Successive linearization NMPC for a class of stochastic nonlinear systems

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Abstract : A receding horizon control methodology is proposed for systems with nonlinear dynamics, additive stochastic uncertainty, and both hard and soft (probabilistic) input/state constraints. Jacobian linearization about predicted trajectories is used to derive a sequence of convex optimization problems. Constraints are handled through the construction of a sequence of tubes and an associated Markov chain model. The parameters defining the tubes are optimized simultaneously with the predicted future control trajectory via online linear programming.

1 Introduction

Constraints handled by predictive control strategies are typically treated as hard (unbreakable) constraints, or as soft constraints, in which case the degree of violation is to be minimized in some sense. This paper considers probabilistic constraints in the form of soft input/state constraints, for which the probability of violation is subject to hard limits. This form of constraint can account for the distribution of model or measurement uncertainty, and thus avoid the conservativeness of a hard-constraint strategy based on the worst-case uncertainty, which may be highly unlikely. The approach also avoids the suboptimality of soft constraint strategies based on adding penalty functions to the MPC cost.

The difficulties of predicting the distributions of model states over a horizon and of ensuring recursive feasibility in closed-loop operation have limited MPC based on probabilistic constraints to highly computationally intensive Monte Carlo methods (e.g. [1]) or to limited problem classes (e.g. linear dynamics [6]). This paper considers nonlinear systems with stochastic disturbances, and proposes a receding horizon control law subject to probabilistic and hard constraints based on tubes [4, 3]. Analysis of a simplified Markov chain model verifies that the probability of constraint violation is within the limits specified by the soft constraints. Linearizations about predicted trajectories allow for an efficient online optimization which may be terminated after a single iteration. The approach is illustrated by a numerical example.

1.1 Problem statement

The system to be controlled is described by a discrete-time nonlinear model with state $x_k \in \mathbb{R}^{n_x}$ and input $u_k \in \mathbb{R}^{n_u}$:

$$x_{k+1} = f(x_k, u_k) + d_k, \quad k = 0, 1, \ldots \tag{1}$$

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and with $f(0,0) = 0$. Here d_k is a random disturbance with a finitely supported, stationary distribution satisfying

$$\mathbb{E}(d_k) = 0, \quad \forall k$$

(where $\mathbb{E}(\cdot)$ denotes expectation). Furthermore d_j, d_k are assumed to be independent for all $j \neq k$. We assume that x_k is available for measurement at time k. The dynamics of (1) are assumed to be continuous throughout the operating region for the state (denoted $\mathcal{X}$) and input (denoted $\mathcal{U}$) in the following sense.

Assumption 1. *$f(x,u)$ is Lipschitz continuous for all $(x,u) \in \mathcal{X} \times \mathcal{U}$.*

The system is subject to two types of constraint on state and input variables. Hard constraints of the form

$$F_H x_k + G_H u_k \le h_H, \quad h_H \in \mathbb{R}^{n_H} \tag{2}$$

must be satisfied at all times $k = 0, 1, \ldots$. Thus, for example, we require the set of feasible (x,u) for (2) to be a subset of the operating region, i.e.

$$\{(x,u) : F_H x + G_H u \le h\} \subset \mathcal{X} \times \mathcal{U}.$$

In addition, we consider soft input/state constraints:

$$F_S x_k + G_S u_k \le h_S, \quad h_S \in \mathbb{R}^{n_S} \tag{3}$$

which may be violated at any given time k, but which are subject to hard bounds on the expected number of constraint violations over a given horizon. To simplify presentation (but with no loss of generality), we consider the case of a single soft constraint ($n_S = 1$). The bound on the expected number of constraint violations can therefore be expressed as a hard constraint:

$$\frac{1}{N_c} \sum_{i=1}^{N_c} \Pr\{F_S x_{k+i} + G_S u_{k+i} > h_S\} \le \frac{N_{\max}}{N_c} \tag{4}$$

which must hold for all $k = 0, 1, \ldots$. Here $\Pr\{A\}$ denotes the probability of event A, and $N_{\max}/N_c$ is the maximum allowable rate of violation of soft constraints averaged over an interval of N_c samples.

The control objective is the optimal regulation of x_k about the origin with respect to the performance index

$$J(\{u_0, u_1, \ldots\}, x_0) = \sum_{k=0}^{\infty} \mathbb{E}_0\big(\mathbf{1}^T |x_k| + \lambda \mathbf{1}^T |u_k|\big) \tag{5}$$

subject to constraints (2) and (4). Here $\mathbb{E}_k(\cdot)$ denotes expectation conditional on information available to the controller at time k, namely the measured state x_k; $\mathbf{1}$ is a vector, $\mathbf{1} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$, with dimension dependent on the context and $\lambda > 0$ is a control weighting. The 1-norm cost defined in (5) is employed for computational convenience but the paper's approach is easily extended to more general stage costs that are convex in (x_k, u_k).

2 Successive linearization MPC

This section describes in outline a method of solving the receding horizon formulation of the control problem defined in Section 1.1. Let $\{u_{k|k}, u_{k+1|k}, \ldots\}$ denote a predicted input sequence at time k and denote $\{x_{k|k}, x_{k+1|k}, \ldots\}$ as the corresponding state trajectory, with $x_{k|k} = x_k$. Following the dual mode prediction paradigm [5], we define the infinite horizon predicted input sequence in terms of a finite number of free variables, $\mathbf{c}_k = \{c_{0|k}, \ldots, c_{N-1|k}\}$, as a perturbed feedback law

$$u_{k+i|k} = Kx_{k+i|k} + c_{i|k} \tag{6}$$

with

$$c_{i|k} = 0, \quad i = N, N+1, \ldots$$

and where the linear feedback law $u = Kx$ is assumed to stabilize the model (1) in a neighbourhood of $x = 0$. Under the control law of (6), state predictions are governed by the model

$$x_{k+i+1|k} = \phi(x_{k+i|k}, c_{i|k}) + d_{k+i}, \qquad x_{k|k} = x_k \tag{7}$$

where $\phi : \mathbb{R}^{n_x \times n_u} \to \mathbb{R}^{n_x}$ is defined by the identity

$$\phi(x, c) = f(x, Kx + c), \quad \forall x \in \mathbb{R}^{n_x},\ c \in \mathbb{R}^{n_u}.$$

In order to account efficiently for the nonlinearity and uncertainty in the prediction system (7), the proposed receding horizon optimization is based on linear models obtained from the Jacobian linearization of (7) around nominal trajectories for the predicted state. Let $\{x^0_{k|k}, \ldots, x^0_{k+N|k}\}$ denote a trajectory for the nominal system associated with the expected value of d_k in (7) and $\mathbf{c}^0_k = \{c^0_{0|k}, \ldots, c^0_{N-1|k}\}$, so that $x^0_{k+i|k}$ evolves according to

$$x^0_{k+i+1|k} = \phi(x^0_{k+i|k}, c^0_k), \qquad x^0_{k|k} = x_k. \tag{8}$$

The combined effects of approximation errors and unknown disturbances can be taken into account through the definition of a sequence of sets centred on the nominal trajectory of (8) at prediction times $i = 1, \ldots, N$ and a terminal set centred at the origin for $i > N$. For computational convenience we define these sets as low complexity polytopes of the form $\{x : |V(x - x^0_{k+i|k})| \leq \bar{z}_{i|k}\}$ for $i = 1, \ldots, N$, and $\{x : |Vx| \leq \bar{z}_t\}$ for the terminal set. Here V is a square full-rank matrix and the parameters $\bar{z}_{i|k}, \bar{z}_t$ determine the relative scaling of the sets. Possible choices for V and K are discussed in Section 3.2.

To simplify presentation, we define a transformed variable $z = Vx$, and express the condition that $x_{k+i|k}$ should belong to the relevant set as

$$z^\delta_{k+i|k} \in \mathcal{Z}_{i|k}, \qquad \mathcal{Z}_{i|k} = \{z^\delta : |z^\delta| \leq \bar{z}_{i|k}\}$$

where

$$z^0_{k+i|k} + z^\delta_{k+i|k} = Vx_{k+i|k}, \qquad z^0_{k+i|k} = Vx^0_{k+i|k}.$$

Then for $c_{i|k} = c^0_{i|k} + c^\delta_{i|k}$, $z_{k+i|k} = z^0_{k+i|k} + z^\delta_{k+i|k}$ evolves according to

$$z^0_{k+i+1|k} + z^\delta_{k+i+1|k} = V\phi\big(V^{-1}(z^0_{k+i|k} + z^\delta_{k+i|k})\,,\ c^0_{i|k} + c^\delta_{i|k}\big) + \varepsilon_{k+i} \tag{9}$$

where $\varepsilon_k = Vd_k$. The linearization of (7) about $\{x^0_{k|k}, \dots, x^0_{k+N|k}\}$ and $\mathbf{c}^0_k$ can therefore be expressed

$$z^\delta_{k+i+1|k} = \Phi_{k+i|k} z^\delta_{k+i|k} + B_{k+i|k} c^\delta_{i|k} + \varepsilon_{k+i} + e_{k+i|k}, \qquad z^\delta_{k|k} = 0 \tag{10}$$

where

$$\Phi_{k+i|k} = V \frac{\partial \phi}{\partial x}\bigg|_{(x^0_{k+i|k}, c^0_{i|k})} V^{-1} \qquad B_{k+i|k} = V \frac{\partial \phi}{\partial c}\bigg|_{(x^0_{k+i|k}, c^0_{i|k})}$$

Similarly, for $i \geq N$ we have $z_{k+i|k} = V x_{k+i|k}$ where

$$z_{k+i+1|k} = V\phi\big(V^{-1} z_{k+i|k}\,,\ 0\big) + \varepsilon_{k+i} \tag{11}$$

and the Jacobian linearization about $z = 0$ therefore gives

$$z_{k+i+1|k} = \Phi z_{k+i|k} + \varepsilon_{k+i} + e_{k+i|k}\ , \qquad \Phi = V \frac{\partial \phi}{\partial x}\bigg|_{(0,0)} V^{-1}. \tag{12}$$

Remark 1. *From Assumption 1 it follows that the linearization error in (10):*

$$\begin{aligned} e_{k+i|k} = V\phi\big(V^{-1}(z^0_{k+i|k} + z^\delta_{k+i|k})\,,\ c^0_{i|k} + c^\delta_{i|k}\big) \\ - V\phi\big(V^{-1} z^0_{k+i|k}, c^0_{i|k}\big) - \Phi_{k+i|k} z^\delta_{k+i|k} - B_{k+i|k} c^\delta_{i|k} \end{aligned}$$

necessarily satisfies the Lipschitz condition

$$|e_{k+i|k}| \leq \Gamma_z |z^\delta_{k+i|k}| + \Gamma_c |c^\delta_{i|k}| \tag{13}$$

for some positive matrices Γ_z, Γ_c, for all $(z^\delta_{k+i|k}, c^\delta_{i|k})$ such that

$$\big(V^{-1}(z^0_{k+i|k} + z^\delta_{k+i|k})\,,\ KV^{-1}(z^0_{k+i|k} + z^\delta_{k+i|k}) + c^0_{i|k} + c^\delta_{i|k}\big) \in \mathcal{X} \times \mathcal{U}.$$

Similarly, for $i \geq N$, the linearization error in (12):

$$e_{k+i|k} = V\phi\big(V^{-1}(z_{k+i|k})\,,\ 0\big) - \Phi z_{k+i|k}$$

is Lipschitz continuous, with

$$|e_{k+i|k}| \leq \Gamma_t |z_{k+i|k}| \tag{14}$$

for some positive matrix Γ_t, for all $z_{k+i|k}$ such that

$$\big(V^{-1} z_{k+i|k}\,,\ KV^{-1} z_{k+i|k}\big) \in \mathcal{X} \times \mathcal{U}.$$

In Section 3 the bounds (13) and (14) are combined with bounds on ε_k to construct sets $Z_{i|k}, i = 0, \dots, N$ that depend on $\mathbf{c}^\delta_k = \{c^\delta_{0|k}, \dots, c^\delta_{N-1|k}\}$, thus defining tubes centred on the nominal trajectory containing the predictions of (7). These tubes provide a means of bounding the receding horizon performance cost and of ensuring satisfaction of constraints. As a result, the process of successively linearizing about $(\{x^0_{k+i|k}\}, \mathbf{c}^0_k)$, optimizing $\mathbf{c}^\delta_k$, and then redefining $(\{x^0_{k+i|k}\}, \mathbf{c}^0_k)$ by setting $\mathbf{c}^0_k \leftarrow \mathbf{c}^0_k + \mathbf{c}^\delta_k$ necessarily converges to a (local) optimum for the original nonlinear dynamics, as discussed in Section 4.

3 Probabilistic tubes

This section describes a method of constructing a series of tubes around a nominal predicted trajectory so that each tube contains the future predicted state with a prescribed probability. This process provides a means of bounding the predicted value of the cost (5) and of ensuring satisfaction of hard constraints (2) and probabilistic constraints (4) along future predicted trajectories. The probabilities of transition between tubes from one sampling instant to the next and the probability of constraint violation within each tube are governed by fixed probabilities that are determined offline. However the parameters determining the size of each tube are retained as optimization variables, and this allows the effects of stochastic model uncertainty and linearization errors (which depend on the predicted input trajectory) to be estimated non-conservatively over the prediction horizon.

Let $\{\mathcal{S}_t^{(1)}, \ldots, \mathcal{S}_t^{(r)}\}$ and $\{\mathcal{S}_{i|k}^{(1)}, \ldots, \mathcal{S}_{i|k}^{(r)}\}$ for $i = 1, \ldots, N$ denote collections of sets in $\mathbb{R}^{n_x}$ with

$$\mathcal{S}_t^{(j)} \cap \mathcal{S}_t^{(m)} = \emptyset, \quad \mathcal{S}_{i|k}^{(j)} \cap \mathcal{S}_{i|k}^{(m)} = \emptyset \quad \forall j \neq m, \tag{15}$$

and let $\mathcal{S}_{0|k}^{(j)} = 0$, $j = 1, \ldots, r$. Denote p_{jm} for $j, m = 1, \ldots, r$ as transition probabilities, with

$$\sum_{j=1}^{r} p_{jm} = 1 \quad j = 1, \ldots, r, \tag{16}$$

and assume that the sequences $\{z_{k+i|k}^{\delta},\ i = 0, \ldots, N\}$ and $\{z_{k+i|k},\ i \geq N\}$ generated respectively by the prediction models of (10) and (12) satisfy

$$\Pr\big(z_{k+i+1|k}^{\delta} \in \mathcal{S}_{i+1|k}^{(j)} \mid z_{k+i|k}^{\delta} \in \mathcal{S}_{i|k}^{(m)}\big) = p_{jm} \quad i = 0, \ldots, N \tag{17a}$$

$$\Pr\big(z_{k+i+1|k} \in \mathcal{S}_t^{(j)} \mid z_{k+i|k} \in \mathcal{S}_t^{(m)}\big) = p_{jm} \quad i = N, N+1, \ldots \tag{17b}$$

(note that the requirement for these probabilities to hold with equality is relaxed in Section 3.1). Assume moreover that the sets $\mathcal{S}_{N|k}^{(j)}$ are linked to the terminal sets $\mathcal{S}_t^{(j)}$ through the conditions:

$$z_{k+N|k}^{\delta} \in \mathcal{S}_{N|k}^{(j)} \implies z_{k+N|k} = z_{k+N|k}^{0} + z_{k+N|k}^{\delta} \in \mathcal{S}_t^{(j)} \quad j = 1, \ldots, r. \tag{18}$$

Then the probabilities of $z_{k+i|k}^{\delta}$ lying in $\mathcal{S}_{i|k}^{(j)}$ for $i = 0, \ldots, N$ and of $z_{k+i|k}$ lying in $\mathcal{S}_t^{(j)}$ for $i = N, N+1, \ldots$ are governed by a Markov chain model with transition matrix Π:

$$\Pi = \begin{bmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & \ddots & \vdots \\ p_{r1} & \cdots & p_{rr} \end{bmatrix}$$

and the distribution of state predictions can be approximated for all $i = 0, 1, \ldots$ using the property that

$$\begin{bmatrix} p_i^{(1)} \\ p_i^{(2)} \\ \vdots \\ p_i^{(r)} \end{bmatrix} = \Pi^i e_1, \quad p_i^{(j)} = \begin{cases} \Pr\big(z_{k+i|k}^{\delta} \in \mathcal{S}_{i|k}^{(j)}\big) & i = 1, \ldots, N \\ \Pr\big(z_{k+i|k} \in \mathcal{S}_t^{(j)}\big) & i = N, N+1, \ldots \end{cases} \tag{19}$$

where $e_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$.

Define p_j as a bound on the one-step-ahead conditional probability of violating the soft constraint (3):

$$\Pr\big(F_S x_{k+i+1|k} + G_S u_{k+i+1|k} > h_S \mid z^{\delta}_{k+i|k} \in \mathcal{S}^{(j)}_{i|k}\big) \leq p_j \quad i = 0, \ldots, N-1 \tag{20a}$$

$$\Pr\big(F_S x_{k+i+1|k} + G_S u_{k+i+1|k} > h_S \mid z_{k+i|k} \in \mathcal{S}^{(j)}_{t}\big) \leq p_j \quad i = N, N+1, \ldots \tag{20b}$$

Then it follows from (15), (16) and (19) that the probability of violating (3) is bounded by

$$\Pr(F_S x_{k+i+1|k} + G_S u_{k+i+1|k} > h_S) \leq \begin{bmatrix} p_1 & p_2 & \cdots & p_r \end{bmatrix} \Pi^i e_1 \quad i = 0, 1, \ldots \tag{21}$$

Assume, in addition, that the hard constraints (2) are satisfied at all points in $\mathcal{S}^{(j)}_{i|k}$ and $\mathcal{S}^{(j)}_{t}$:

$$z^{\delta}_{k+i|k} \in \mathcal{S}^{(j)}_{i|k} \implies F_H x_{k+i|k} + G_H u_{k+i|k} \leq h_H \quad i = 0, \ldots, N-1 \tag{22a}$$

$$z_{k+i|k} \in \mathcal{S}^{(j)}_{t} \implies F_H x_{k+i|k} + G_H u_{k+i|k} \leq h_H \quad i = N, N+1, \ldots \tag{22b}$$

for $j = 1, \ldots, r$. Then sufficient conditions for satisfaction of both hard and probabilistic constraints are given by the following lemma.

Lemma 3.1. *The constraints of (2) and (4) are necessarily satisfied along predicted state and input trajectories of (6)-(7) if the conditions on: transition probabilities (17a,b), terminal sets (18), probabilities of soft constraint violation (20a,b), and hard constraints (22a,b), are satisfied for Π and p_j, $j = 1, \ldots, r$ such that:*

$$\frac{1}{N_c} \sum_{i=0}^{N_c - 1} \begin{bmatrix} p_1 & p_2 & \cdots & p_r \end{bmatrix} \Pi^i e_1 \leq \frac{N_{\max}}{N_c}. \tag{23}$$

Proof. This is a direct consequence of (21) and (22a,b). $\square$

Throughout the following development we assume that Π and p_j satisfy (23).

3.1 Tube constraints

We next construct constraints that ensure that the conditions of Lemma 3.1 are satisfied, and which are suitable for incorporation in an online receding horizon optimization. Consider the sequences of nested sets:

$$\mathcal{Z}^{(1)}_t \subseteq \mathcal{Z}^{(2)}_t \subseteq \cdots \subseteq \mathcal{Z}^{(r)}_t, \quad \mathcal{Z}^{(1)}_{i|k} \subseteq \mathcal{Z}^{(2)}_{i|k} \subseteq \cdots \subseteq \mathcal{Z}^{(r)}_{i|k} \tag{24}$$

defined for $i = 1, \ldots, N$ as low-complexity polytopes:

$$\mathcal{Z}^{(j)}_t = \{z : |z| \leq \bar{z}^{(j)}_t\} \qquad \mathcal{Z}^{(j)}_{i|k} = \{z : |z| \leq \bar{z}^{(j)}_{i|k}\}. \tag{25}$$

Define $\mathcal{S}^{(j)}_t$ and $\mathcal{S}^{(j)}_{i|k}$ in terms of $\mathcal{Z}^{(j)}_t$ and $\mathcal{Z}^{(j)}_{i|k}$ via the relations:

$$\mathcal{S}^{(j)}_t = \begin{cases} \mathcal{Z}^{(1)}_t & j = 1 \\ \mathcal{Z}^{(j)}_t - \mathcal{Z}^{(j-1)}_t & j = 2, \ldots, r \end{cases} \qquad \mathcal{S}^{(j)}_{i|k} = \begin{cases} \mathcal{Z}^{(1)}_{i|k} & j = 1 \\ \mathcal{Z}^{(j)}_{i|k} - \mathcal{Z}^{(j-1)}_{i|k} & j = 2, \ldots, r \end{cases}$$

for $i = 1, \ldots, N$. The transition probabilities in (17a,b) are assumed to hold with equality, which places a strong and unrealistic restriction on the distribution of the uncertain disturbance in (7). Here we remove this assumption by instead imposing constraints on transition probabilities for $\mathcal{Z}_t^{(j)}$ and $\mathcal{Z}_{i|k}^{(j)}$. These constraints have the additional advantage over constraints invoked directly on $\mathcal{S}_t^{(j)}$ and $\mathcal{S}_{i|k}^{(j)}$ that they are convex (in fact linear in the degrees of freedom). We show that, when combined with conditions on the violation of system constraints (2) and (3), this formulation provides sufficient conditions for the conditions of Lemma 3.1 for disturbances d_k with general (continuous, finitely supported) distributions.

Accordingly, let

$$\tilde{p}_{jm} = \sum_{l=1}^{j} p_{lm}, \quad j, m = 1, \ldots, r$$

(so that $\tilde{p}_{rm} = 1$, $m = 1, \ldots, r$) and define

$$\tilde{\Pi} = T\Pi, \quad \tilde{\Pi} = \begin{bmatrix} \tilde{p}_{11} & \cdots & \tilde{p}_{1r} \\ \vdots & \ddots & \vdots \\ \tilde{p}_{r1} & \cdots & \tilde{p}_{rr} \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

For $j = 1, \ldots, r-1$ and $m = 1, \ldots, r$, we impose the transition probabilities

$$\Pr\bigl(z^{\delta}_{k+i+1|k} \in \mathcal{Z}^{(j)}_{i+1|k} \mid z^{\delta}_{k+i|k} \in \mathcal{Z}^{(m)}_{i|k}\bigr) \geq \tilde{p}_{jm} \quad i = 0, \ldots, N \tag{26a}$$

$$\Pr\bigl(z_{k+i+1|k} \in \mathcal{Z}^{(j)}_{t} \mid z_{k+i|k} \in \mathcal{Z}^{(m)}_{t}\bigr) \geq \tilde{p}_{jm} \quad i = N, N+1, \ldots \tag{26b}$$

whereas for $m = 1, \ldots, r$ we require

$$\Pr\bigl(z^{\delta}_{k+i+1|k} \in \mathcal{Z}^{(r)}_{i+1|k} \mid z^{\delta}_{k+i|k} \in \mathcal{Z}^{(m)}_{i|k}\bigr) = 1 \quad i = 0, \ldots, N \tag{27a}$$

$$\Pr\bigl(z_{k+i+1|k} \in \mathcal{Z}^{(r)}_{t} \mid z_{k+i|k} \in \mathcal{Z}^{(m)}_{t}\bigr) = 1 \quad i = N, N+1, \ldots \tag{27b}$$

The required probabilities on soft constraints are invoked for $j = 1, \ldots, r$ by

$$\Pr\bigl(F_S x_{k+i+1|k} + G_S u_{k+i+1|k} > h_S \mid z^{\delta}_{k+i|k} \in \mathcal{Z}^{(j)}_{i|k}\bigr) \leq p_j, \ i = 0, \ldots, N-1 \tag{28a}$$

$$\Pr\bigl(F_S x_{k+i+1|k} + G_S u_{k+i+1|k} > h_S \mid z_{k+i|k} \in \mathcal{Z}^{(j)}_{t}\bigr) \leq p_j, \ i = N, N+1, \ldots \tag{28b}$$

while the hard constraints are invoked via

$$z^{\delta}_{k+i|k} \in \mathcal{Z}^{(r)}_{i|k} \implies F_H x_{k+i|k} + G_H u_{k+i|k} \leq h_H \quad i = 0, \ldots, N-1 \tag{29a}$$

$$z_{k+i|k} \in \mathcal{Z}^{(r)}_{t} \implies F_H x_{k+i|k} + G_H u_{k+i|k} \leq h_H \quad i = N, N+1, \ldots \tag{29b}$$

Lemma 3.2. *If p_j and $\tilde{p}_{jm}$ satisfy*

$$p_j \leq p_{j+1}, \qquad j = 1, \ldots, r-1 \tag{30a}$$

$$\tilde{p}_{jm} \geq \tilde{p}_{j\,m+1}, \quad j = 1, \ldots, r-1. \tag{30b}$$

then constraints (26), (27), (28) and (29), together with the terminal constraints that $z^0_{k+N|k} + z^{\delta}_{k+N|k} \in \mathcal{Z}^{(j)}_t$ for all $z^{\delta}_{k+N|k} \in \mathcal{Z}^{(j)}_{N|k}$, $j = 1, \ldots, r$, are sufficient to ensure that (2) and (4) hold along predicted trajectories of (6)-(7).

Proof. Satisfaction of the hard constraint (2) is trivially ensured by (29a,b) and (27a,b) due to the nested property (24). On the other hand, satisfaction of (21), and hence also the probabilistic constraint (4), can be shown using (30a,b). For $i = 0$ this is obvious from (28), whereas for $i = 1$ we have from (28):

$$\begin{aligned}
&\Pr(F_S x_{k+2|k} + G_S u_{k+2|k} > h_S)\\
&\qquad \leq \begin{bmatrix} p_1 & \cdots & p_r \end{bmatrix} \begin{bmatrix} \Pr\big(x_{k+1|k} \in \mathcal{S}^{(1)}_{1|k}\big) & \cdots & \Pr\big(x_{k+1|k} \in \mathcal{S}^{(r)}_{1|k}\big) \end{bmatrix}^T\\
&\qquad = \begin{bmatrix} p_1 - p_2 & \cdots & p_r \end{bmatrix} \begin{bmatrix} \Pr\big(x_{k+1|k} \in \mathcal{Z}^{(1)}_{1|k}\big) & \cdots & \Pr\big(x_{k+1|k} \in \mathcal{Z}^{(r)}_{1|k}\big) \end{bmatrix}^T\\
&\qquad \leq \begin{bmatrix} p_1 - p_2 & \cdots & p_r \end{bmatrix} \tilde{\Pi}\, e_1\\
&\qquad = \begin{bmatrix} p_1 & \cdots & p_r \end{bmatrix} \Pi\, e_1
\end{aligned}$$

where the last inequality follows from (30a) and (26a). Similarly, for $i = 2$:

$$\begin{aligned}
&\Pr(F_S x_{k+3|k} + G_S u_{k+3|k} > h_S)\\
&\quad \leq \begin{bmatrix} p_1 & \cdots & p_r \end{bmatrix} \begin{bmatrix} \Pr\big(x_{k+2|k} \in \mathcal{S}^{(1)}_{2|k}\big) & \cdots & \Pr\big(x_{k+2|k} \in \mathcal{S}^{(r)}_{2|k}\big) \end{bmatrix}^T\\
&\quad = \begin{bmatrix} p_1 - p_2 & \cdots & p_r \end{bmatrix} \begin{bmatrix} \Pr\big(x_{k+2|k} \in \mathcal{Z}^{(1)}_{2|k}\big) & \cdots & \Pr\big(x_{k+2|k} \in \mathcal{Z}^{(r)}_{2|k}\big) \end{bmatrix}^T\\
&\quad \leq \begin{bmatrix} p_1 - p_2 & \cdots & p_r \end{bmatrix} \tilde{\Pi} \begin{bmatrix} \Pr\big(x_{k+1|k} \in \mathcal{S}^{(1)}_{1|k}\big) & \cdots & \Pr\big(x_{k+1|k} \in \mathcal{S}^{(r)}_{1|k}\big) \end{bmatrix}^T\\
&\quad = \begin{bmatrix} p_1 - p_2 & \cdots & p_r \end{bmatrix} \tilde{\Pi} T^{-1} \begin{bmatrix} \Pr\big(x_{k+1|k} \in \mathcal{Z}^{(1)}_{1|k}\big) & \cdots & \Pr\big(x_{k+1|k} \in \mathcal{Z}^{(r)}_{1|k}\big) \end{bmatrix}^T\\
&\quad \leq \begin{bmatrix} p_1 - p_2 & \cdots & p_r \end{bmatrix} \tilde{\Pi} T^{-1} \tilde{\Pi}\, e_1\\
&\quad = \begin{bmatrix} p_1 & \cdots & p_r \end{bmatrix} \Pi^2\, e_1
\end{aligned}$$

where the last inequality follows from (30b) (which implies the matrix $\tilde{\Pi} T^{-1}$ has non-negative elements in the first $r-1$ rows and $[0 \;\; 0 \;\; \cdots \;\; 1]$ in the last row) and (26a). The same arguments show that (21) also holds for all $i > 2$. □

Remark 2. *The condition (30a) is equivalent to requiring that the probability of soft constraint violation should decrease towards the centre of the tube. Furthermore, due to the nested property (24), the convexity of $\mathcal{Z}^{(j)}_t$ and $\mathcal{Z}^{(j)}_{i|k}$, and the linearity of (10) and (12), condition (30b) can be assumed to hold without loss of generality.*

To invoke (26)-(29) we use confidence intervals for the elements of $\varepsilon = Vd$ in (10) inferred from the distribution for d:

$$\Pr(|\varepsilon| \leq \xi_j) = \tilde{p}_j, \quad \Pr(|\varepsilon| \leq \xi_{jm}) = \tilde{p}_{jm}, \quad j, m = 1, \ldots, r \tag{31a}$$

$$\Pr(|\varepsilon| \leq \bar{\xi}) = 1. \tag{31b}$$

From (10) and (13) we obtain the bounds

$$|z^{\delta}_{k+i+1|k}| \leq |\Phi_{k+i|k} z^{\delta}_{k+i|k} + B_{k+i|k} c^{\delta}_{i|k}| + \Gamma_z |z^{\delta}_{k+i|k}| + \Gamma_c |c^{\delta}_{i|k}| + |\varepsilon_{k+i}|$$

and, since $\mathcal{Z}^{(j)}_{i|k}$ has vertices $D_p \bar{z}^{(m)}_{i|k}$ (where D_p, $p = 1, \ldots, 2^{n_x}$ are appropriate diagonal matrices), (26a) is therefore implied by the condition

$$\bar{z}^{(j)}_{i+1|k} \geq |\Phi_{k+i|k} D_p \bar{z}^{(m)}_{i|k} + B_{k+i|k} c^{\delta}_{i|k}| + \Gamma_z \bar{z}^{(m)}_{i|k} + \Gamma_c |c^{\delta}_{i|k}| + \xi_{jm} \tag{32}$$

for $p = 1, \dots, 2^{n_x}$, while (27a) is implied by

$$\bar{z}^{(r)}_{i+1|k} \geq |\Phi_{k+i|k} D_p \bar{z}^{(m)}_{i|k} + B_{k+i|k} c^{\delta}_{i|k}| + \Gamma_z \bar{z}^{(m)}_{i|k} + \Gamma_c |c^{\delta}_{i|k}| + \bar{\xi} \tag{33}$$

for $p = 1, \dots, 2^{n_x}$. Similarly, from (10) and (13) it follows that sufficient conditions for (28a) are given by

$$(F_S + G_S K) V^{-1} (z^0_{k+i+1|k} + \Phi_{k+i|k} D_p \bar{z}^{(j)}_{i|k} + B_{k+i|k} c^{\delta}_{i|k}) + G_S (c^0_{i|k} + c^{\delta}_{i|k}) \\ + |(F_S + G_S K) V^{-1}| (\Gamma_z \bar{z}^{(j)}_{i|k} + \Gamma_c |c^{\delta}_{i|k}| + \xi_j) \leq h_S \tag{34}$$

for $p = 1, \dots, 2^{n_x}$, whereas (29a) is implied by

$$(F_H + G_H K) V^{-1} (z^0_{k+i|k} + D_p \bar{z}^{(r)}_{i|k}) + G_H (c^0_{i|k} + c^{\delta}_{i|k}) \leq h_H. \tag{35}$$

for $p = 1, \dots, 2^{n_x}$. Note that the conditions (32)-(35) are linear in $\bar{z}^{(j)}_{i|k}$ and $c_{i|k}$, which are retained as variables in the online optimization described in Section 4.

3.2 Terminal sets and terminal cost

In the interests of optimizing predicted performance, K in (6) should be optimal for the cost (5) when constraints are inactive. However the constraint (27b) also requires that $\mathcal{Z}^{(r)}_t$ is robustly invariant under (12), and this may conflict with the requirement for unconstrained optimality. We therefore specify K as optimal for the linearized model $(\partial f/\partial x|_{(0,0)}, \partial f/\partial u|_{(0,0)})$ with a suitable quadratic cost, and define V in (12) as the transformation matrix such that $\Phi = V \partial\phi/\partial x|_{(0,0)} V^{-1}$ is in modal form (see [4] for more details of this approach).

To maximize the region of attraction of the resulting receding horizon control law, it is desirable to maximize the terminal sets $\mathcal{Z}^{(j)}_t$. This suggests the following offline optimization problem:

$$
\begin{aligned}
(\bar{z}^{(1)}_t, \dots, \bar{z}^{(r)}_t) &= \arg \max_{(\bar{z}^{(1)}_t, \dots, \bar{z}^{(r)}_t)} \prod_{j=1}^{r} \mathrm{vol}(\mathcal{Z}^{(j)}_t) && \text{(36a)} \\
\text{s.t.} \quad & \bar{z}^{(r)}_t \geq \bar{z}^{(r-1)}_t \geq \cdots \geq \bar{z}^{(1)}_t > 0 && \text{(36b)} \\
& \bar{z}^{(j)}_t \geq (|\Phi| + \Gamma_t) \bar{z}^{(m)}_t + \xi_{jm}, \quad m = 1, \dots, r, \ j = 1, \dots, r-1 && \text{(36c)} \\
& \bar{z}^{(r)}_t \geq (|\Phi| + \Gamma_t) \bar{z}^{(m)}_t + \bar{\xi}, \quad m = 1, \dots, r && \text{(36d)} \\
& |(F_S + G_S K) V^{-1}| \big\{ (|\Phi| + \Gamma_t) \bar{z}^{(j)}_t + \xi_j \big\} \leq h_S, \quad j = 1, \dots, r && \text{(36e)} \\
& |(F_H + G_H K) V^{-1}| \bar{z}^{(r)}_t \leq h_H && \text{(36f)}
\end{aligned}
$$

where (36b) ensures (24), (36c) and (36d) are sufficient for (26b) and (27b) respectively, while (36e) and (36f) are sufficient for (28b) and (29b) respectively. The objective (36a) is chosen so that the optimization problem is convex, but could be modified by introducing weights in order to obtain a more favourable solution for $\mathcal{Z}^{(j)}_t$, $j = 1, \dots, r$.

To obtain a finite value for the infinite horizon predicted cost despite the presence of non-decaying disturbances, we subtract a bound on the steady-state value of the stage cost under (6), and hence redefine the performance index as

$$J(\mathbf{c}_k, x_k) = \sum_{i=0}^{\infty} \mathbb{E}_k\big(\mathbf{1}^T|V^{-1}z_{k+i|k}| + \lambda\mathbf{1}^T|KV^{-1}z_{k+i|k} + c_{i|k}| - l_{ss}\big) \tag{37a}$$

$$l_{ss} = \mathbf{1}^T\big(|V^{-1}| + \lambda|KV^{-1}|\big)\big(I - |\Phi| - \Gamma_t\big)^{-1}\bar{\xi} \tag{37b}$$

The following result enables the cost over the prediction interval $i = N, N+1, \ldots$ to be bounded in terms of a function of $z_{k+N|k}$.

Lemma 3.3. *If q satisfies*

$$q^T(|z| - |\Phi z| - \Gamma_t|z| - \bar{\xi}) \geq \mathbf{1}^T|V^{-1}z| + \lambda\mathbf{1}^T|KV^{-1}z| - l_{ss} \tag{38}$$

for all $z \in \mathcal{Z}_t^{(r)}$, then

$$q^T|z_{k+N|k}| \geq \sum_{i=N}^{\infty} \mathbb{E}_k\big(\mathbf{1}^T|V^{-1}z_{k+i|k}| + \lambda\mathbf{1}^T|KV^{-1}z_{k+i|k}| - l_{ss}\big). \tag{39}$$

Proof. From (12), (14) and (31b), the inequality (38) implies

$$q^T|z_{k+i|k}| - \mathbb{E}_{k+i}(q^T|z_{k+i+1|k}|) \geq \mathbf{1}^T|V^{-1}z| + \lambda\mathbf{1}^T|KV^{-1}z| - l_{ss}.$$

Taking expectations and summing over $i = N, N+1, \ldots$ yields (39). □

Using Lemma 3.3 we determine an optimal bound on the cost-to-go for the case that $z_{k+N|k} \in \mathcal{S}_t^{(j)}$ by solving the following LPs for $q^{(j)}$, $j = 1, \ldots, r$:

$$\begin{aligned} q^{(j)} = \arg\min_q \quad & q^T\bar{z}_t^{(j)} \\ \text{s.t.} \quad & q^T(\bar{z}_t^{(r)} - |\Phi D_p\bar{z}_t^{(r)}| - \Gamma_t\bar{z}_t^{(r)} - \bar{\xi}) \geq \\ & \quad \mathbf{1}^T|V^{-1}D_p\bar{z}_t^{(r)}| + \lambda\mathbf{1}^T|KV^{-1}D_p\bar{z}_t^{(r)}| - l_{ss}, \quad p = 1, \ldots, 2^{n_x} \end{aligned} \tag{40}$$

Given the distribution of predictions (19), this implies the following bound

$$\sum_{j=1}^{r} q^{(j)T}\bar{z}_{k+N|k}^{(j)}p_N^{(j)} \geq \sum_{i=N}^{\infty} \mathbb{E}_k\big(\mathbf{1}^T|V^{-1}z_{k+i|k}| + \lambda\mathbf{1}^T|KV^{-1}z_{k+i|k}| - l_{ss}\big).$$

4 Receding horizon control law

Let V be a bound on the cost J in (37a)

$$\begin{aligned} & V(\mathbf{c}_k^\delta, \{\bar{z}_{i|k}^{(j)}, i = 1, \ldots, N, j = 1, \ldots, r\}, \{x_{k+i|k}^0\}, \mathbf{c}_k^0) = \\ & \quad \sum_{j=1}^{r}\Bigg\{\sum_{i=0}^{N-1} \max_{z_{i|k}^\delta \in \mathcal{Z}_{i|k}^{(j)}} \big(\mathbf{1}^T|V^{-1}z_{k+i|k}| + \lambda\mathbf{1}^T|KV^{-1}z_{k+i|k} + c_{i|k}^0 + c_{i|k}^\delta| - l_{ss}\big)p_i^{(j)} \\ & \qquad\qquad + q^{(j)T}\bar{z}_{k+N|k}^{(j)}p_N^{(j)}\Bigg\} \end{aligned} \tag{41}$$

and consider the following receding horizon control strategy.

Algorithm 1. At times $k = 0, 1, \ldots$:
1. Given $\mathbf{c}_k^0$, determine $x_{k+i|k}^0$, and $\Phi_{k+i|k}, B_{k+i|k}$, $i = 0, \ldots, N$ and solve:

$$\mathbf{c}_k^{\delta *} = \arg \min_{\mathbf{c}_k^\delta, \{\bar{z}_{i|k}^{(j)}\}} \quad V(\mathbf{c}_k^\delta, \{\bar{z}_{i|k}^{(j)}\}, \{x_{k+i|k}^0\}, \mathbf{c}_k^0) \tag{42a}$$

$$\text{s.t.} \quad (32),(33),(34),(35) \tag{42b}$$

$$\mathcal{Z}_{k+N|k}^{(j)} + z_{k+N|k}^0 \subseteq \mathcal{Z}_t^{(j)}, \quad j = 1, \ldots, r \tag{42c}$$

2. Set $u_k = Kx_k + c_{0|k}^0 + c_{0|k}^{\delta *}$ and $\mathbf{c}_{k+1}^0 = \{c_{1|k}^0 + c_{1|k}^{\delta *}, \ldots, c_{N-1|k}^0 + c_{N-1|k}^{\delta *}, 0\}$.

Theorem 4.1. *In closed-loop operation, Algorithm 1 has the properties:*
(i). the optimization (42) is feasible for all $k > 0$ if feasible at $k = 0$
(ii). the optimal value $V^*(\{x_{k+i|k}^0\}, \mathbf{c}_k^0)$ of the objective (42a) satisfies

$$\mathbb{E}_k\big[V^*(\{x_{k+i+1|k+1}^0\}, \mathbf{c}_{k+1}^0)\big] - V^*(\{x_{k+i|k}^0\}, \mathbf{c}_k^0) \leq l_{ss} - \mathbf{1}^T|x_k| - \lambda \mathbf{1}^T|u_k| \tag{43}$$

(iii). constraints (2) and (4) are satisfied at all times k and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}_0\big(\mathbf{1}^T|x_k| + \lambda \mathbf{1}^T|u_k|\big) \leq l_{ss}. \tag{44}$$

Proof. (i) and (ii) follow from feasibility of $\mathbf{c}_k^\delta = 0$, $\bar{z}_{i|k}^{(j)} = \bar{z}_{i+1|k-1}^{(j)}$ in (42). Constraint satisfaction in (iii) follows from (i), and (44) results from summing (43) over $0 \leq k \leq n$ and noting that $V^*(\{x_{i|0}^0\}, \mathbf{c}_0^0)$ is finite. □

Remark 3. *The optimization (42) can be formulated as a LP.*

Remark 4. *If the constraints on online computation allow for more than one optimization at each sample, then setting $\mathbf{c}_k^0 \leftarrow \mathbf{c}_k^0 + \mathbf{c}_k^{\delta *}$ and repeating step 1 results in non-increasing optimal cost values $V^*(\{x_{k+i|k}^0\}, \mathbf{c}_k^0)$. This process generates a sequence of iterates $\mathbf{c}_k^{\delta *}$ that converges to an optimum point for the problem of minimizing (41) for the nonlinear dynamics (7) at time k.*

5 Example

The levels $h_1 = x_1 + x_1^r$ and $h_2 = x_2 + x_2^r$ of fluid in a pair of coupled tanks are governed by the discrete-time system:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} - Ta_1\sqrt{h_{1,k} - h_{2,k}} \\ x_{2,k} + Ta_1\sqrt{h_{1,k} - h_{2,k}} - Ta_2\sqrt{h_{2,k}} \end{bmatrix} + \frac{T}{C_1}\begin{bmatrix} u_k + u^r \\ 0 \end{bmatrix} + \begin{bmatrix} d_{1,k} \\ d_{2,k} \end{bmatrix}$$

with $a_1 = 0.0690$, $a_2 = 0.0518$, $C_1 = 159.3\text{cm}^2$, sampling interval $T = 10$s, and where $x_1^r = 30.14$cm and $x_2^r = 19.29$cm are setpoints corresponding to the steady state input flow-rate $u_r = 35\text{cm}^3$/s. The manipulated variable is the flow-rate u_k into tank 1, and d_{1k}, d_{2k} are zero-mean random disturbances with normal distributions truncated at the 95% confidence level. The system has probabilistic constraints: $\Pr(|x_{1k}| > 16) \leq 0.2$ and hard constraints: $|x_{1k}| < 16$, $0 \leq u_k \leq 70$. For the operating region: $|x_i| < 30$, $i = 1, 2$, the Lipschitz constants were obtained as $\Gamma_z = \left[\begin{smallmatrix} 0.79 & 0.14 \\ 0.04 & 0.87 \end{smallmatrix}\right]$. Choosing $r = 2$ and $(p_{11}, p_{12}, p_{21}, p_{22}) = (0.8, 0.1, 0.2, 0.9)$, terminal sets $\mathcal{Z}_t^{(1)}, \mathcal{Z}_t^{(2)}$ were computed offline (Fig. 1). The sequence of sets $\mathcal{Z}_{i|k}^{(1)}, \mathcal{Z}_{i|k}^{(2)}$, $i = 0, \ldots, 5$, obtained with one iteration of Algorithm 1 are shown in Figure 2.

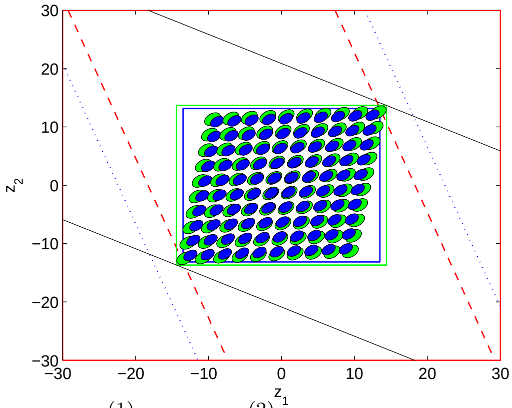

Figure 1: Terminal sets $\mathcal{Z}_t^{(1)}$ (blue), $\mathcal{Z}_t^{(2)}$ (green); confidence regions corresponding to $p_{11} = 0.8$ (blue) and $p_{22} = 0.9$ (green); hard constraints on u (black); soft constraints on x_1 (red, dashed); hard constraints on x_1 (blue, dotted)

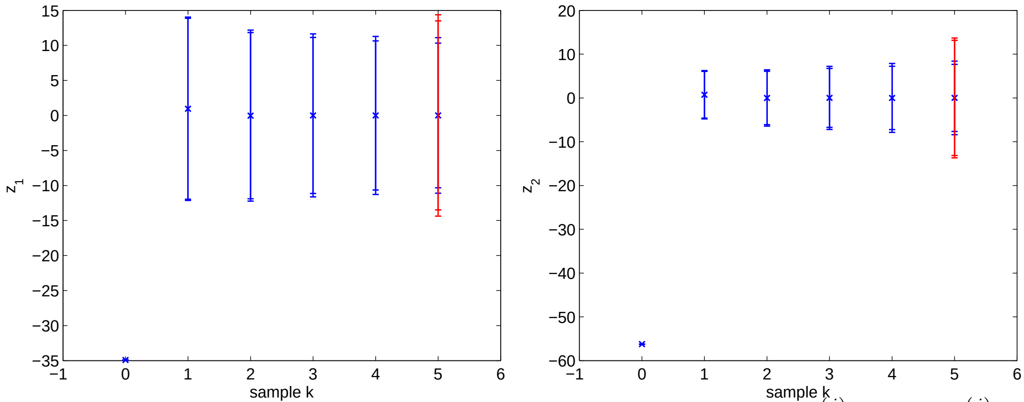

Figure 2: Evolution of bounds on $e_1^T z$ (left) and $e_2^T z$ (right): $\mathcal{Z}_{i|k}^{(j)}$ (blue), $\mathcal{Z}_t^{(j)}$ (red), for $j = 1, 2$, and the nominal trajectory $z_{k+i|k}^0$ (blue x)

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