c_{k+i|k} + \hat{d}^{(j)},\ j = 1, \ldots, l\} \subseteq \mathcal{X}^{(m)}_{i+1|k},\ \ r = 1, \ldots, s$$

where $\mathcal{X}^{(m)}_{i|k} = \mathrm{Co}\{x^{(m,r)}_{i|k},\ r = 1, \ldots, s\}$

$$\Downarrow$$

finite set of linear constraints in the variables $\mathbf{c}_k, \{x^{(m,j)}_{i|k}\}$
online optimization via QP

Affine uncertainty: Approximate tubes

Example

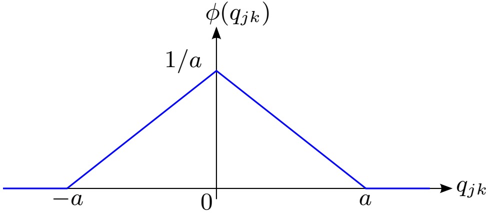
- Model parameters:

$$A^{(0)} = \begin{bmatrix} 1.2 & 0.1 \\ 0.1 & 1.26 \end{bmatrix} \quad A^{(1)} = \frac{10^{-2}}{a} \begin{bmatrix} -1 & -0.5 \\ -1 & 0.2 \end{bmatrix}, \quad A^{(2)} = \frac{10^{-2}}{a} \begin{bmatrix} -0.6 & 0.7 \\ -0.3 & 0.3 \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0.5 \\ 0.21 \end{bmatrix}, \quad B^{(1)} = \frac{10^{-3}}{a} \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad B^{(2)} = \frac{10^{-3}}{a} \begin{bmatrix} 2 \\ -9 \end{bmatrix}$$

$$d^{(1)} = \frac{1}{a} \begin{bmatrix} 0.1 \\ 0.01 \end{bmatrix}, \quad d^{(2)} = \frac{1}{a} \begin{bmatrix} 0.5 \\ 0.12 \end{bmatrix} \quad a = \sqrt{6}.$$

- Uncertainty distribution:



- Transition and constraint violation probabilities for $r = 2$:

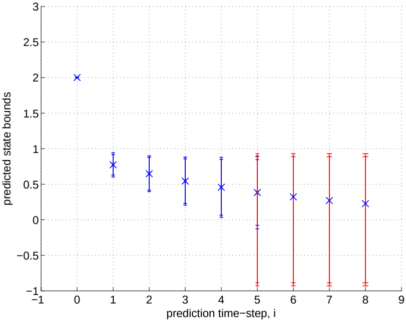
$$\Pi = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 1 & 1 \end{bmatrix}$$

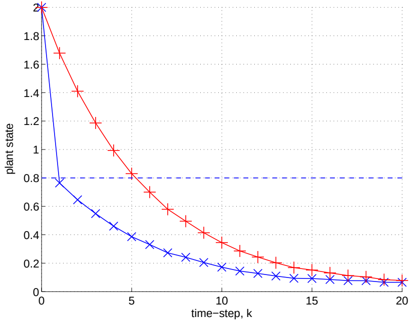
$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \quad p = 0.4$$

Affine uncertainty: Approximate tubes

Constraint: $\Pr\{x_1 > 0.8\} < 0.4$

Horizon: $N = 5$





— : $\{\mathrm{Proj}_{x_1}(\mathcal{X}^{(j)}_{i|0}),\ i = 1,\dots,5,\ j = 1,2\}$

— : $\{\mathrm{Proj}_{x_1}(\mathcal{X}^{(j)}_{\mathbb{T}}),\ \ j = 1,2\}$

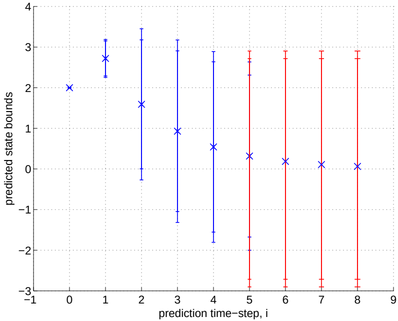
— : $\mathbb{E}(x_1(k))$, constrained MPC

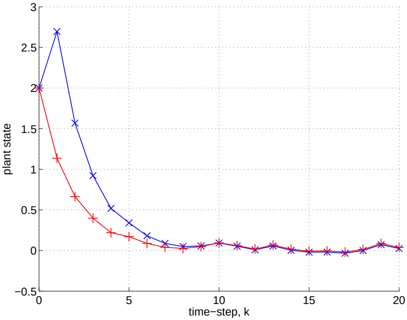
— : $\mathbb{E}(x_1(k))$, unconstrained optimal

Affine uncertainty: Approximate tubes

Constraint: $\Pr\{x_1 > 0.8\} < 0.4$

Horizon: $N = 5$





— : $\{\mathrm{Proj}_{x_2}(\mathcal{X}^{(j)}_{i|0}),\ i = 1, \dots, 5,\ j = 1, 2\}$

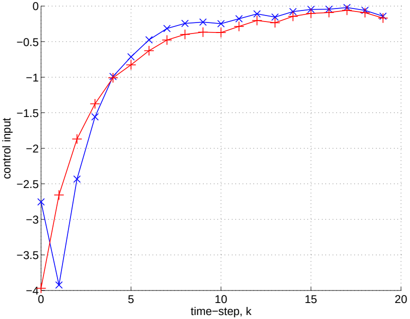
— : $\{\mathrm{Proj}_{x_2}(\mathcal{X}^{(j)}_{\mathbb{T}}),\ j = 1, 2\}$

— : $\mathbb{E}(x_2(k))$, constrained MPC

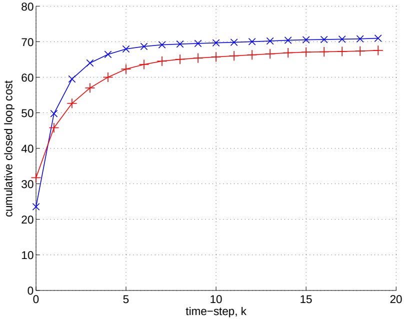
— : $\mathbb{E}(x_2(k))$, unconstrained optimal

Affine uncertainty: Approximate tubes

$\mathbb{E}(u(k))$:



$\mathbb{E}(\|x(k)\|_Q^2 + \|u(k)\|_R^2)$:



— : constrained MPC

— : unconstrained optimal

Target constraint violation rate: 0.4, actual: 0.2 (conservative!)

Affine uncertainty: Exact tubes

Affine uncertainty: Exact tubes

Invoke probabilistic constraints by sampling uncertainty distributions directly

[Tempo, Calafiore, Campi]

- e.g. constraints: $x(k) \in \mathcal{T}_0$ w.p. 1 & $F(x(k), u(k), \mathfrak{D}) \leq 1$ w.p. p invoked via the constraints:

$$\forall x \in \mathcal{X}_k : \quad \frac{1}{N_\sigma} \sum_{j=1}^{N_\sigma} \sigma_j(x) \geq p, \quad \sigma_j(x) = \begin{cases} 1, & F(x, u(k), q_j) \leq 1 \\ 0, & F(x, u(k), q_j) > 1 \end{cases}$$

  where $\{x(0), \mathcal{X}_1, \mathcal{X}_2, \ldots\}$ = robust uncertain tube given $q \in \mathcal{Q} = \operatorname{supp}(\mathfrak{D})$ w.p. 1

- Bounds exist on N_σ to ensure satisfaction of $\Pr\big\{F(x, u, \mathfrak{D}) \leq 1\big\} \geq p$ w.p. $1 - \epsilon$

  [Campi]

- Avoids conservative parametric confidence bounds and computationally expensive numerical convolutions

Affine uncertainty: Exact tubes

Assume:

(i) $\mathcal{Q}$: convex, bounded, poytopic

(ii) $F(x,u,q)$: convex in (x,u) [Prekopa, 1995]

then: $\mathcal{X}_k = \mathrm{Co}\{x^{(k,l)},\ l = 1,\ldots,r\}$,

and $x(k) \in \mathcal{T}_0$ & $\Pr\big\{F(x,u,\mathfrak{D}) \leq 1\big\} \geq p$ invoked via

$$\boxed{\forall l = 1,\ldots,r: \quad \frac{1}{N_\sigma}\sum_{j=1}^{N_\sigma} \sigma_j(x^{(k,l)}) \geq p, \quad \sigma_j(x^{(k,l)}) = \begin{cases} 1, & F(x^{(k,l)}, u^{(k,l)}, q_j) \leq 1 \\ 0, & F(x^{(k,l)}, u^{(k,l)}, q_j) > 1 \end{cases}}$$

↑

Finite number of mixed integer linear constraints

[e.g. Blackmore, 2006]

Application to terminal set calculation

Given r.p.i. Ω set such that

$$F(x, K_{\mathbb{T}}x, \mathfrak{D}) \leq 1 \text{ w.p. } p \;\; \forall x \in \Omega$$

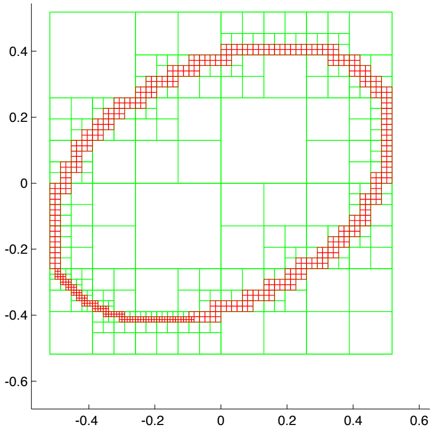
(i) cover $\partial\Omega$ with boxes Π_i

(ii) If:

$$\left.\begin{array}{l} f(x, K_{\mathbb{T}}x, \mathfrak{D}) \in \Omega \text{ w.p. } 1 \\ F(x, K_{\mathbb{T}}x, \mathfrak{D}) \leq 1 \text{ w.p. } p \end{array}\right\} \forall x \in \Pi_i$$

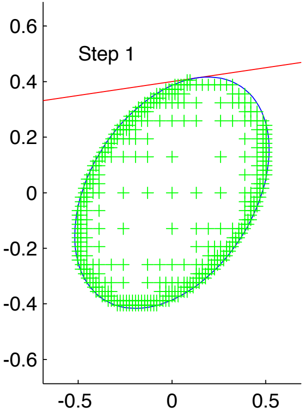
then $\Pi_i \in \partial\hat{\Omega}$

(iii) set $\Omega = \mathrm{Co}\{\partial\hat{\Omega}\}$ and return to (i).



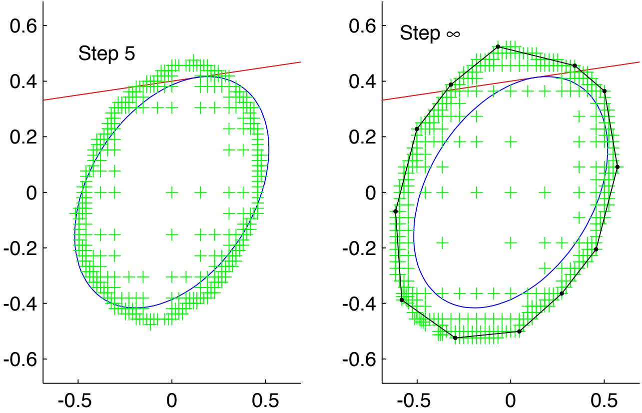
Application to terminal set calculation

Example



Application to terminal set calculation

Example



Additive uncertainty: Exact tubes

Additive uncertainty: problem formulation

Linear uncertain system:

plant model	$x_{k+1} = Ax_k + Bu_k + w_k$	$x_k \in \mathbb{R}^n$
disturbance	$w_k = Dq_k, \quad q_k \in \mathcal{Q}$	$\mathcal{Q} \subset \mathbb{R}^m$

$\{q_0, q_1, q_2, \ldots\}$ assumed iid, with

- $q_{k,i}$: zero-mean, independent, with known distributions:

$$\Pr\{q_{k,i} \le \xi_i\} = \mathcal{F}_i(\xi_i), \quad i = 1, \ldots, m$$

- $\mathcal{F}_i$: right-continuous, with finitely many discontinuities and

$$\mathcal{F}_i(\xi) = \begin{cases} 1 & \text{for } \xi \ge \alpha_i \\ 0 & \text{for } \xi < -\alpha_i \end{cases}$$

$$\Downarrow$$

$\mathcal{Q}$ compactly supported, $\quad \mathcal{Q} = \{q : |q| \le \alpha\}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$

Additive uncertainty: problem formulation

- Linear probabilistic constraints:

$$\Pr\{f_x^T x_k + g^T u_k \le h\} \ge p$$

- Quasi-closed loop input predictions: $u_{k+i|k} = K_{\mathbb{T}} x_{k+i|k} + c_{k+i|k}$,

$$c_{k+i|k} = 0, \quad i \ge N$$

State decomposition: $x_{k+i|k} = z_{k+i|k} + e_{k+i|k}$ $\begin{cases} z_{k+i|k}: & \text{nominal} \\ e_{k+i|k}: & \text{uncertain} \end{cases}$

$$\boxed{\begin{aligned} z_{k+i+1|k} &= \Phi z_{k+i|k} + Bc_{k+i|k} & z_{k|k} &= x_k \\ e_{k+i+1|k} &= \Phi e_{k+i|k} + Dq_{k+i|k} & e_{k|k} &= 0 \end{aligned}}$$

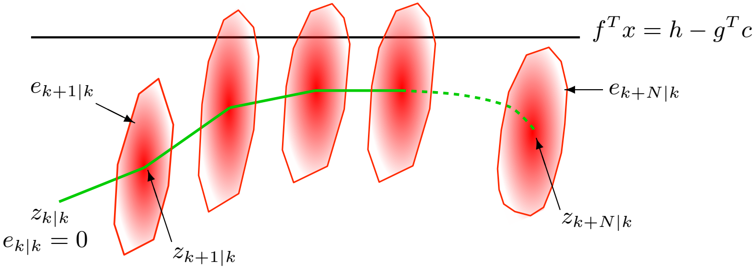
$$\Phi = A + BK_{\mathbb{T}}$$

- Probabilistic constraints on predictions:

$$\Pr\{f^T x_{k+i|k} + g^T c_{k+i|k} \le h\} \ge p$$

Additive uncertainty: tubes

Future trajectories $\{x_{k+i|k},\ i = 0, 1, \ldots\}$ belong to a tube centred on $\{z_{k+i|k},\ i = 0, 1, \ldots\}$:



Compute the distributions of the projections $f^T e_{k+i|k}$ directly via a sequence of 1-dimensional convolutions

Additive uncertainty: tubes

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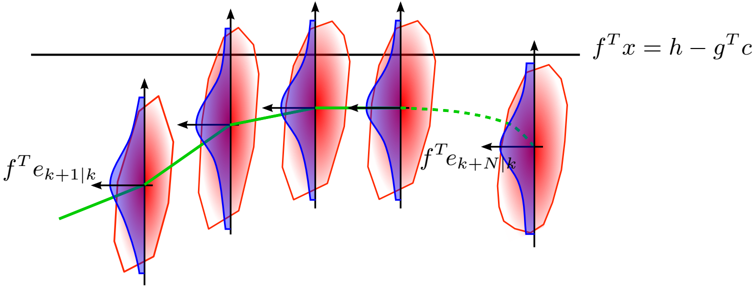

Compute the distributions of the projections $f^Te_{k+i|k}$ directly
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Additive uncertainty: tubes

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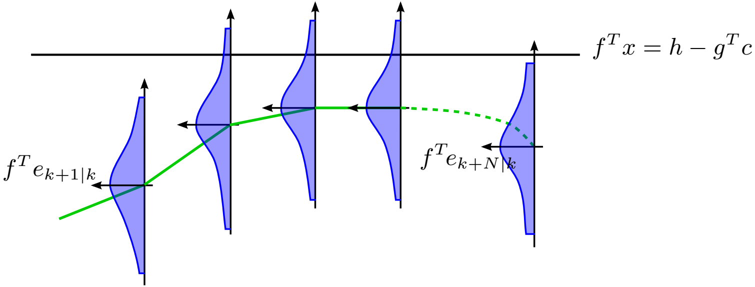

Compute the distributions of the projections $f^T e_{k+i|k}$ directly
via a sequence of 1-dimensional convolutions

Additive uncertainty: probabilistic constraints

Define γ_i as the minimum value such that

$$\Pr\{f^T e_{k+i|k} \leq \gamma_i\} = p$$

then $\Pr\{f^T x_{k+i|k} + g^T c_{k+i|k} \leq h\} \geq p$ iff

$$\boxed{f^T z_{k+i|k} + g^T c_{k+i|k} \leq h - \gamma_i}$$

- Tightened linear constraints on nominal input/state predictions
- Given the distribution of $\{w_0, w_1, w_2, \ldots\}$,

  compute γ_i for $i = 1, 2, \ldots$ using

$$\begin{aligned} f^T e_{k+i|k} &= f^T \Phi^{i-1} D q_{k|k} + \cdots + f^T D q_{k+i-1|k} \\ &= \text{sum of independent, scalar r.v.'s} \end{aligned}$$

- γ_i can be computed offline

Additive uncertainty: numerical convolution

For scalar r.v.'s X, Y with densities

$$\begin{aligned} f_X(x): &\quad \Pr\{X \le x\} = \int^x f_X \,\mathrm{d}x \\ f_Y(y): &\quad \Pr\{Y \le y\} = \int^y f_Y \,\mathrm{d}y \end{aligned}$$

the pdf of $X + Y$ is the convolution

$$f_{X+Y} = f_X * f_Y$$

f_{X+Y} can be approximated to required accuracy via discrete convolution

Additive uncertainty: numerical convolution

For scalar r.v.'s X, Y with densities

$$f_X(x): \quad \Pr\{X \le x\} = \int^x f_X \,\mathrm{d}x$$
$$f_Y(y): \quad \Pr\{Y \le y\} = \int^y f_Y \,\mathrm{d}y$$

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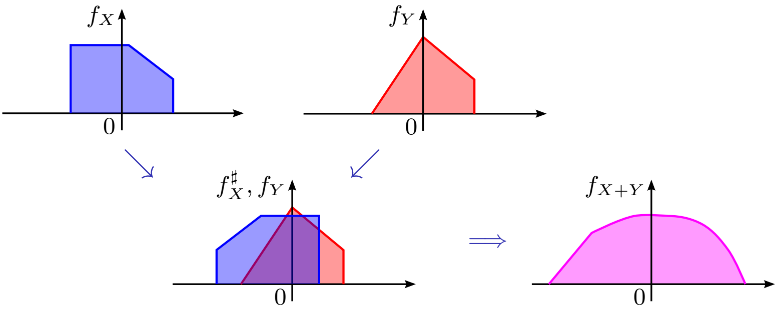

f_{X+Y} can be approximated to required accuracy via discrete convolution

Additive uncertainty: numerical convolution

For scalar r.v.'s X, Y with densities
$$f_X(x): \quad \Pr\{X \le x\} = \int^x f_X \,\mathrm{d}x$$
$$f_Y(y): \quad \Pr\{Y \le y\} = \int^y f_Y \,\mathrm{d}y$$

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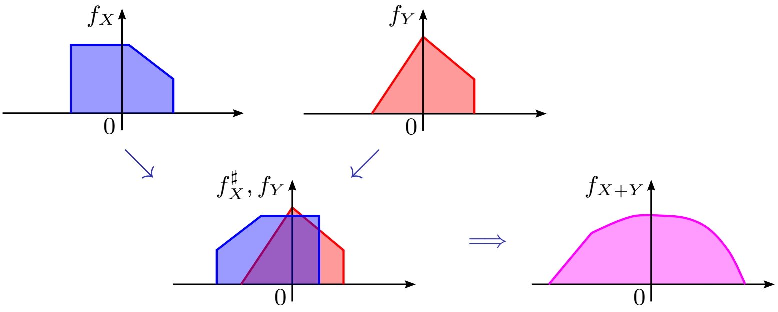

f_{X+Y} can be approximated to required accuracy via discrete convolution

Additive uncertainty: numerical convolution

- Discretization of distribution functions $F_{f^T\Phi^i Dq}(\cdot)$ on r intervals implies:
  - ⋆ approximation error: $O(1/r^2)$ (e.g. trapezoidal integration)
  - ⋆ computation: $O(r^2)$ multiplications/additions per convolution

- $\mathrm{supp}(dF_{f^T\Phi^i Dq})$ increases monotonically with i, but is finite for all i.

N.B. γ_i is bounded $\forall i$ (since Φ is strictly stable)

$$\Uparrow$$

e.g. Chebychev's one-sided inequality gives $\gamma_i \leq \alpha\sqrt{f^T\Sigma_i f}$

where

$$\begin{aligned}
\alpha^2 &= p/(1-p) \\
\Sigma_1 &= D\mathbb{E}(qq^T)D^T \\
\Sigma_{i+1} &= \Phi\Sigma_i\Phi^T + D\mathbb{E}(qq^T)D^T, \ i = 1, 2, \ldots
\end{aligned}$$

Additive uncertainty: recursive feasibility

Consider the i-step-ahead prediction at time k:

$$f^T e_{k+i|k} = f^T \Phi^{i-1} D q_k + f^T \Phi^{i-2} D q_{k+1} + \cdots + f^T D q_{k+i-1}$$

Additive uncertainty: recursive feasibility

Consider the i-step-ahead prediction at time k:

$$f^T e_{k+i|k} = f^T \Phi^{i-1} D q_k + f^T \Phi^{i-2} D q_{k+1} + \cdots + f^T D q_{k+i-1}$$

$\uparrow$

at time $k+1$, this term has already been realized

$\Downarrow$

Best bound on $f^T e_{k+i|k+1}$ given information on q_k available at time k:

$$f^T e_{k+i|k+1} \le a_{i-1} + f^T \Phi^{i-2} D q_{k+1} + \cdots + f^T D q_{k+i-1}$$

where

$$a_{i-1} = \max_{q \in \mathcal{Q}} f^T \Phi^{i-1} D q$$

Additive uncertainty: recursive feasibility

Consider the i-step-ahead prediction at time k:

$$f^T e_{k+i|k} = \underset{\substack{\uparrow \\ \text{worst case bound}}}{f^T \Phi^{i-1} D q_k} + \underset{\substack{\uparrow \\ \text{probabilistic bound}}}{f^T \Phi^{i-2} D q_{k+1} + \cdots + f^T D q_{k+i-1}}$$

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$$\Downarrow$$

Best bound on $f^T e_{k+i|k+1}$ with probability p given information available at k:

$$\boxed{f^T e_{k+i|k+1} \leq a_{i-1} + \gamma_{i-1} \quad \text{w.p. } p}$$

where

$$a_{i-1} = \max_{q \in \mathcal{Q}} \; f^T \Phi^{i-1} D q$$

and γ_{i-1} is the minimum value such that

$$\Pr\{f^T \Phi^{i-2} D q_{k+1} + \cdots + f^T D q_{k+i-1} \leq \gamma_{i-1}\} = p$$

Additive uncertainty: recursive feasibility

- Predictions at time k must ensure feasibility at $k+1$, $k+2$, ...
- Hence tighten constraints on nominal i-step-ahead prediction by β_i,

  where β_i = maximum element of ith column of:

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots \\ 0 & \gamma_1 + a_1 & \gamma_2 + a_2 & \gamma_3 + a_3 & \cdots \\ 0 & 0 & \gamma_1 + a_1 + a_2 & \gamma_2 + a_2 + a_3 & \cdots \\ 0 & 0 & 0 & \gamma_1 + a_1 + a_2 + a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Satisfaction of probabilistic constraints and recursive feasibility is ensured if

$$\boxed{f^T z_{k+i|k} + g^T c_{k+i|k} \le h - \beta_i}$$

for $i = 0, 1, 2, \ldots$, at each time k

Additive uncertainty: recursive feasibility

Properties of β_i

(i) $\beta_i = \gamma_1 + \sum_{j=1}^{i-1} a_j$ for $i = 1, 2, \ldots$

$$\Uparrow$$

Largest element of each column lies on the diagonal:

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots \\ 0 & \gamma_1 + a_1 & \gamma_2 + a_2 & \gamma_3 + a_3 & \cdots \\ 0 & 0 & \gamma_1 + a_1 + a_2 & \gamma_2 + a_2 + a_3 & \cdots \\ 0 & 0 & 0 & \gamma_1 + a_1 + a_2 + a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

- ▷ Follows from $\gamma_{i+1} \le \gamma_i + a_i$
- ▷ Intuitively: future feasibility depends on worst case bounds on disturbances that have already been realized

$$\Downarrow$$

$\{\beta_1, \beta_2, \beta_3, \ldots\}$ is monotonically increasing

Additive uncertainty: recursive feasibility

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$\Downarrow$

$\{\beta_1, \beta_2, \beta_3, \ldots\}$ is monotonically increasing

Additive uncertainty: recursive feasibility

Properties of β_i

(ii) $\lim_{i\to\infty} \beta_i \leq \bar{\beta}_\nu$, where $\bar{\beta}_\nu$ is defined for any integer $\nu \geq 1$ by

$$\bar{\beta}_\nu = \gamma_1 + \sum_{j=1}^{\nu-1} a_j + \frac{\rho^\nu}{1-\rho}\|g\|_S$$

where $\|g\|_S = \sqrt{g^T S g}$ and ρ, S satisfy

$$\max_{q\in\mathcal{Q}} \|Dq\|_{S^{-1}} \leq 1$$

$$\Phi S \Phi^T \leq \rho^2 S, \quad \rho \in (0,1)$$

Additive uncertainty: recursive feasibility

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Follows from

- ⋆ $a_j = \max_{q\in\mathcal{Q}} g^T \Phi^j Dq \leq \max_{\|v\|_{S^{-1}}\leq 1} g^T \Phi^j v \leq \|\Phi^{j^T} g\|_S$
- ⋆ $\max_{q\in\mathcal{Q}} g^T \Phi^j Dq \leq \rho \|\Phi^{j-1^T} g\|_S$

Additive uncertainty: recursive feasibility

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$$\Downarrow$$

$\lim_{i\to\infty} \beta_i$ lies in the interval (since $a_i \geq 0 \ \forall i$):

$$\bar{\beta}_\nu - \frac{\rho^\nu}{1-\rho}\|g\|_S \leq \lim_{i\to\infty} \beta_i \leq \bar{\beta}_\nu$$

Additive uncertainty: recursive feasibility

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$$\Downarrow$$

$\lim_{i\to\infty} \beta_i$ may be determined to any desired accuracy $\epsilon > 0$

if ν is chosen to be sufficiently large that $\dfrac{\rho^\nu}{1-\rho}\|g\|_S < \epsilon$

Additive uncertainty: SMPC algorithm

Constraints on predicted $\{z_{k+i|k}, c_{k+i|k}\}$ in MPC optimization at time k:

$$f^T z_{k+i|k} + g^T c_{k+i|k} \le h - \beta_i, \quad i = 0, \ldots, N-1$$
$$z_{k+N|k} \in \mathcal{S}_\nu$$

where the terminal constraint set $\mathcal{S}_\nu$ is defined by

$$\begin{aligned} \mathcal{S}_\nu = \{z : f^T \Phi^{i-N} z &\le h - \beta_i, & i &= N, \ldots, \nu - 1 \\ f^T \Phi^{i-N} z &\le h - \bar{\beta}_\nu, & i &= \nu, \nu + 1, \ldots\} \end{aligned}$$

$$\uparrow$$

These constraints are sufficient to ensure recursive feasibility since

$$\mathcal{S}_\nu \supseteq \mathcal{S}_\infty$$

where $\mathcal{S}_\infty$ is the maximal admissible set:

$$\mathcal{S}_\infty = \{z : \; f^T \Phi^{i-N} z \le h - \beta_i, \quad i = N, N+1, \ldots\}$$

Additive uncertainty: SMPC algorithm

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$$z_{k+N|k} \in \mathcal{S}_\nu$$

Properties of $\mathcal{S}_\nu$:

(i) $\mathcal{S}_\nu$ is compact and non-empty iff $h \ge \bar{\beta}_\nu$ (assuming (Φ, g) observable)

Additive uncertainty: SMPC algorithm

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$$z_{k+N|k} \in \mathcal{S}_\nu$$

Properties of $\mathcal{S}_\nu$:

(ii) $\mathcal{S}_\nu$ is finitely determined:

$$\mathcal{S}_\nu = \{z : f^T \Phi^{i-N} z \leq h - \beta_i, \quad i = N, \ldots, \nu - 1$$
$$f^T \Phi^{i-N} z \leq h - \bar{\beta}_\nu, \quad i = \nu, \nu+1, \ldots, \nu + N^*\}$$

- ⋆ N^* can be computed using e.g. [Gilbert & Tan, 1991]
- ⋆ $\mathcal{S}_\nu$ is assumed to be compact

Additive uncertainty: SMPC algorithm

Algorithm:

Offline Determine parameters $\gamma_1,\ \beta_1, \ldots, \beta_{\nu-1},\ \bar{\beta}_\nu,\ N^*$
defining the recursively feasible probabilistic constraints

Online At each time $k = 0, 1, \ldots$:

1. Obtain the current state x_k
2. Solve the QP:

$$\boxed{\begin{aligned} \mathbf{c}_k^* = \arg\min_{\mathbf{c}}\ & V(x_k, \mathbf{c}) \\ \text{subject to } & f^T z_{k+i|k} + g^T c_{k+i|k} \le h - \beta_i, \quad i = 0, \ldots, N-1 \\ & z_{k+N|k} \in \mathcal{S}_\nu \end{aligned}}$$

3. Set $u_k = K_{\mathbb{T}} x_{k+i|k} + c^*_{k|k}$.

Additive uncertainty: State estimation example

- Output feedback MPC formulation:

$$\text{plant model} \quad x_{k+1} = Ax_k + Bu_k + Dw_k$$

$$\text{output} \quad y_k = Cx_k + Fv_k$$

disturbance w_k and measurement noise v_k iid & compactly supported

- Linear observer:

$$\text{estimate update} \quad \hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k)$$

$$\text{with} \quad \hat{y}_k = C\hat{x}_k$$

state estimate: $\hat{x}$, observer gain: L such that $(A - LC)$ is strictly stable

- Estimation error $\epsilon_k := x_k - \hat{x}_k$ has dynamics:

$$\epsilon_{k+1} = (A - LC)\epsilon_k + Dw_k + Fv_k$$

ϵ_0: r.v. with known & compactly supported distribution and $\epsilon_0 \in \Pi_0$

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Additive uncertainty: State estimation example

Model parameters

$$A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\ C = \begin{bmatrix} 0.9 & 0.2 \end{bmatrix},\ D = I,\ F = 1$$

- Noise & uncertainty derived from truncated normal distrubtions:

$$\begin{aligned} v_k &\sim \{\ \mathcal{N}(0, 1/24^2) \quad \text{truncated so that} \quad |v_k| \le 0.12\ \} \\ w_k &\sim \{\ \mathcal{N}(0, I/24^2) \quad \text{truncated so that} \quad \|w_k\|_\infty \le 0.12\ \} \\ \epsilon_0 &\sim \{\ \mathcal{N}(0, I/24^2) \quad \text{truncated so that} \quad \|\epsilon_0\|_\infty \le 0.12\ \} \end{aligned}$$

- Probabilistic state constraints:

$$\eta_x^T = \begin{bmatrix} \pm 1 & \pm 0.3 \end{bmatrix},\quad \eta_u = 0,\quad h = 1.5,\quad p = 0.8.$$

- $$\left.\begin{array}{ll} K_{\mathbb{T}} = -[1.03\ \ 1.07]: & \text{unconstrained LQ-optimal} \\ L = -[0.83\ \ 1.22]: & \text{steady Kalman filter gain} \end{array}\right\} \implies |\lambda_{\max}(\Psi)| = 0.12$$

Additive uncertainty: State estimation example

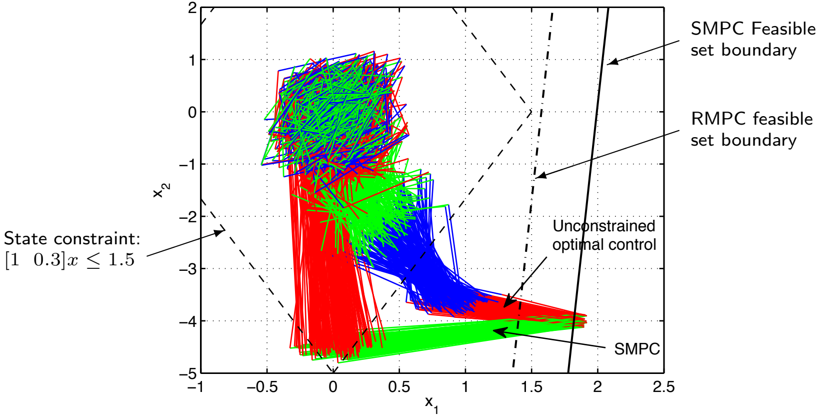
Prediction parameters

- Dof in predictions: $N = 6$
- Horizon for transients: $\nu = 13$
- Probabilistic bounds $\gamma_{0|k}$ computed for $k = 0, \ldots, \nu - 1$
  using rectangular integration with grid spacing 10^{-4}
- Terminal sets: non-empty for $h \geq \bar{\beta} = 1.31, 1.20$
  finitely determined, with $N_k^* = 10$ for all k

Simulation parameters

- 10^4 realizations of disturbance and noise sequences
- $x_0 = \hat{x}_0 + \epsilon_0$, for 10^4 realizations of ϵ_0
  and fixed $\hat{x}_0 = (1.8, -4.0)$

Additive uncertainty: Example

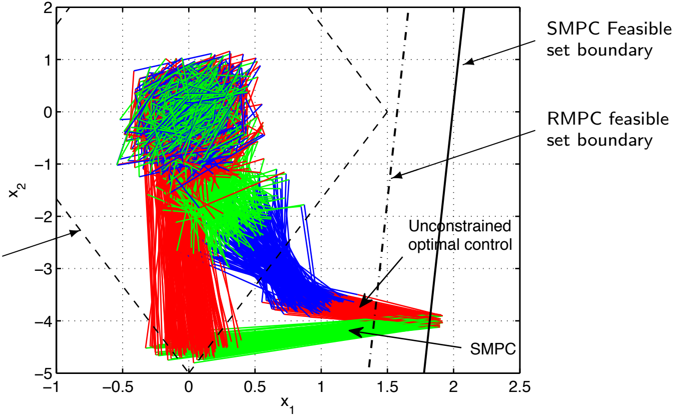


$\hat{x}_0 = (1.8, -4.0)$:

Constraint violation frequency:

	target	observed (at $k = 1$)
SMPC	20%	20.0%
LQ-optimal	–	100%

Additive uncertainty: Example



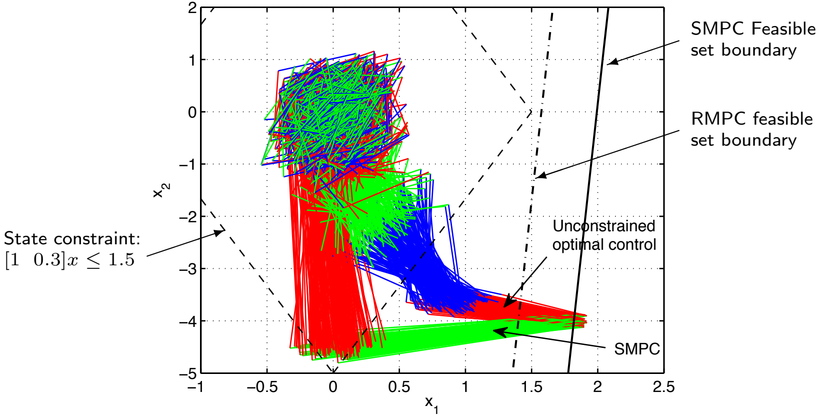
$\hat{x}_0 = (1.8, -4.0)$:

Cumulative cost $\displaystyle\sum_{k=0}^{12} \mathbb{E}(\|x_k\|_Q^2 + \|u_k\|_R^2)$:

SMPC	3.56
LQ-optimal	2.38

(RMPC infeasible)

Additive uncertainty: Example



$\hat{x}_0 = (1.35, -4.0)$:

Cumulative cost $\sum_{k=0}^{12} \mathbb{E}(\|x_k\|_Q^2 + \|u_k\|_R^2)$:

SMPC	1.72
RMPC	2.42

Conclusions

1. Example: fatigue control
2. Stochastic MPC: basic formulations
   - Probabilistic constraints & recursive feasibility
   - Performance costs and stability analyses
3. Implementation
   - Affine model uncertainty: approximate and exact tubes
   - Additive model uncertainty: exact tubes