Online Computation of Backwards-Reachable Sets for Robust Linear Discrete Time MPC

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Outline

1. Active set solver for robust optimal control with quadratic ($\mathcal{H}_\infty$-type) cost
   - Riccati recursion
   - constraint degeneracy
   - computation and properties
2. Active set solver for robust optimal control with piecewise linear cost
   - prediction tree structure
   - computation and properties

Motivation

- Robust Model Predictive Control (RMPC)
  open loop min-max problem solved repeatedly online [Witsenhausen 1968]
- Closed loop RMPC
  online optimization over feedback policies [Lee & Yu 1997]
  - large computational loads [Scokaert & Mayne 1998]
  - optimality often sacrificed to gain computational efficiency
    [e.g. Goulart, Kerrigan & Alamo 2009; Rakovic, Kouvaritakis, Cannon & Panos 2012]
- Parametric solution methods
  optimal feedback structure computed offline [Bemporad, Borelli & Morari 2003]
  - requires solution at all points in state space
  - requires online search over polyhedral subsets of state space

Motivation

- Active set methods

  local solution based on homotopy

  - efficient technique for nominal MPC [Cannon 2008; Ferreau 2008]
  - DP solution for:

    min-max input-constrained RMPC [Buerger, Cannon, Kouvaritakis 2012]

    min-max input/state-constrained RMPC [Buerger, Cannon, Kouvaritakis 2013]
  - computation per iteration depends linearly on horizon length

  ... but

  - requires (offline) solution of robust controllable k-step sets, $k = 1, \ldots, N$

- Objective

  devise an active set method to compute robust controllable sets locally (online)

Problem statement

- Model $x^+ = Ax + Bu + Dw$
- System constraints $x \in \mathcal{X}$, $u \in \mathcal{U}$, $w \in \mathcal{W}$,
  $\mathcal{X}, \mathcal{U}, \mathcal{W}$ compact, convex, polytopic sets
- Problem A: min-max optimal control with quadratic cost

$$\big(u_m^*(x), w_m^*(x,u)\big) = \arg \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} J_m(x,u,w)$$

$$\text{subject to } Ax + Bu \in \mathcal{X}_{m-1} \ominus D\mathcal{W}$$

where J_m : $\mathcal{H}_\infty$-type cost for m-stage problem

$$J_m(x,u,w) = \tfrac{1}{2}\big(\|x\|_Q^2 + \|u\|_R^2 - \gamma^2\|w\|^2\big) + J_{m-1}^*(x^+)$$

$\mathcal{X}_m$: robustly controllable set to terminal set $\mathcal{X}^f$ in m steps

$$\mathcal{X}_m = \mathcal{X} \cap \{x : \exists u \in \mathcal{U},\ Ax + Bu \in \mathcal{X}_{m-1} \ominus D\mathcal{W}\}$$

Problem statement

- Model $\qquad x^+ = Ax + Bu + Dw$

- System constraints $\qquad x \in \mathcal{X}$, $u \in \mathcal{U}$, $w \in \mathcal{W}$,
  $\mathcal{X}, \mathcal{U}, \mathcal{W}$ compact, convex, polytopic sets

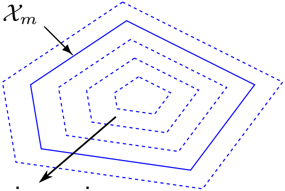
- Problem B: min-max optimal control with piecewise linear cost

$$\big(u_m^*(x), w_m^*(x,u)\big) = \arg \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} V_m(x,u,w)$$

where $\quad V_m^*(x) \leq 1 \quad \Longleftrightarrow \quad x \in \mathcal{X}_m$

$$V_m(x,u,w) = \max\{\|Ex\|_\infty, V_{m-1}(x^+)\}$$

with $\quad \mathcal{X} = \{x : Ex \leq \mathbf{1}\}$



Problem statement

- Problem A: terminal conditions

$$J_0^*(x) = \tfrac{1}{2}\|x\|_P^2$$
$$\mathcal{X}_0 = \mathcal{X}^f$$

where $\quad \|x_0\|_P^2 = \sum_{k=0}^{\infty}(\|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2\|w_k\|^2)$ under $\begin{cases} u_k = u_\infty^f(x_k) \\ w_k = w_\infty^f(x_k, u_k) \end{cases}$

$\mathcal{X}^f$: r.p.i. under $u_\infty^f(x) \quad \Longleftrightarrow \quad A\mathcal{X}^f + Bu_\infty^f(x) \subseteq \mathcal{X}^f \ominus D\mathcal{W}$

- Problem B: terminal conditions

$$V_0^*(x) = \|E^f x\|_\infty$$

where $\quad \mathcal{X}^f = \{x : E^f x \leq \mathbf{1}\}$

Problem statement

- Assume
  - (A, B) controllable
  - $Q \succeq 0, \; R \succ 0$
  - $(A, Q^{1/2})$ observable
  - γ sufficiently large to ensure convexity

- Sequential min-max optimization performed over arbitrary feedback laws:

$$\begin{aligned}\mathbf{u} &= \{u_N(x_0), \ldots, u_1(x_{N-1})\} \\ \mathbf{w} &= \{w_N(x_0, u_0), \ldots, w_1(x_{N-1}, u_{N-1})\} \\ \mathbf{x} &= \{x_0, \ldots, x_N\}\end{aligned}$$

- Receding horizon control law at time t for N-step horizon:

$$u_t = u_N^*(x_t^p)$$

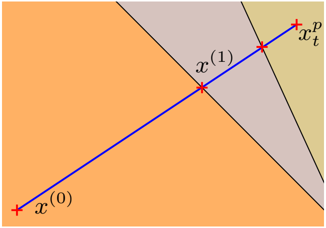
($x_t^p =$ plant state at t)

Active set solution

Outline of method:

- ▷ For a given active set, use Riccati recursions to solve KKT conditions
  $\Longrightarrow$ sequence of optimal control laws and worst case disturbances

- ▷ Line search through polyhedral partitions of x-space



- ▷ Update the set of active constraints

Problem A: 1st order optimality conditions

- Derive 1st order optimality (KKT) conditions
  by using the Wolfe dual to combine Lagrangian functions for $k = N-1, \ldots, 1, 0$
  assuming convexity

- kth stage subproblem ($k = N - m$)
  - ▷ Inequality constraints

$$\begin{array}{rcll} & & \text{Primal} & \text{Dual} \\ u \in \mathcal{U} & \Longleftrightarrow & Fu \leq \mathbf{1} & \mu \geq 0 \\ w \in \mathcal{W} & \Longleftrightarrow & Gw \leq \mathbf{1} & \eta \geq 0 \\ Ax + Bu \in \mathcal{X}_{m-1} \ominus D\mathcal{W} & \Longleftrightarrow & E^{N-m}\hat{x} \leq \mathbf{1} & \nu \geq 0 \end{array}$$

  - ▷ Equality constraints — Multipliers

$$\begin{array}{ll} \hat{x} = Ax + Bu & \hat{\lambda} \\ x^{+} = \hat{x} + Dw & \lambda \\ C\hat{x} = \mathbf{1} & \zeta \\ \hat{C}x = -\mathbf{1} & \hat{\zeta} \end{array}$$

Problem A: 1st order optimality conditions

N-stage problem KKT conditions

$$\gamma^2 w_k + G^T \eta_k - D^T \lambda_k = 0$$

$$\eta_k \geq 0, \quad \eta_k^T (\mathbf{1} - G w_k) = 0, \quad \mathbf{1} - G w_k \geq 0$$

$$\hat{C}_k x_{k+1} = -\mathbf{1}$$

$$R u_k + F^T \mu_k + B^T \hat{\lambda}_k = 0$$

$$\nu_k \geq 0, \quad \nu_k^T (\mathbf{1} - E^k \hat{x}_{k+1}) = 0, \quad \mathbf{1} - E^k \hat{x}_{k+1} \geq 0$$

$$\mu_k \geq 0, \quad \mu_k^T (\mathbf{1} - F u_k) = 0, \quad \mathbf{1} - F u_k \geq 0$$

$$C_k \hat{x}_{k+1} = \mathbf{1}$$

$$\hat{x}_{k+1} = A x_k + B u_k, \quad x_{k+1} = \hat{x}_{k+1} + D w_k$$

$$\lambda_k = A^T \hat{\lambda}_{k+1} + Q x_{k+1} + \hat{C}_k^T \zeta_k$$

$$\hat{\lambda}_k = \lambda_k + {E^k}^T \nu_k + C_k^T \hat{\zeta}_k$$

for $k = 0, \ldots, N-1$

Initial and terminal conditions:

$$x_0 = x^p$$

$$\lambda_{N-1} = P x_N$$

Problem A: 1st order optimality conditions

N-stage problem KKT conditions for a given active set

$$\left.\begin{aligned} \gamma^2 w_k + G^T \eta_{a,k} - D^T \lambda_k = 0 \\ \eta_{a,k} \geq 0, \quad \mathbf{1} - G_k w_k = 0 \\ \hat{C}_k x_{k+1} = -\mathbf{1} \end{aligned}\right\} \implies w^*_{N-k}(x_k, u_k)$$

$$\left.\begin{aligned} R u_k + F^T \mu_{a,k} + B^T \hat{\lambda}_k = 0 \\ \nu_{a,k} \geq 0, \quad \mathbf{1} - E_k \hat{x}_{k+1} = 0 \\ \mu_{a,k} \geq 0, \quad \mathbf{1} - F u_{a,k} = 0 \\ C_k \hat{x}_{k+1} = \mathbf{1} \end{aligned}\right\} \implies u^*_{N-k}(x_k)$$

$$\left.\begin{aligned} \hat{x}_{k+1} = A x_k + B u_k, \quad x_{k+1} = \hat{x}_{k+1} + D w_k \\ \lambda_k = A^T \hat{\lambda}_{k+1} + Q x_{k+1} + \hat{C}_k^T \zeta_k \\ \hat{\lambda}_k = \lambda_k + {E_k}^T \nu_{a,k} + C_k^T \hat{\zeta}_k \end{aligned}\right\} \implies \begin{aligned} &\lambda_k(x_{k+1}), \\ &\hat{\lambda}_k(\hat{x}_{k+1}) \end{aligned}$$

for $k = 0, \ldots, N-1$

Initial and terminal conditions:

$$x_0 = x^p$$

$$\lambda_{N-1} = P x_N$$

Problem A: Riccati recursion

State/co-state dependence: $$\left.\begin{aligned} \lambda_k &= P_k x_{k+1} + q_k \\ \hat{\lambda}_k &= \hat{P}_k \hat{x}_{k+1} + \hat{q}_k + E_k^T \nu_{a,k} \end{aligned}\right\} k = 0, \ldots, N-1$$

- kth stage maximization:

$$\begin{bmatrix} \gamma^2 I - D^T P_k D & G_k^T \\ G_k & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \eta_{a,k} \end{bmatrix} = \begin{bmatrix} D^T P_k \\ 0 \end{bmatrix} \hat{x}_{k+1} + \begin{bmatrix} D^T q_k \\ \mathbf{1} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} w_k \\ \eta_{a,k} \end{bmatrix} = \begin{bmatrix} M_k^w \\ M_k^\eta \end{bmatrix} \hat{x}_{k+1} + \begin{bmatrix} m_k^w \\ m_k^\eta \end{bmatrix}$$

giving $\begin{bmatrix} \hat{P}_k & \hat{q}_k \end{bmatrix} = \begin{bmatrix} P_k & q_k \end{bmatrix} + P_k D \begin{bmatrix} M_k^w & m_k^w \end{bmatrix}$

Problem A: Riccati recursion

State/co-state dependence: $$\left.\begin{aligned} \lambda_k &= P_k x_{k+1} + q_k \\ \hat{\lambda}_k &= \hat{P}_k \hat{x}_{k+1} + \hat{q}_k + E_k^T \nu_{a,k} \end{aligned}\right\} k = 0, \dots, N-1$$

- kth stage minimization:

$$\begin{bmatrix} R + B^T \hat{P}_k B & B^T E_k^T & F_k^T \\ E_k B & 0 & 0 \\ F_k & 0 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ \nu_{a,k} \\ \mu_{a,k} \end{bmatrix} = - \begin{bmatrix} B^T \hat{P}_k \\ E_k \\ 0 \end{bmatrix} A x_k + \begin{bmatrix} -B^T \hat{q}_k \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} u_k \\ \nu_{a,k} \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} L_k^u \\ L_k^\nu \\ L_k^\mu \end{bmatrix} x_k + \begin{bmatrix} l_k^u \\ l_k^\nu \\ l_k^\mu \end{bmatrix}$$

giving $\begin{bmatrix} P_{k-1} & q_{k-1} \end{bmatrix} = \begin{bmatrix} Q + A^T \hat{P}_k A & A^T \hat{q}_k \end{bmatrix} + A^T \begin{bmatrix} \hat{P}_k B & E_k^T \end{bmatrix} \begin{bmatrix} L_k^u & l_k^u \\ L_k^\nu & l_k^\nu \end{bmatrix}$

Problem A: Riccati recursion

State/co-state dependence:
$$\left.\begin{aligned} \lambda_k &= P_k x_{k+1} + q_k \\ \hat{\lambda}_k &= \hat{P}_k \hat{x}_{k+1} + \hat{q}_k + E_k^T \nu_{a,k} \end{aligned}\right\} k = 0, \ldots, N-1$$

- ▷ Introduce equality (compatibility) constraints into preceding stage whenever constraints are degenerate
- ▷ Set $\zeta = 0$, $\hat{\zeta} = 0$ to preserve continuity

Problem A: active set method

- Forward simulation: $x^+ = Ax_k + Bu^*_{N-k}(x_k) + Dw^*_{N-k}(x_k, u^*_{N-k}(x_k))$

$$\Downarrow$$

$$x_k = \Phi_k x_0 + \phi_k$$

- For given active set $\mathcal{A}$,

  $\mathbf{p}(x_0, \beta_0, \mathcal{A}) = \left\{\mathbf{u}(x_0, \mathcal{A}), \mathbf{w}(x_0, \mathcal{A}), \boldsymbol{\mu}(x_0, \beta_0, \mathcal{A}), \boldsymbol{\nu}(x_0, \beta_0, \mathcal{A}), \boldsymbol{\eta}(x_0, \beta_0, \mathcal{A})\right\}$

  is affine in x_0 (and possibly β_0)

- Define $\mathcal{X}(\mathcal{A}) = \{x_0 : \mathcal{A}^*(x_0) = \mathcal{A},\ \text{for some } \beta_0\}$, then
  - $\mathcal{X}(\mathcal{A})$ is convex and polyhedral
  - $\bigcup_{\mathcal{A}\in\Omega} \mathcal{X}(\mathcal{A})$ covers the set of feasible initial conditions, $\mathcal{X}_N$

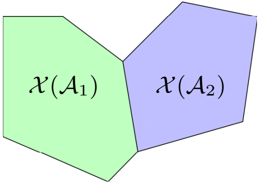
Problem A: geometric properties of solution

(a) If $\mathcal{A}_1, \mathcal{A}_2$ are non-degenerate, then

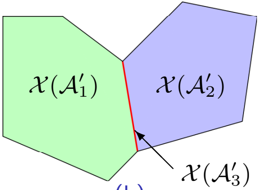
- $\star$ $\dim\big(\mathcal{X}(\mathcal{A}_i)\big) = n_x \quad i = 1, 2$
- $\star$ $\mathbf{p}(x_0, \mathcal{A}_1) = \mathbf{p}(x_0, \mathcal{A}_2) \quad \forall x_0 \in \partial\mathcal{X}(\mathcal{A}_1) \cap \partial\mathcal{X}(\mathcal{A}_2)$

(b) If $\mathcal{A}_1', \mathcal{A}_2'$ are non-degenerate and $\mathcal{A}_3'$ is degenerate at stage $k = 0$, then

- $\star$ $\dim\big(\mathcal{X}(\mathcal{A}_3')\big) < n_x$
- $\star$ $\mathcal{X}(\mathcal{A}_3') \subset \partial\mathcal{X}(\mathcal{A}_1') \cap \partial\mathcal{X}(\mathcal{A}_2')$
- $\star$ $\mathbf{p}(x_0, \beta_-, \mathcal{A}_3) = \mathbf{p}(x_0, \mathcal{A}_1)$ and $\mathbf{p}(x_0, \beta_+, \mathcal{A}_3) = \mathbf{p}(x_0, \mathcal{A}_2)$



(a)



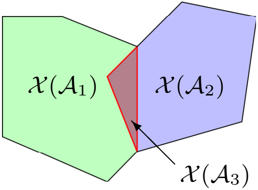
(b)

$\Longrightarrow$ $\mathbf{p}$ is continuous in x_0, β_0

Problem A: geometric properties of solution

(c) If $\mathcal{A}_1, \mathcal{A}_2$ are non-degenerate and $\mathcal{A}_3$ is degenerate at $k > 0$, then

- $\star$ $\dim\big(\mathcal{X}(\mathcal{A}_i)\big) = n_x \quad i = 1, 2, 3$
- $\star$ $\mathcal{X}(\mathcal{A}_3) \subset \mathcal{X}(A_1) \cap \mathcal{X}(A_2)$
- $\star$ regions overlap so $\mathcal{A}^*(x_0)$ non-unique for $x_0 \in \mathcal{X}(\mathcal{A}_3)$

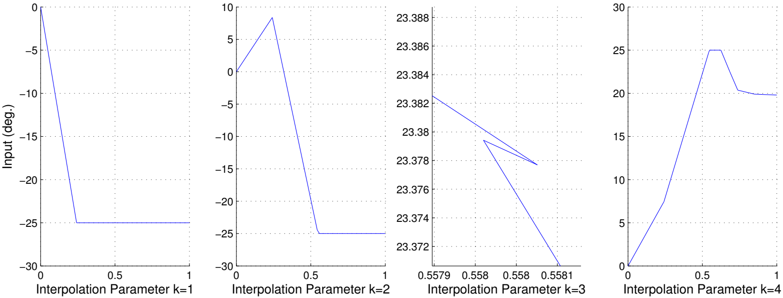


(c)

but $\mathbf{p}$ is still continuous in x_0

Problem A: example of degeneracy

Example: variation of $\mathbf{u} = \{u_0, u_1, u_2, u_3\}$ with linesearch parameter



degenerate constraints at $k = 3$

Problem A: active set algorithm

Initialize with $x_0^{(0)}$ and $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathcal{A}^{(0)})$.

At iteration $i = 0, 1, \ldots$:

1. backwards recursion

$k = 0$ $\qquad\qquad$ $k = N - 1$

$\hat{P}_k, \hat{q}_k$

w_k, η_k

P_k, q_k

u_k, μ_k, ν_k

$P_N = P$

$q_N = 0$

computation: $\quad O(N(6n_x^3 + an_w^3 + bn_u^3))$

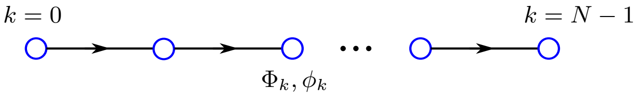
a, b: constants depending on QP subproblem factorization

Problem A: active set algorithm

Initialize with $x_0^{(0)}$ and $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathcal{A}^{(0)})$.

At iteration $i = 0, 1, \ldots$:

2. forward simulation



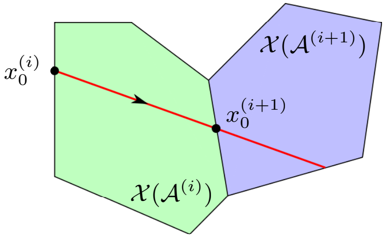
computation: $O(Nn_x^2)$

Problem A: active set algorithm

Initialize with $x_0^{(0)}$ and $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathcal{A}^{(0)})$.

At iteration $i = 0, 1, \ldots$:

3. perform line search & update $\mathcal{A}^{(i)}$



inequalities: $O(N)$

$\Downarrow$

Overall complexity is $O(N)$ per iteration,

but bounds on number of iterations could be large

Problem A: active set algorithm

- Initialization
  - ▷ cold start: $x_0^{(0)} = 0$, $\mathcal{A}^{(0)} = \emptyset$
  - ▷ warm start: $x_0^{(0)} = x_1^*$ at time $t-1$, $\mathcal{A}^{(0)}$ computed from $\mathcal{A}^*(x_{t-1}^p)$

- Convergence: $\mathcal{A}^{(i)} = \mathcal{A}^*(x^p)$ in finite number of iterations

  follows from
  - ▷ uniqueness of transition $\mathcal{A}^{(i)} \to \mathcal{A}^{(i+1)}$ at each iteration
  - ▷ finite number of admissible active sets $\mathcal{A}$

- Closed loop stability: $u_t = u_N^*(x_t)$ ensures
  - ▷ $\mathcal{X}_N$ robustly invariant
  - ▷ l^2-gain bound:

$$\sum_{t=0}^{\infty} (\|x_t\|_Q^2 + \|u_t\|_R^2) \leq \gamma^2 \sum_{t=0}^{\infty} \|w_t\|^2 + 2J_N^*(x_0)$$

Problem A: example

Computation and performance of:

DP active set algorithm

vs

Disturbance Affine (DA) feedback policy [Goulart, Kerrigan, Alamo 2009]

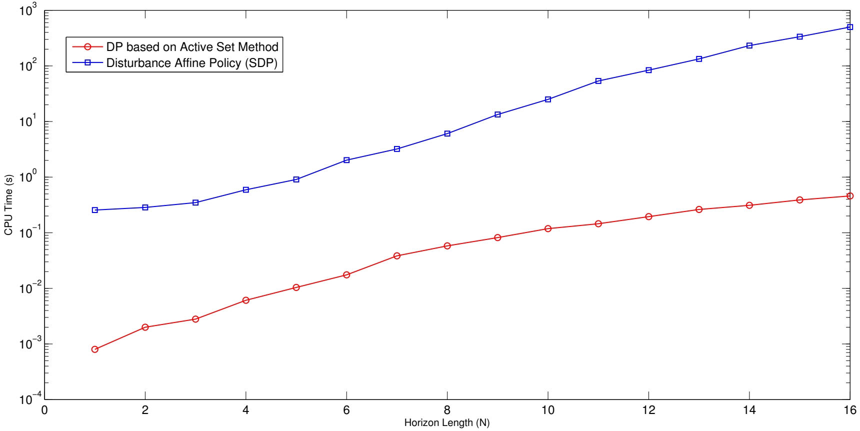
(i) double integrator plant model: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

with $\mathcal{U} = \{u \in \mathbb{R} : -1 \leq u \leq 1\}$
$\mathcal{W} = \{w \in \mathbb{R}^2 : -0.3 \leq w_i \leq 0.3 \text{ for } i = 1, 2\}$,

(ii) 4th order plant (aircraft pitch dynamics) with input and state constraints
disturbance: uncertainty in wing and elevator lift forces

Problem A: example

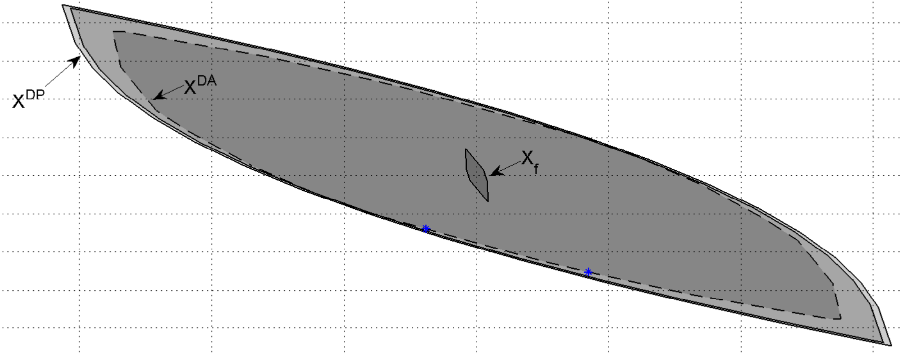
Double integrator: Execution time of DP active set algorithm and DA policy



- ▷ average data for 50 plant states $x^p \in \mathcal{X}_N - \mathcal{X}_{N-1}$
- ▷ DP active set is around 100 times faster despite being optimal

Problem A: example

Region of attraction of DP and DA policies with $N = 16$

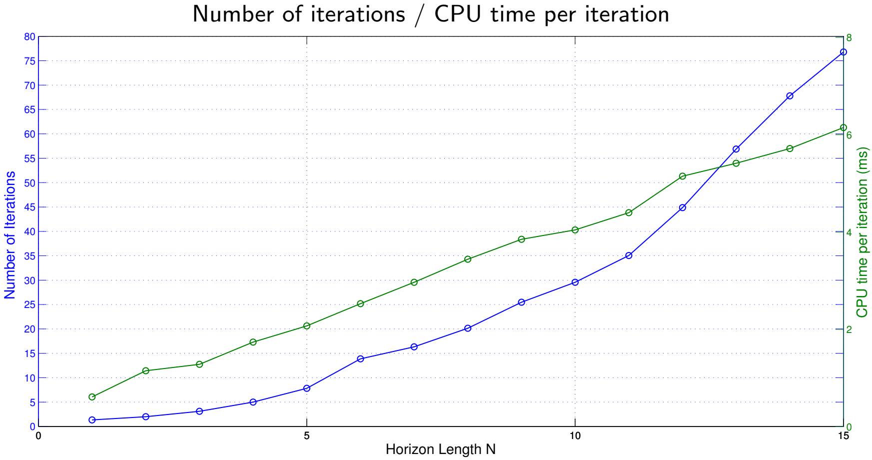


Predicted cost for 50 plant states on $\partial\mathcal{X}_{16}^{DA}$:

- ▷ average suboptimality of DA policy: 4.2%
- ▷ maximum suboptimality of DA policy: 22%

Problem A: example

4th order plant (aircraft pitch dynamics) with input and state constraints:



- ▷ average data for 50 plant states $x^p \in \mathcal{X}_N - \mathcal{X}_{N-1}$

Problem B: solution structure

Maximization problem is NP-hard:

$$\max_{w\in\mathcal{W}} V_m(x,u,w) = \max_{w\in\mathcal{W}} \|E_k(Ax+Bu+Dw)+e_k\|_\infty$$

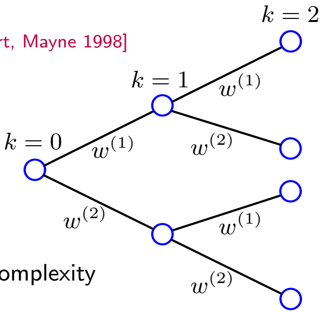
- If $\mathcal{W} = \mathrm{co}\{w^{(j)},\ j=1,\dots,n_\mathcal{W}\}$, then

$$w^{(j_k^*)} = \arg\max_{w\in\mathcal{W}} \|E_k(Ax+Bu+Dw)+e_k\|_\infty$$

$$j_k^* = \arg\max_{j=1,\dots,n_\mathcal{W}} \|E_k D w^{(j)}\|_\infty$$

- results in a tree of predictions [Scokaert, Mayne 1998]

  e.g. for $n_\mathcal{W}=2$, $N=3$:



- ...leading to unavoidable limits on complexity

Problem B: continuity and uniqueness of solution

Equivalent maximization problem at stage k:

$$\begin{aligned}\max_{w\in\mathcal{W}} V_m(x,u,w) &= \max_{j=1,\ldots,n_{\mathcal{W}}} \left\{ E_k^{(j)}(Ax+Bu) + e_k^{(j)} \right\} \\ &= E_k^{(j_k^*)}(Ax+Bu) + e_k^{(j_k^*)}\end{aligned}$$

▷ min-max problem is a parametric LP:

$$\begin{aligned} V_m^*(x) = \min_{\alpha,u} \quad & \alpha \\ \text{subject to} \quad & \alpha \geq E_k^{(j)}(Ax+Bu) + e_k^{(j)},\ j = 1,\ldots n_{\mathcal{W}} \\ & Fu \leq \mathbf{1} \end{aligned}$$

▷ hence: $u_m^*(x)$ is continuous and piecewise affine
$V_m^*(x)$ is continuous, convex and piecewise affine

[Gal 1995; Bemporad, Borrelli, Morari, 2003]

$\Downarrow$

active set regions $\mathcal{X}(\mathcal{A})$ are non-overlapping

Problem B: 1st order optimality conditions

- KKT system for kth stage subproblem:

Primal:
$$\begin{bmatrix} \mathbf{1} & -E \\ 0 & F \end{bmatrix} \begin{bmatrix} \alpha_k \\ u_k \end{bmatrix} = \begin{bmatrix} EA \\ 0 \end{bmatrix} x + \begin{bmatrix} e \\ \mathbf{1} \end{bmatrix}$$

Dual:
$$\begin{bmatrix} \mathbf{1}^T & 0 \\ B^T E^T & F^T \end{bmatrix} \begin{bmatrix} \nu_{a,k} \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - ▷ primal and dual problems linked only by active set
  - ▷ dual variables are piecewise constant functions of x_0

- $E_{N-1} = E^f$, $e_{N-1} = \max_j E^f D w^{(j)}$ (row-wise max)

  E_{k-1}, e_{k-1} obtained from $\alpha_k(x)$ $\implies$ backwards recursion along path within tree

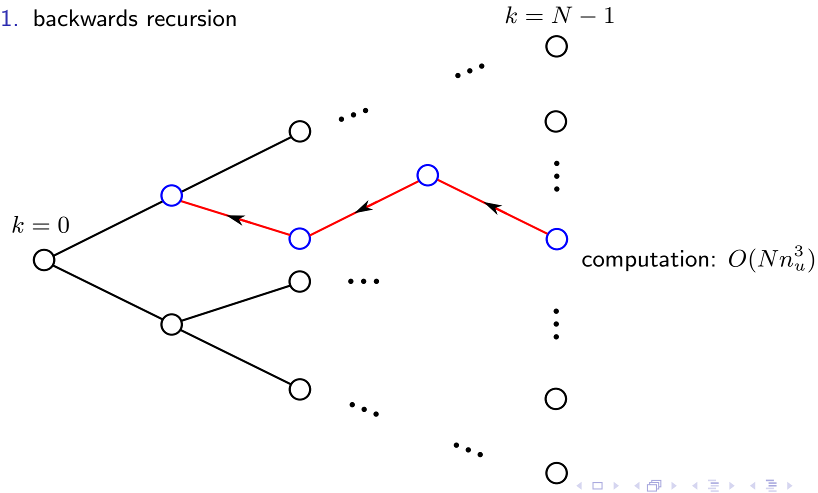
- Forwards simulation computes $\alpha_k(x_0), u_k(x_0)$ at all nodes in a sub-tree

Problem B: active set algorithm

Initialize with $x_0^{(0)}$ and $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathcal{A}^{(0)})$.

At iteration $i = 0, 1, \ldots$:

1. backwards recursion

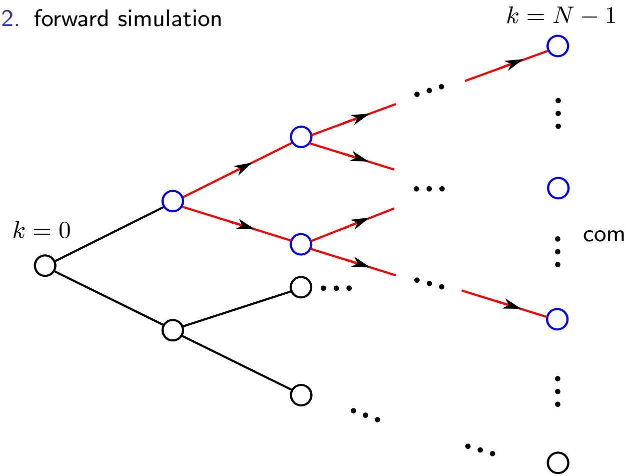


Problem B: active set algorithm

Initialize with $x_0^{(0)}$ and $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathcal{A}^{(0)})$.

At iteration $i = 0, 1, \ldots$:

2. forward simulation



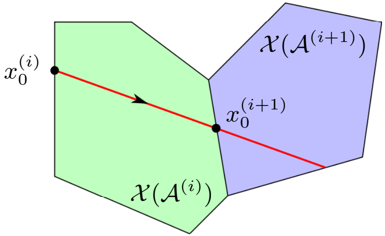
computation: $O(n_{\mathcal{W}}^{N-1} n_x^2)$
or $O(N n_x^2)$ (parallel)

Problem B: active set algorithm

Initialize with $x_0^{(0)}$ and $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathcal{A}^{(0)})$.

At iteration $i = 0, 1, \ldots$:

3. perform line search & update $\mathcal{A}^{(i)}$



\# inequalities: $O(n_{\mathcal{W}}^{N-1})$

Overall complexity is $O(Nn_u^3 + n_{\mathcal{W}}^{N-1} n_x^2)$ per iteration

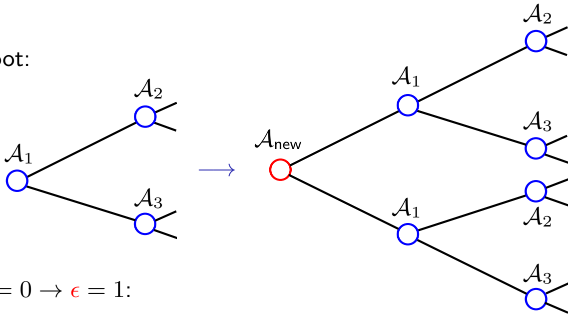
or $O\big(N(n_u^3 + n_x^2)\big)$ per iteration using parallel processing

- ▷ number of matrix inversions required per iteration is linear in N
- ▷ number of constraints grows exponentially with N

Problem B: growing the prediction tree

- Optimal terminal feedback law is not available in closed form hence initialization (cold start) requires tree growth

- Grow tree from root:



  ▷ linesearch $\epsilon = 0 \to \epsilon = 1$:

$$u^*_{N+1}(\epsilon) = \arg\min_{u \in \epsilon\mathcal{U}} \max_{w \in \epsilon\mathcal{W}} E_1(\hat{x}_1 + Dw)$$

  with $\hat{x}_1 = x_0 + \epsilon(A - I)x_0 + Bu$ and $x_0 =$ constant

  ▷ $\mathcal{A}_{\text{new}}$ obtained by inspection for $\epsilon = 0$

Problem B: growing and shrinking the prediction tree

- During linesearch over x_0,

$$\text{if } V_N^*(x_0) \geq 1, \quad \text{then: } \begin{array}{l} \text{set } N := N+1 \\ \text{grow the tree} \\ \text{and perform linesearch } \epsilon = 0 \to \epsilon = 1 \end{array}$$

$$\Downarrow$$

$$u_N^*(x) = \text{optimal minimum-time control law}$$

- Minimum-time optimal shrinking of tree requires computation of $V_{N-1}(x_0)$

- Shrink the tree suboptimally during linesearch over x_0:

$$\text{if } V_N^*(x_0) < v_{\min}, \quad \text{then: } \begin{array}{l} \text{perform linesearch } \epsilon = 1 \to \epsilon = 0 \\ \text{prune the tree} \\ \text{and set } N := N-1 \end{array}$$

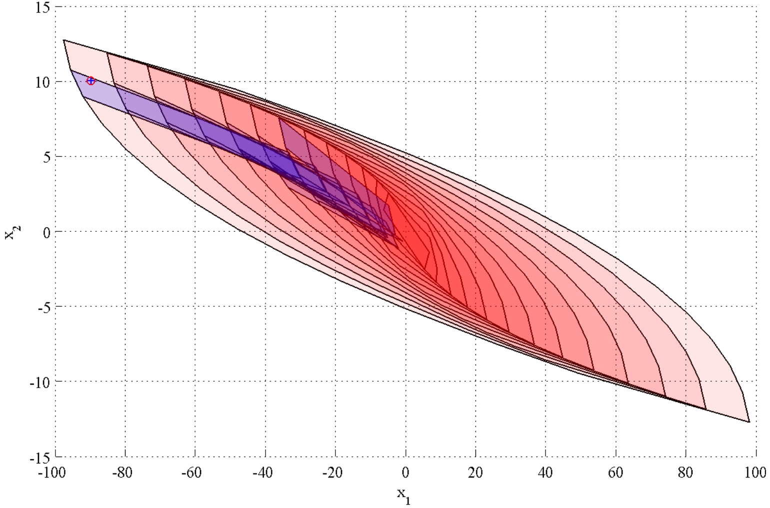
where $V_N^*(x) < v_{\min} \implies x \in \mathcal{X}_{N-1}$

Problem B: example

Double integrator with input and terminal constraints:

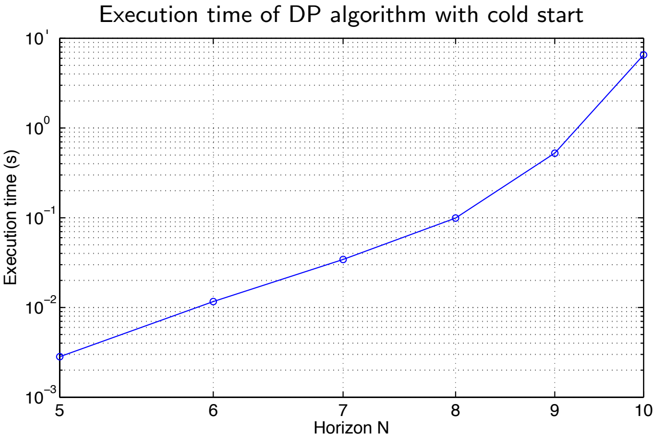
k-steps controllable sets to $\mathcal{X}^f$, $k = 1, \ldots, 12$ (red)

active set regions $\mathcal{X}(\mathcal{A}^{(i)})$ at each iteration of DP algorithm (blue)



Problem B: example

4th order plant (aircraft pitch dynamics) with input and state constraints



- ▷ average times for 30 plant states $x^p \in \mathcal{X}_N - \mathcal{X}_{N-1}$
- ▷ forward simulation <u>not</u> implemented using parallel processing

Conclusions

- Active set method for DP based on homotopy of optimal trajectories
- Quadratic $\mathcal{H}_\infty$ cost:
  - computational load per iteration depends linearly on horizon N

    overall load approx. quadratic in N
  - requires knowledge of robust controllable k-step sets, $k = 1, \ldots, N$
- Piecewise linear cost:
  - enables local exploration of robust controllable sets
  - number of matrix inversions per iteration depends linearly on N
  - number of constraints in line search grows exponentially with N
- Future work: multiplicative uncertainty & stochastic problems