Robust tubes in nonlinear model predictive control

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Abstract: Nonlinear model predictive control (NMPC) strategies based on linearization about predicted system trajectories enable the online NMPC optimization to be performed by a sequence of convex optimization problems. The approach relies on bounds on linearization errors in order to ensure constraint satisfaction and convergence of the performance index during the optimization at each sampling instant and along closed loop system trajectories. This paper proposes bounds based on robust tubes constructed around predicted trajectories. To ensure local optimality, the bounds are non-conservative for the case of zero linearization error, which requires the tube cross-sections to vary along predicted trajectories. The feasibility, stability and convergence properties of the algorithm are established without the need for predictions to satisfy local optimality criteria. The strategy is applied to a simulated fixed-rotor helicopter.

Keywords: nonlinear systems, optimization, constrained control

1. INTRODUCTION

The online computation that must be performed in order to implement nonlinear model predictive control strategies remains a significant barrier to their application to real processes. The literature on numerical methods for nonlinear MPC can be broadly divided into methods that attempt to exploit the optimal control structure of the MPC optimization, and methods that aim to solve sequences of smaller problems (see e.g. the survey in Diehl et al., 2009). This paper is concerned with the latter approach, which is particularly attractive for MPC since the online optimization problems solved at successive sampling instants are often closely related. A number of nonlinear programming (NLP) approaches based on perturbing previously computed trajectories have therefore been developed specifically for MPC, for example De Oliveira and Biegler (1995); Lee et al. (2002); Diehl et al. (2005a); Ohtsuka (2004).

Methods based on linearizing nonlinear model dynamics around previously predicted trajectories (known as Newton-type methods (De Oliveira and Biegler, 1995)), are arguably the most successful of the perturbation-based approaches. These have the advantages that the optimization can be split into convex subproblems that are variants of linear MPC, which are therefore efficiently solvable, and furthermore they benefit from the robustness properties and robust techniques of linear MPC strategies (e.g. Diehl et al., 2005b). However, to ensure stability and convergence of these approaches, it becomes necessary to limit the perturbations to regions within which the model approximation is meaningful.

In this context, Zavala and Biegler (2009) proposes a method for bounding the effects of model approximations through a NLP sensitivity analysis based on the Lagrange multipliers for the constrained optimization problem. We consider an alternative approach, which was developed in Lee et al. (2002) and Cannon et al. (2009), of bounding linearization errors by constructing tubes containing the predicted trajectories. Tubes provide a computationally convenient means of bounding the effects of uncertainty over several time-steps through a sequence of single-step conditions (Mayne et al., 2005). This approach enables a feasible solution estimate to be retained at every stage of the optimization, thus allowing the optimization to be terminated early before convergence to the solution without compromising closed loop stability.

To ensure closed loop stability given that the predicted trajectories are only feasible at each sampling instant (and not necessarily optimal for the nonlinear MPC problem), the error bounds that are used to bound the predicted cost and robustify constraints must be non-conservative for the case of zero perturbation. This requires variable tubes, and although Lee et al. (2002) proposed to construct and optimize the tubes simultaneously with the MPC optimization in a single linear program, the computation of the tubes involved large numbers of optimization variables. This was largely due to the use of polytopic tubes, and, to obtain more efficient formulation we propose instead to use ellipsoidal tube cross-sections in the current paper, making use of recently developed techniques (Cannon et al., 2010) for constructing ellipsoidal tube cross-sections online.

2. PROBLEM FORMULATION

Consider the nonlinear system with model

$$x_{k+1} = f(x_k, u_k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and input, f is continuous and differentiable for all (x, u) in an operating region, and $f(0,0)=0$. The control problem is optimal regulation with respect to the quadratic cost:

$$\sum_{k=0}^{\infty}\left(\|x_k\|_Q^2+\|u_k\|_R^2\right) \tag{2}$$

($\|x\|_Q^2=x^TQx$), subject to linear constraints of the form:

$$Fx_k+Gu_k\le h,\quad k=0,1,\ldots \tag{3}$$

for $F\in\mathbb{R}^{n_c\times n}$, $G\in\mathbb{R}^{n_c\times m}$. The state x_k is assumed to be measured at each time k.

The proposed NMPC strategy takes a feasible but suboptimal trajectory for the future predicted states and inputs and, by successively linearizing the nonlinear plant model about predicted trajectories, computes locally optimal predicted trajectories which are used to define a receding horizon control law. Let $\{(x^0_{k+i|k},u^0_{k+i|k}),i=0,\ldots,N-1\}$ denote state and input trajectories predicted at time k over an N-step horizon according to the model (1)

$$x^0_{k+i+1|k}=f(x^0_{k+i|k},u^0_{k+i|k})$$

for $i=0,\ldots,N-1$, with $x^0_{k|k}=x_k$. Also let

$$x_{k+i|k}=x^0_{k+i|k}+x^\delta_{k+i|k},\quad u_{k+i|k}=u^0_{k+i|k}+u^\delta_{k+i|k},$$

where $\{(x^\delta_{k+i|k},u^\delta_{k+i|k}),i=0,\ldots,N-1\}$ satisfy

$$x^0_{k+i+1|k}+x^\delta_{k+i+1|k}=f(x^0_{k+i|k}+x^\delta_{k+i|k},u^0_{k+i|k}+u^\delta_{k+i|k})$$

for $i=0,\ldots,N-1$, with $x^\delta_{k|k}=0$. Parameterizing u^δ_k as the sum of a linear feedback law and a feedforward term v,

$$u^\delta_{k+i|k}=K_{k+i|k}x^\delta_{k+i|k}+v_{k+i|k}, \tag{4}$$

the NMPC law is determined by minimizing at each time k an upper bound on the cost (2) over $\{v_{k+i|k},\ i=0,\ldots,N-1\}$, subject to constraints (3).

The cost bound and constraints are computed using a time-varying linear model:

$$x^\delta_{k+i+1|k}=\Phi_{k+i|k}x^\delta_{k+i|k}+B_{k+i|k}v_{k+i|k}+w_{k+i|k} \tag{5}$$

which is derived by linearizing the nonlinear model (1) about $\{x^0_{k+i|k},u^0_{k+i|k}\}$:

$$\Phi_{k+i|k}=A_{k+i|k}+B_{k+i|k}K_{k+i|k}$$

$$A_{k+i|k}=\frac{\partial f}{\partial x}\Big|_{(x^0_{k+i|k},u^0_{k+i|k})},\ B_{k+i|k}=\frac{\partial f}{\partial u}\Big|_{(x^0_{k+i|k},u^0_{k+i|k})}$$

and where $w_{k+i|k}$ is the linearization error. The predictions for $i\ge N$ steps ahead are determined from the linearization of (1) about the target set-point $(x,u)=(0,0)$, and a pre-determined linear feedback law $\hat{K}x$:

$$x_{k+i+1|k}=\hat{\Phi}x_{k+i|k}+\hat{w}_{k+i|k} \tag{6}$$

$$u_{k+i|k}=\hat{K}x_{k+i|k} \tag{7}$$

for $i=N,N+1,\ldots$, where $\hat{\Phi}=\hat{A}+\hat{B}\hat{K}$, with

$$\hat{A}=\frac{\partial f}{\partial x}\Big|_{(0,0)},\ \hat{B}=\frac{\partial f}{\partial u}\Big|_{(0,0)}.$$

Bounds on the linearization errors w in (5) and $\hat{w}$ in (6) can be used to bound the predicted cost and to determine robustly feasible constraints. Since f is continuous and differentiable, it is easy to show (e.g. using the mean-value theorem, Boyd et al., 1994) that there necessarily exists a convex set $\Omega\subset\mathbb{R}^{n\times(n+m)}$ such that $w_k\in\Omega[x^{\delta T}_k\ u^{\delta T}_k]^T$. In the following development we assume that Ω is polytopic with vertices $[C_j\ \ D_j]$, $j=1,\ldots,p$:

$$w_k\in\mathrm{Co}\,\{C_jx^\delta_k+D_ju^\delta_k,\ j=1,\ldots,p\} \tag{8}$$

(where Co denotes the convex hull). Similarly the errors $\hat{w}_{k+i|k}$ in the approximate linear dynamics (6) employed for $i\ge N$ necessarily lie within a convex set:

$$\hat{w}_k\in\mathrm{Co}\,\{\hat{C}_jx_k+\hat{D}_ju_k,\ j=1,\ldots,p\}. \tag{9}$$

Remark 1. If f satisfies a Lipschitz condition of the form

$$|f(x^0+x^\delta,u^0+u^\delta)-f(x^0,u^0)|\le\Gamma_x|x^\delta|+\Gamma_u|u^\delta|\ ,$$

then $[C_j\ D_j]=[\Gamma_x\ \Gamma_u]S_j$, where $\{S_j,\ j=1,\ldots,2^{n+m}\}$ denotes the collection of diagonal matrices with diagonal elements equal to ±1.

3. TUBES FOR LINEARIZATION ERRORS

To bound the effects of linearization errors on predicted trajectories, we propose to construct, using one-step-ahead predictions, a tube containing the component of the predicted state that arises due to linearization errors. The tube is used to derive bounds on the cost and constraints in the NMPC optimization, and to ensure that these are not conservative when $v_{k+i|k}=0$ for all i (and hence also $x^\delta_{k+i|k}=0$ for all i), the scaling of the tube cross-section is retained as a variable in the NMPC online optimization.

To simplify notation, we split the prediction of $x^\delta_{k+i|k}$ into a nominal component $z_{k+i|k}$ and a component $e_{k+i|k}$ which depends only on the linearization errors $w_{k+i|k}$:

$$x^\delta_{k+i|k}=z_{k+i|k}+e_{k+i|k} \tag{10a}$$

$$z_{k+i+1|k}=\Phi_{k+i|k}z_{k+i|k}+B_{k+i|k}v_{k+i|k} \tag{10b}$$

$$e_{k+i+1|k}=\Phi_{k+i|k}e_{k+i|k}+w_{k+i|k} \tag{10c}$$

with $z_{k|k}=e_{k|k}=0$. In this section we describe a method of determining tubes with ellipsoidal cross-sections:

$$e_{k+i|k}\in\mathcal{E}(V_{k+i|k},\beta^2_{k+i|k}),\qquad i=1,\ldots,N \tag{11a}$$

$$x_{k+i|k}\in\mathcal{E}(\hat{V},1),\qquad i\ge N \tag{11b}$$

where $\mathcal{E}(P,\rho)$, for $P\succ0$, $\rho>0$, denotes the ellipsoidal set $\mathcal{E}(P,\rho)=\{x:x^TPx\le\rho\}$. Ellipsoidal sets are chosen (over polyhedral sets for example) for the definition of the tube cross-sections since the scalings $\{\beta_{k+i|k},\ i=1,\ldots,N\}$ can be conveniently incorporated into an online optimization expressed as a second order cone program.

In (11b), $\mathcal{E}(\hat{V},1)$ is a terminal constraint set which must clearly be invariant under (6)-(7), requiring that

$$\hat{\Phi}x+\hat{w}\in\mathcal{E}(\hat{V},1),\ \forall\hat{w}\in\mathrm{Co}\,\{(\hat{C}_j+\hat{D}_j\hat{K})x\},\ \forall x\in\mathcal{E}(\hat{V},1). \tag{12}$$

We also require $\mathcal{E}(\hat{V},1)$ to be feasible with respect to (3) in order that input/state constraints are satisfied over an infinite prediction horizon, i.e.

$$(F+G\hat{K})x\le h,\ \forall x\in\mathcal{E}(\hat{V},1). \tag{13}$$

The matrices $\hat{K}$ and $\hat{V}$ can be computed offline so as to maximize the terminal set $\mathcal{E}(\hat{V},1)$ (hence maximizing the set of feasible initial conditions) subject to (12)-(13) by solving a semidefinite program (SDP). For example, if $\hat{S},\hat{Y}$ are the solutions of the following SDP:

$$\max_{\hat{S},\hat{Y}}\det(\hat{S})$$

$$\text{s.t.}\ \begin{bmatrix}\hat{S} & (\hat{A}+\hat{C}_j)\hat{S}+(\hat{B}+\hat{D}_j)\hat{Y}\\ * & \hat{S}\end{bmatrix}\succeq0,\ j=1,\ldots,p \tag{14}$$

$$\begin{bmatrix}h_q^2 & F_q\hat{S}+G_q\hat{Y}\\ * & \hat{S}\end{bmatrix}\succeq0,\ q=1,\ldots,n_c$$

(where $[\cdot]_q$ denotes the qth row of $[\cdot]$), then the volume of $\mathcal{E}(\hat{V}, 1)$ maximized with $\hat{V} = \hat{S}^{-1}$ and $\hat{K} = \hat{Y}\hat{S}^{-1}$ (see e.g. Kothare et al., 1996, for details).

Two procedures for ensuring the membership condition (11a) are described below. In the first approach, the feedback gains $K_{k+i|k}$ and matrices $V_{k+i|k}$ specifying the shapes of tube cross-sections are re-computed online via a sequence of semidefinite programs each time the linearization trajectory $\{(x^0_{k+i|k}, u^0_{k+i|k}),\ i = 0, \dots, N-1\}$ is updated. The second approach achieves reduced online computation by fixing $K_{k+i|k}$ and $V_{k+i|k}$ offline. Both procedures are based on the recursive membership condition,

$$\Phi_{k+i|k} e + w \in \mathcal{E}(V_{k+i+1|k}, \beta^2_{k+i+1|k}) \tag{15}$$

for all $w \in \mathrm{Co}\{C_j x^\delta_{k+i|k} + D_j u^\delta_{k+i|k},\ j = 1, \dots, p\}$, for all $e \in \mathcal{E}(V_{k+i|k}, \beta^2_{k+i|k})$. The terminal constraint that $x_{k+N|k} \in \mathcal{E}(\hat{V}, 1)$ is similarly ensured by the condition

$$x^0_{k+N|k} + z_{k+N|k} + e \in \mathcal{E}(\hat{V}, 1) \tag{16}$$

for all $e \in \mathcal{E}(V_{k+N|k}, \beta^2_{k+N|k})$.

3.1 Time-varying feedback gains and tube cross-sections

Using (4) and (10a), and applying the triangle inequality, a sufficient condition to ensure the recursive membership condition of (15) is given by

$$\beta' \geq \|(C_j + D_j K)z + D_j v\|_{V'} + \max_{e \in \mathcal{E}(V, \beta^2)} \|(\Phi + C_j + D_j K)e\|_{V'} \tag{17}$$

for $j = 1, \dots, p$, where $(V', \beta') = (V_{k+i+1|k}, \beta_{k+i+1|k})$ and subscripts $k+i|k$ have been omitted to simplify notation.

Lemma 2. Condition (17) is equivalent to

$$V \succeq (\Phi + C_j + D_j K)^T V' (\Phi + C_j + D_j K) \tag{18a}$$

$$\beta' \geq \|(C_j + D_j K)z + D_j v\|_{V'} + \beta \tag{18b}$$

for $j = 1, \dots, p$.

Proof: This follows from squaring both sides of (17) and applying the S-procedure to the resulting condition. $\square$

Using the triangle inequality to derive (17) from (15) would be non-conservative (i.e. (17) would also be necessary for (15)) if $V = (\Phi + C_j + D_j K)^T V' (\Phi + C_j + D_j K)$, since then the direction of the maximizing e in (17) would be arbitrary. But (17) must hold for all $j = 1, \dots, p$, so this is clearly not possible in general. However, to reduce the degree of conservativeness in (17), it is desirable to choose K and the shape of $\mathcal{E}(V, \beta^2)$ so that the LMI (18a) is tight. For given $V_{k+i+1|k}$, this can be achieved for example by solving the following SDP in variables S, Y, γ:

$$\begin{aligned} &\max_{S,Y,\gamma}\ \gamma \\ &\text{s.t. } S \succeq \gamma I, \\ &\begin{bmatrix} S & [(A_{k+i|k} + C_j)S + (B_{k+i|k} + D_j)Y]^T \\ * & V^{-1}_{k+i+1|k} \end{bmatrix} \succeq 0,\ j = 1, \dots, p \end{aligned} \tag{19}$$

and then setting $V_{k+i|k} = S^{-1}$, $K_{k+i|k} = YS^{-1}$.

By a similar argument to that used in Lemma 2, if $V_{k+N|k} = \hat{V}$ is assumed, then the terminal condition (16) is equivalent to

$$1 \geq \|x^0_{k+N|k} + z_{k+N|k}\|_{\hat{V}} + \beta_{k+N|k}.$$

The preceding discussion suggests a method of defining $V_{k+i|k}$ and $K_{k+i|k}$ for $i = 1, \dots, N-1$ so as to minimize the degree of conservativeness of the bound (17), by solving the $N-1$ SDP problems given by (19) for $i = N-1, N-2, \dots, 1$, and initialized with $V_{k+N|k} = \hat{V}$.

3.2 Fixed feedback gains and tube cross-sections

In order to optimize $V_{k+i|k}$ and $K_{k+i|k}$ for $i = 1, \dots, N-1$, the sequence of SDP's described in section 3.1 must be solved after each update of the trajectory $\{(x^0_{k+i|k}, u^0_{k+i|k})\}$ about which the linearization of the nonlinear model is performed. By minimizing the degree of conservativeness of the conditions that are used to invoke the recursive membership condition in (17), this optimization procedure increases the feasible initial condition sets of the MPC law described in section 5, and is also likely to increase the convergence rate of the associated online optimization. However it has to be performed online and may therefore be computationally intractable for large systems and problems that require fast sampling.

We therefore propose an alternative approach in which much of the computation involved in invoking the membership condition (17) is performed offline. In this approach we set $V_{k+i|k}$ and $K_{k+i|k}$ equal to the terminal values:

$$V_{k+i|k} = \hat{V},\ K_{k+i|k} = \hat{K},\ i = 1, \dots, N-1. \tag{20}$$

Lemma 3. A sufficient condition for (17) is

$$\begin{aligned} \beta' \geq{}& \|(C_j + D_j \hat{K})z + D_j v\|_{\hat{V}} \\ &+ \beta\big(\|\Phi \hat{V}^{-1/2}\|_{\hat{V}} + \|(C_j + D_j \hat{K})\hat{V}^{-1/2}\|_{\hat{V}}\big) \end{aligned} \tag{21}$$

for $j = 1, \dots, p$.

Proof: The bound in (21) is obtained by replacing the maximization on the RHS of (17) by the associated induced norm and then applying the triangle inequality. $\square$

Remark 4. The factor $\|(C_j + D_j \hat{K})\hat{V}^{-1/2}\|_{\hat{V}}$ in (21) can be computed for $j = 1, \dots, p$ offline, leaving only the determination of $\|\Phi_{k+i|k} \hat{V}^{-1/2}\|_{\hat{V}}$ for $i = 1, \dots, N-1$ (which requires finding the maximum eigenvalues of $N-1$ matrices) to be performed online.

Although the use of the triangle inequality in deriving (21) incurs a degree of conservativeness, the analysis of section 5 shows that this does not affect the stability properties of the resulting MPC law.

4. COST AND CONSTRAINT BOUNDS

The tubes constructed in section 3 to bound linearization errors can be used to determine an upper bound on the predicted cost-to-go function, and also to construct robust constraints that ensure that predicted states and inputs satisfy (3) over an infinite prediction horizon.

Using the dual mode prediction paradigm (Mayne et al., 2000), we define a performance index as a cost-to-go which is computed explicitly only for the first N stages:

$$J(\mathbf{x}_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \big(\|x_{k+i|k}\|^2_Q + \|u_{k+i|k}\|^2_R\big) + \|x_{k+N|k}\|^2_P \tag{22}$$

where $\mathbf{u}_k = \{u_{k+i|k},\ i = 0,\dots,N-1\}$ is a predicted input trajectory and $\mathbf{x}_k = \{x_{k+i|k},\ i = 0,\dots,N\}$ is the corresponding sequence of states satisfying the nonlinear model (1). The weight P on the terminal state is computed so as to bound the cost-to-go over prediction times $i \geq N$:

$$\sum_{i=N}^{\infty}(\|x_{k+i|k}\|_Q^2 + \|u_{k+i|k}\|_R^2) \leq \|x_{k+N|k}\|_P^2.$$

Therefore P can be optimized by solving the SDP

$$\begin{aligned}&\min_P \ \mathrm{tr}(P)\\ &\text{s.t. } P - (\hat{\Phi} + \hat{C}_j + \hat{D}_j\hat{K})^T P(\hat{\Phi} + \hat{C}_j + \hat{D}_j\hat{K})\\ &\qquad\qquad \succeq Q + \hat{K}^T R\hat{K},\ j = 1,\dots,p\end{aligned} \tag{23}$$

which is performed offline.

In order to define a convex MPC optimization to be performed online, we minimize a bound on $J(\mathbf{x},\mathbf{u})$ which is derived from bounds on the individual terms in (22). Define $l_{x,i}$ and $l_{u,i}$ for $i = 0,\dots,N-1$ by

$$l_{x,i} = \|x^0_{k+i|k} + z_{k+i|k}\|_Q + \beta_{k+i|k}\|V^{-1/2}_{k+i|k}\|_Q$$

$$l_{u,i} = \|u^0_{k+i|k} + K_{k+i|k}z_{k+i|k} + v_{k+i|k}\|_R + \beta_{k+i|k}\|KV^{-1/2}_{k+i|k}\|_R$$

and let

$$l_{x,N} = \|x^0_{k+N|k} + z_{k+N|k}\|_P + \beta_{k+N|k}\|\hat{V}^{-1/2}\|_P.$$

Consider the cost $\bar{J}(\mathbf{v}_k, \boldsymbol{\beta}_k, \mathbf{x}^0_k, \mathbf{u}^0_k)$, which is defined for given sequences $\mathbf{v}_k = \{v_{k+i|k}, i = 0,\dots,N-1\}$ and $\boldsymbol{\beta}_k = \{\beta_{k+i|k},\ i = 0,\dots,N\}$ by

$$\bar{J}(\mathbf{v}_k, \boldsymbol{\beta}_k, \mathbf{x}^0_k, \mathbf{u}^0_k) = \sum_{i=0}^{N-1}(l^2_{x,i} + l^2_{u,i}) + l^2_{x,N}. \tag{24}$$

Lemma 5. Given the feedback law (4) and the state decomposition (10a), we have

$$J(\mathbf{x}_k, \mathbf{u}_k) \leq \bar{J}(\mathbf{v}_k, \boldsymbol{\beta}_k, \mathbf{x}^0_k, \mathbf{u}^0_k) \tag{25}$$

for any $\mathbf{x}_k$ and $\boldsymbol{\beta}_k$ satisfying the membership conditions (11a-b).

Proof: The state decomposition (10a) and membership conditions (11a) give $x = x^0 + z + e$ and $e \in \mathcal{E}(V, \beta^2)$ (where the subscripts $k+i|k$ have been omitted for simplicity). The triangle inequality therefore implies the bound

$$\|x\|_Q \leq \|x^0 + z\|_Q + \beta\|V^{-1/2}\|_Q.$$

Similarly, (10a), (11a), and the predicted feedback law (4) give $u = u^0 + K(z+e) + v$, so the triangle inequality implies

$$\|u\|_R \leq \|u^0 + Kz + v\|_R + \beta\|KV^{-1/2}\|_R. \qquad \square$$

The constraints (3) can be imposed on the predicted state and input trajectories through constraints applied to $z_{k+i|k}$, the nominal component of predicted trajectories, using the following result.

Lemma 6. Given the feedback law (4) and state decomposition (10a), sufficient conditions for $Fx_{k+i|k} + Gu_{k+i|k} \leq h$ are

$$\begin{aligned}&(F_q + G_qK_{k+i|k})z_{k+i|k} + G_qv_{k+i|k}\\ &\qquad + \beta_{k+i|k}\|(F_q + G_qK_{k+i|k})\|_{V^{-1}_{k+i|k}}\\ &\qquad\qquad \leq h_q - (F_qx^0_{k+i|k} + G_qu^0_{k+i|k})\end{aligned} \tag{26}$$

for $q = 1,\dots,n_c$, for any $\beta_{k+i|k}$ such that the membership condition (11a) holds.

Proof: The bound in (26) follows from

$$\max_{e\in\mathcal{E}(V,\beta^2)} |(F_q + G_qK)e| = \beta\|(F_q + G_qK)\|_{V^{-1}}$$

for any row $q = 1,\dots,n_c$ of F, G. $\square$

The procedure described in section 5 splits the online MPC optimization into a sequence of iterations, each of which consists of minimizing the objective $\bar{J}(\mathbf{v}_k, \boldsymbol{\beta}_k, \mathbf{x}^0_k, \mathbf{u}^0_k)$ over $\mathbf{v}_k, \boldsymbol{\beta}_k$ subject to the robust constraints of Lemma 6. Combining this with the conditions defining the ellipsoidal tubes derived in section 3, the resulting optimization can be expressed as follows:

$$(\mathbf{v}^*_k, \boldsymbol{\beta}^*_k) = \min_{\mathbf{v}_k, \boldsymbol{\beta}_k} \ \sum_{i=0}^{N-1}(l^2_{x,i} + l^2_{u,i}) + l^2_{x,N} \tag{27a}$$

subject to

$$z_{k+i+1|k} = \Phi_{k+i|k}z_{k+i|k} + B_{k+i|k}v_{k+i|k} \tag{27b}$$

$$\begin{aligned}\beta_{k+i+1|k} &\geq \lambda_{i,j}\beta_{k+i|k}\\ &\quad + \|(C_j + D_jK_{k+i|k})z_{k+i|k} + D_jv_{k+i|k}\|_{V_{k+i+1|k}}\end{aligned} \tag{27c}$$

$$l_{x,i} \geq \|x^0_{k+i|k} + z_{k+i|k}\|_Q + \beta_{k+i|k}\|V^{-1/2}_{k+i|k}\|_Q \tag{27d}$$

$$\begin{aligned}l_{u,i} &\geq \|u^0_{k+i|k} + K_{k+i|k}z_{k+i|k} + v_{k+i|k}\|_R\\ &\qquad + \beta_{k+i|k}\|K_{k+i|k}V^{-1/2}_{k+i|k}\|_R\end{aligned} \tag{27e}$$

$$\begin{aligned}&h_q - (F_qx^0_{k+i|k} + G_qu^0_{k+i|k})\\ &\quad \geq (F_q + G_qK_{k+i|k})z_{k+i|k} + G_qv_{k+i|k}\\ &\qquad + \beta_{k+i|k}\|(F_q + G_qK_{k+i|k})\|_{V^{-1}_{k+i|k}}\end{aligned} \tag{27f}$$

for $i = 0,\dots,N-1$, and

$$z_{k|k} = 0 \tag{27g}$$

$$\beta_{k|k} = 0 \tag{27h}$$

$$1 \geq \|x^0_{k+N|k} + z_{k+N|k}\|_{\hat{V}} + \beta_{k+N|k} \tag{27i}$$

$$l_{x,N} \geq \|x^0_{k+N|k} + z_{k+N|k}\|_P + \beta_{k+N|k}\|\hat{V}^{-1/2}\|_P. \tag{27j}$$

Here $\lambda_{i,j} = 1$ if the time-varying tubes and feedback gains of section 3.1 are employed, whereas

$$\lambda_{i,j} = \|\Phi_{k+i|k}\hat{V}^{-1/2}\|_{\hat{V}} + \|(C_j + D_j\hat{K})\hat{V}^{-1/2}\|_{\hat{V}}$$

$$V_{k+i|k} = \hat{V}, \quad K_{k+i|k} = \hat{K}$$

if the approach of section 3.2 is used.

Remark 7. The optimization in (27) can be expressed as a second-order cone program (SOCP), which is convex and efficiently solvable (Lobo et al., 1998).

The following result shows that the cost bounds and constraint bounds employed in (27) are non-conservative whenever the perturbation sequence $\mathbf{v}^*_k$ is zero.

Lemma 8. For any $\mathbf{x}^0_k$ and $\mathbf{u}^0_k$ satisfying (1), we have

$$\bar{J}(0, 0, \mathbf{x}^0_k, \mathbf{u}^0_k) = J(\mathbf{x}^0_k, \mathbf{u}^0_k). \tag{28}$$

Furthermore, if $\mathbf{v}^*_k = 0$, then $\boldsymbol{\beta}^*_k = 0$, and the constraints of (27) are equivalent to

$$Fx^0_{k+i|k} + Gu^0_{k+i|k} \leq h, \quad i = 0,\dots,N-1 \tag{29a}$$

$$x^0_{k+N|k} \in \mathcal{E}(\hat{V}, 1). \tag{29b}$$

Proof: The equality of costs in (28) follows directly from (27d,e,j) with $\boldsymbol{\beta}_k = 0$. Furthermore, if $\mathbf{v}^*_k = 0$, then

(27b,g) imply that $z_{k+i|k} = 0$ for all i and hence $\boldsymbol{\beta}_k^* = 0$ due to (27c,h) and the objective (27a). The equivalence of the constraints of (27) and (29) is then due to (27f,i). $\square$

5. RECEDING HORIZON CONTROL LAW

This section describes the proposed MPC optimization procedure and discusses the properties of the associated receding horizon control law. The optimization procedure computes an optimal sequence of perturbations on a feasible predicted trajectory by solving the SOCP (27), which is based on the Jacobian linearization about this trajectory, then updates the feasible trajectory using the optimal perturbations and repeats the process. Since a feasible predicted trajectory is available at each step, the procedure can be terminated after any chosen maximum number of iterations (denoted *Maxiters*), or when the input perturbation vector falls below any given tolerance (denoted by *solution-tolerance*).

Algorithm 1. Offline: Compute $\hat{V}$, $\hat{K}$ defining the terminal set and feedback law by solving (14), and the terminal cost weight P by solving (23). Find an initial trajectory $\mathbf{u}_0^0$ such that $\mathbf{u}_0^0, \mathbf{x}_0^0$ satisfies (29a,b).
Online: At times $k = 0, 1, \ldots$:
- (i) Set *iter* $= 1$. Given $\mathbf{u}_k^0$, compute $\mathbf{x}_k^0$ satisfying (1) with $x_{k|k}^0 = x_k$.
- (ii) Linearize the model (1) about $\mathbf{x}_k^0, \mathbf{u}_k^0$ to determine $A_{k+i|k}, B_{k+i|k}$ for $i = 0, \ldots, N-1$.
- (iii) If time-varying feedback gains and tube shapes are used, compute $V_{k+i|k}$ and $K_{k+i|k}$ by solving (19) for $i = N-1, \ldots, 1$, with $V_{k+N|k} = \hat{V}$.
- (iv) Solve (27) to determine $\mathbf{v}_k^*$.
- (v) Compute $\mathbf{x}_k, \mathbf{u}_k$ satisfying (1) and (4) with $\mathbf{v}_k = \mathbf{v}_k^*$.
- (vi) If *iter* $<$ *Maxiters* and $\|\mathbf{v}_k^*\| \geq$ *solution-tolerance*, set $\mathbf{x}_k^0 = \mathbf{x}_k$, $\mathbf{u}_k^0 = \mathbf{u}_k$, *iter* := *iter* $+ 1$ and return to step (ii).
- (vii) Otherwise set
$$\mathbf{u}_{k+1}^0 = \{u_{k+1|k}, \ldots, u_{k+N-1|k}, \hat{K}x_{k+N|k+1}\} \quad (30)$$
and implement $u_k = u_{k|k}^0 + v_{k|k}^*$.

Lemma 9. If $\mathbf{x}_0^0, \mathbf{u}_0^0$ is feasible with respect to constraints (29a,b) at time $k = 0$, then the SOCP (27) in step (iv) of Algorithm 1 is feasible at each iteration and for all $k \geq 0$.

Proof: By Lemma 8, if $\mathbf{x}_k^0, \mathbf{u}_k^0$ is feasible for (29a,b), then $(\mathbf{v}_k, \boldsymbol{\beta}_k) = (0, 0)$ is feasible for (27). The recursive guarantee of feasibility follows from Lemma 6, the tube membership conditions of section 3 and the robust invariance of $\mathcal{E}(\hat{V}, 1)$. Specifically: by Lemmas 2, 3, and 6, the constraints in (27) ensure that the updated trajectory $\mathbf{x}_k^0, \mathbf{u}_k^0$ computed in step (vi) and employed in the subsequent iteration is feasible with respect to (29a,b). Similarly, the robust invariance conditions (12)-(13) ensure the feasibility of $\mathbf{x}_{k+1}^0, \mathbf{u}_{k+1}^0$ in step (i) at the following time-step. $\square$

We show next that the iteration in steps (i)-(vi) of Algorithm 1 results in a monotonically non-increasing cost bound.

Theorem 10. Let $\bar{J}_k^{(j)}$ denote the optimal value of the objective of (27) in step (iv) of Algorithm 1 after j iterations at time k. Then for all $j \geq 1$ we have
$$\bar{J}_k^{(j+1)} \leq \bar{J}_k^{(j)}. \quad (31)$$

Proof: Lemma 5 implies that the trajectory $\mathbf{x}_k^0, \mathbf{u}_k^0$ generated in step (vi) of the jth iteration of Algorithm 1 necessarily satisfies $J(\mathbf{x}_k^0, \mathbf{u}_k^0) \leq \bar{J}^{(j)}$. But from the optimality of $\bar{J}^{(j+1)}$ and Lemma 8 we have $\bar{J}^{(j+1)} \leq \bar{J}(0, 0, \mathbf{x}_k^0, \mathbf{u}_k^0)$ and $\bar{J}(0, 0, \mathbf{x}_k^0, \mathbf{u}_k^0) = J(\mathbf{x}_k^0, \mathbf{u}_k^0)$, so that $\bar{J}_k^{(j+1)} \leq \bar{J}_k^{(j)}$. $\square$

Remark 11. Since $\bar{J}_k$ is bounded from below, and since (31) holds with equality only if $\mathbf{v}_k^* = 0$ at iteration $j+1$, Theorem 10 implies that the iteration of Algorithm 1 converges asymptotically as $j \to \infty$ to $(\mathbf{v}_k^*, \boldsymbol{\beta}_k^*) = 0$ at time k. This implies convergence to a (possibly locally) optimal point for the problem of minimizing $\bar{J}(0, 0, \mathbf{x}_k^0, \mathbf{u}_k^0)$ over $\mathbf{u}_k^0$ subject to constraints (29a,b). Note that a minimum point of $J(\mathbf{x}_k^0, \mathbf{u}_k^0)$ is necessarily also a minimum of $\bar{J}(0, 0, \mathbf{x}_k^0, \mathbf{u}_k^0)$ w.r.t. $\mathbf{u}_k^0$ (because $(\mathbf{v}_k^*, \boldsymbol{\beta}_k^*) = (0, 0)$ if $\mathbf{u}_k^0$ is locally optimal for $J(\mathbf{x}_k^0, \mathbf{u}_k^0)$ due to (28) and (25)). However a local minimum of $\bar{J}(0, 0, \mathbf{x}_k^0, \mathbf{u}_k^0)$ may not be optimal for $J(\mathbf{x}_k^0, \mathbf{u}_k^0)$ due to the piecewise-linearity of the bounds imposed on the linearization errors. The degree of suboptimality this causes is likely to decrease if tighter bounds, such as those developed in section 3.1, are used to derive (27).

Theorem 12. If $Q \succ 0$ (or if $R \succ 0$ and the state x_k of (1) is observable from $Q^{1/2}x_k$), then $x = 0$ is an asymptotically stable equilibrium of (1) under the MPC law of Algorithm 1, with a region of attraction equal to the set of feasible initial conditions for (29a,b).

Proof: The LMI constraint on P in (23) and (9) imply
$$\|x\|_P^2 \geq \|x\|_Q^2 + \|\hat{K}x\|_R^2 + \|f(x, \hat{K}x)\|_P^2.$$
From the definition (30) of $\mathbf{u}_{k+1}^0$, it follows that the trajectory $\mathbf{x}_{k+1}^0, \mathbf{u}_{k+1}^0$ computed in step (i) of Algorithm 1 at time $k+1$ satisfies
$$J(\mathbf{x}_{k+1}^0, \mathbf{u}_{k+1}^0) \leq J(\mathbf{x}_k, \mathbf{u}_k) - \|\mathbf{x}_k\|_Q^2 - \|\mathbf{u}_k\|_R^2$$
where $\mathbf{x}_k, \mathbf{u}_k$ is the trajectory computed in step (v) at time k. Lemma 5 therefore implies
$$J(\mathbf{x}_{k+1}^0, \mathbf{u}_{k+1}^0) \leq \bar{J}(\mathbf{v}_k^*, \boldsymbol{\beta}_k^*, \mathbf{x}_k^0, \mathbf{u}_k^0) - \|\mathbf{x}_k\|_Q^2 - \|\mathbf{u}_k\|_R^2.$$
But $(\mathbf{v}_{k+1}, \boldsymbol{\beta}_{k+1}) = (0, 0)$ is feasible for the optimization in step (v) at time $k+1$, and Lemma 8 therefore gives
$$\bar{J}(\mathbf{v}_k^*, \boldsymbol{\beta}_k^*, \mathbf{x}_{k+1}^0, \mathbf{u}_{k+1}^0) \leq \bar{J}(\mathbf{v}_k^*, \boldsymbol{\beta}_k^*, \mathbf{x}_k^0, \mathbf{u}_k^0) - \|\mathbf{x}_k\|_Q^2 - \|\mathbf{u}_k\|_R^2.$$
This establishes that $x = 0$ is stable since $\bar{J}(\cdot, \cdot, \mathbf{x}_{k+1}^0, \mathbf{u}_{k+1}^0)$ is a positive definite function of x_k under the assumptions of the theorem. It also follows that
$$\sum_{k=0}^{\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2) \leq \bar{J}(\mathbf{v}_0^*, \boldsymbol{\beta}_0^*, \mathbf{x}_0^0, \mathbf{u}_0^0),$$
which implies that $(x_k, u_k) \to (0, 0)$ under the conditions of the theorem. $\square$

Remark 13. Algorithm 1 must be initialized with a trajectory that is feasible with respect to constraints (29a,b). In practice this trajectory could be computed by modifying Algorithm 1 to solve the following feasibility problem: given a trajectory $\mathbf{x}_0^0, \mathbf{u}_0^0$ satisfying (1), determine an optimal perturbation $\mathbf{x}_0^\delta, \mathbf{u}_0^\delta$ for the problem of minimizing the maximum violation of constraints (27f,i). This could be formulated as a SOCP problem, which, analogously to Lemma 9 and Theorem 10, is guaranteed to give a non-increasing bound on the maximum constraint violation.

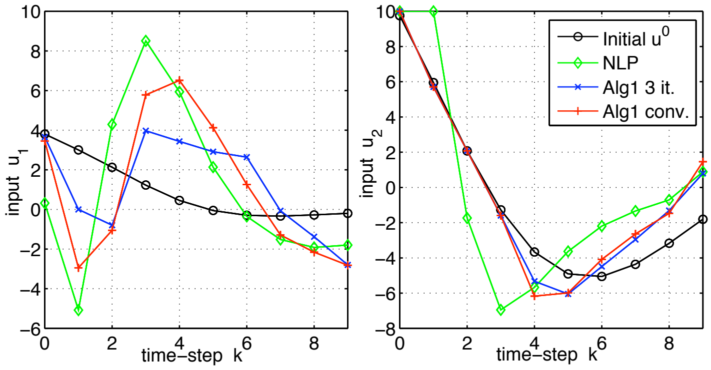

Fig. 1. Input sequences predicted at $k = 0$

6. EXAMPLE

The control law of Algorithm 1 is applied to a simplified planar model of a fixed-rotor helicopter (Lee et al., 2002):

$$\ddot{y} = (u_1 + g)\sin\alpha, \quad \ddot{z} = (u_1 + g)\cos\alpha - g, \quad \ddot{\alpha} = u_2.$$

Here g is the acceleration due to gravity y, z, α represent horizontal, vertical and angular displacement, and the inputs u_1, u_2 are proportional to the net thrust and torque acting on the aircraft. The MPC cost is defined with $Q = I$, $R = 10^{-3}I$, and the system has input constraints:

$$|u_{1,k}| \le 10, \quad |u_{2,k}| \le 10. \tag{32}$$

For a sampling interval of $T = 0.1$ s, a discrete time model with state $x_k = \big(y(kT), z(kT), \dot{y}(kT), \dot{z}(kT), \alpha(kT), \dot{\alpha}(kT)\big)$ was computed online by numerical integration. The linearization error bounds in (9) were computed offline using the bounds (32) on u_k and operating region constraints:

$$|\alpha_k| \le \pi/4, \quad |\dot{\alpha}_k| \le 2.$$

The offline computation of $\hat{K}, \hat{V}$ was performed by modifying (14) to include a bound: $\max_{x\in\mathcal{E}(\hat{V},1)} \|x\|_P^2 \le 10$ on the mode 2 cost. For simplicity $K_{k+i|k}, V_{k+i|k}$ were set equal to $\hat{K}, \hat{V}$ for all i, k, as described in section 3.2.

Figure 1 shows predicted input trajectories for horizon $N = 10$, initial condition $x_0 = (0, -1, 0, 0, -0.5, 0)$, and with the initial $\mathbf{u}_0^0, \mathbf{x}_0^0$ computed as described in Remark 13. Clearly Alg. 1 has not converged (with *solution-tolerance* $= 10^{-3}$) to the optimum solution of the underlying NLP (as computed by Matlab's fmincon). Likewise, the costs reported Table 1 show that the predicted costs at $k = 0$ are 13% and 7% suboptimal for *Maxiters* $= 1$ and 3 respectively, however Alg. 1 (with one iteration) requires only 28% of the CPU time of the NLP solver (Table 1 reports average times for the online optimization at $k = 0$, implemented in Matlab on a 2.4 GHz processor). The degree of suboptimality is however significantly reduced by the receding horizon implementation of Alg. 1, which is only about 5% suboptimal in closed loop operation.

	$J(\mathbf{x}_0, \mathbf{u}_0)$	CPU time	J(closed loop)
Alg. 1 (*Maxiters* $= 1$)	22.89	2.89	21.14
Alg. 1 (*Maxiters* $= 3$)	21.70	6.75	20.95
NLP (fmincon)	20.26	10.14	19.97

Table 1. Predicted and closed loop costs, and CPU times

7. CONCLUSIONS AND FURTHER WORK

A nonlinear MPC algorithm, based on successive approximation of an underlying NLP, is proposed for a class of nonlinear systems. The effects of approximation errors on predicted trajectories are bounded using tubes with ellipsoidal cross-sections, which are optimized online simultaneously with the MPC cost by solving a SOCP. The approach is shown to have a recursive guarantee of feasibility, which implies that successive iterations converge to a minimum point of an upper bound on the cost. Asymptotic stability of the closed loop system is established and the optimization can be terminated early (at the expense of suboptimality) in order to reduce computational load.

The approach suffers from the disadvantages that convergence to a local optimum of the underlying NLP is not guaranteed, and the assumed bounds on linearization errors are linear whereas the actual error in the approximate model derived through Jacobian linearization grows quadratically. Both problems are due to the use of linear error bounds, and future work will focus on the use of quadratic error bounds in the context of ellipsoidal tubes.

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