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Stochastic Model Predictive Control

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Abstract Model Predictive Control (MPC) is a control strategy that has been used successfully in numerous and diverse application areas. The aim of the present article is to discuss how the basic ideas of MPC can be extended to problems involving random model uncertainty with known probability distribution. We discuss cost indices, constraints, closed loop properties and implementation issues.

1 Introduction

Stochastic Model Predictive Control (SMPC) refers to a family of numerical optimization strategies for controlling stochastic systems subject to constraints on the states and inputs of the controlled system. In this approach, future performance is quantified using a cost function evaluated along predicted state and input trajectories. This leads to a stochastic optimal control problem, which is solved numerically to determine an optimal open-loop control sequence or alternatively a sequence of feedback control laws. In MPC, only the first element of this optimal sequence is applied to the controlled system, and the optimal control problem is solved again at the next sampling instant on the basis of updated information on the system state. The numerical nature of the approach makes it applicable to systems with nonlinear dynamics and constraints on states and inputs, while the repeated computation of optimal predicted trajectories introduces feedback to compensate for the effects of uncertainty in the model.

Robust MPC (RMPC) tackles problems with hard state and input constraints, which are to be satisfied for all realizations of model uncertainty. However RMPC is too conservative in many applications and Stochastic MPC (SMPC) provides less conservative solutions by handling a wider class of constraints which are to be satisfied in mean or with a specified probability. This is achieved by taking explicit account of the probability distribution of the stochastic model uncertainty in the optimization of predicted performance. Constraints limit performance and an advantage of MPC is that it allows systems to operate close to constraint boundaries. Stochastic MPC is similarly advantageous when model uncertainty is stochastic with known probability distribution and the constraints are probabilistic in nature.

Applications of SMPC have been reported in diverse fields, including finance and portfolio management, risk management, sustainable development policy assessment, chemical and process industries, electricity generation and distribution, building climate control and telecommunications network traffic control. This article aims to summarise the theoretical framework underlying SMPC algorithms.

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2 Stochastic MPC

Consider a system with discrete time model

$$x^+ = f(x,u,w) \tag{1}$$
$$z = g(x,u,v) \tag{2}$$

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$ are the system state and control input, and x^+ is the successor state (i.e. if x_i is the state at time i, then $x^+ = x_{i+1}$ is the state at time $i{+}1$). Inputs $w \in \mathbb{R}^{n_w}$ and $v \in \mathbb{R}^{n_v}$ are exogenous disturbances with unknown current and future values but known probability distributions, and $z \in \mathbb{R}^{n_z}$ is a vector of output variables that are subject to constraints.

The optimal control problem that is solved online at each time step in SMPC is defined in terms of a performance index $J_N(x, \hat{\mathbf{u}}, \hat{\mathbf{w}})$ evaluated over a future horizon of N time steps. Typically in SMPC $J_N(x, \hat{\mathbf{u}}, \hat{\mathbf{w}})$ is a quadratic function of the following form (in which $\|x\|_Q^2 = x^T Q x$)

$$J_N(x, \hat{\mathbf{u}}, \hat{\mathbf{w}}) = \sum_{i=0}^{N-1} (\|\hat{x}_i\|_Q^2 + \|\hat{u}_i\|_R^2) + V_f(\hat{x}_N) \quad (3)$$

for positive definite matrices Q and R, and a terminal cost $V_f(x)$ defined as discussed in Section 4. Here $\hat{\mathbf{u}} := \{\hat{u}_0, \ldots, \hat{u}_{N-1}\}$ is a postulated sequence of control inputs and $\hat{\mathbf{x}}(x, \hat{\mathbf{u}}, \hat{\mathbf{w}}) := \{\hat{x}_0, \ldots, \hat{x}_N\}$ is the corresponding sequence of states such that $\hat{x}_i$ is the solution of (1) at time i with initial state $\hat{x}_0 = x$, for a given sequence of disturbance inputs $\hat{\mathbf{w}} := \{\hat{w}_0, \ldots, \hat{w}_{N-1}\}$. Since $\hat{\mathbf{w}}$ is a random sequence, $J_N(x, \hat{\mathbf{u}}, \hat{\mathbf{w}})$ is a random variable, and the optimal control problem is therefore formulated as the minimization of a cost $V_N(x, \hat{\mathbf{u}})$ derived from $J_N(x, \hat{\mathbf{u}}, \hat{\mathbf{w}})$ under specific assumptions on $\hat{\mathbf{w}}$. Common definitions of $V_N(x, \hat{\mathbf{u}})$ are as follows.

(a) Expected value cost:

$$V_N(x, \hat{\mathbf{u}}) := \mathbb{E}_x\big(J(x, \hat{\mathbf{u}}, \hat{\mathbf{w}})\big)$$

where $\mathbb{E}_x(\cdot)$ denotes the conditional expectation of a random variable $(\cdot)$ given the model state x.

(b) Worst-case cost, assuming $\hat{w}_i \in \mathcal{W}$ for all i with probability 1, for some compact set $\mathcal{W} \subset \mathbb{R}^{n_w}$:

$$V_N(x, \hat{\mathbf{u}}) := \max_{\hat{\mathbf{w}} \in \mathcal{W}^N} J(x, \hat{\mathbf{u}}, \hat{\mathbf{w}}).$$

(c) Nominal cost, assuming $\hat{w}_i$ is equal to some nominal value, e.g. if $\hat{w}_i = 0$ for all i, then

$$V_N(x, \hat{\mathbf{u}}) := J(x, \hat{\mathbf{u}}, \mathbf{0}),$$

where $\mathbf{0} = \{0, \ldots, 0\}$.

The minimization of $V_N(x, \hat{\mathbf{u}})$ is performed subject to constraints on the sequence of outputs $\hat{z}_i := g(\hat{x}_i, \hat{u}_i, \hat{v}_i)$, $i \geq 0$. These constraints may be formulated in various ways, summarised as follows, where for simplicity we assume $n_z = 1$.

(A) Expected value constraints: for all i,

$$\mathbb{E}_x(\hat{z}_i) \leq 1.$$

(B) Probabilistic constraints pointwise in time:

$$\Pr_x(\hat{z}_i \leq 1) \geq p,$$

for all i and for a given probability p.

(C) Probabilistic constraints over a future horizon:

$$\Pr_x(\hat{z}_i \leq 1,\ i = 0, 1, \ldots, N) \geq p$$

for a given probability p.

In (B) and (C), $\Pr_x(\mathcal{A})$ represents the conditional probability of an event $\mathcal{A}$ that depends on the sequence $\hat{\mathbf{x}}(x, \hat{\mathbf{u}}, \hat{\mathbf{w}})$, given that the initial model state is $\hat{x}_0 = x$; for example the probability $\Pr_x(\hat{z}_i \leq 1)$ depends on the distribution of $\{\hat{w}_0, \ldots, \hat{w}_{i-1}, \hat{v}_i\}$.

The important special case of state constraints can also be handled by (A)-(C) through appropriate choice of the function $g(x, u, v)$. For example the constraint $\Pr_x\big(h(x) \leq 1\big) \geq p$, for a given function $h : \mathbb{R}^n \to \mathbb{R}$, can be expressed in the form (B) with $z = g(x, u, v) := h\big(f(x, u, w)\big)$ and $v := w$ in (2).

In common with other receding horizon control strategies, SMPC is implemented via the following **Algorithm**. At each discrete time-step:

(i) Minimize the cost index $V_N(x, \hat{\mathbf{u}})$ over $\hat{\mathbf{u}}$ subject to the constraints on $\hat{z}_i$, $i \geq 0$, given the current system state x.

(ii) Apply the control input $u = \hat{u}_0^*(x)$ to the system, where $\hat{\mathbf{u}}^*(x) = \{\hat{u}_0^*(x), \ldots, \hat{u}_{N-1}^*(x)\}$ is the minimizing sequence given x.

If the system dynamics (1) are unstable, then performing the optimization in step (i) directly over future control sequences can result in a small set of feasible states x. To avoid this difficulty the elements of the control sequence $\hat{\mathbf{u}}$ are usually expressed in the form $\hat{u}_i = u_T(\hat{x}_i) + s_i$, where $u_T(x)$ is a locally stabilizing feedback law, and $\{s_0, \ldots, s_{N-1}\}$ are optimization variables in step (i).

3 Constraints and recursive feasibility

The constraints in (B) and (C) include hard constraints ($p = 1$) as a special case, but in general the conditions (A)-(C) represent soft constraints that are not required to hold for all realizations of model uncertainty. However, these constraints can only be satisfied if the state belongs to a subset of state space, and the requirement (common in MPC) that the optimization in step (i) of the SMPC algorithm should remain feasible if it is initially feasible therefore implies additional constraints. For example the condition $\Pr_x(\hat{z}_0 \leq 1) \geq p$ can be satisfied only if x belongs to the set for which there exists $\hat{u}_0$ such that $\Pr_x\big(g(x, \hat{u}_0, \hat{v}_0) \leq 1\big) \geq p$. Hence soft constraints implicitly impose hard constraints on the model state.

SMPC algorithms typically handle the conditions relating to feasibility of constraint sets in one of two ways. Either the SMPC optimization is allowed to become infeasible (often with penalties on constraint violations included in the cost index), or conditions ensuring robust feasibility of the SMPC optimization at all future times are imposed as extra constraints in the SMPC optimization.

The first of these approaches has been used in the context of constraints (C) imposed over a horizon, for which conditions ensuring future feasibility are generally harder to characterise in terms of algebraic conditions on the model state than (A) or (B). A disadvantage of this approach is that the closed loop system may not satisfy the required soft constraints, even if these constraints are feasible when applied to system trajectories predicted at initial time.

The second approach treats conditions for feasibility as hard constraints and hence requires a guarantee of recursive feasibility, namely that the SMPC optimization must remain feasible for the closed loop system if it is feasible initially. This can be achieved by requiring, similarly to RMPC, that the conditions for feasibility of the SMPC optimization problem should be satisfied for all realizations of the sequence $\hat{\mathbf{w}}$. For example, for given $\hat{x}_0 = x$, there exists $\hat{\mathbf{u}}$ satisfying that the conditions of (B) if

$$\Pr_{\hat{x}_i}(g(\hat{x}_i, \hat{u}_i, \hat{v}_i) \leq 1) \geq p,\ i = 0, 1, \dots \tag{4a}$$

$$\hat{x}_i \in X\ \forall \{\hat{w}_0, \dots, \hat{w}_{i-1}\} \in \mathcal{W}^i,\ i = 1, 2, \dots \tag{4b}$$

where X is the set

$$X = \{x : \exists u \text{ such that } \Pr_x\big(g(x, u, v) \leq 1\big) \geq p\}.$$

Furthermore an SMPC optimization that includes the constraints of (4) must remain feasible at subsequent times (since (4) ensures the existence of $\hat{\mathbf{u}}^+$ such that each element of $\hat{\mathbf{x}}\big(f(x, \hat{u}_0, \hat{w}_0), \hat{\mathbf{u}}^+, \hat{\mathbf{w}}^+\big)$ lies in X for all $\hat{w}_0 \in \mathcal{W}$ and all $\hat{\mathbf{w}}^+ \in \mathcal{W}^N$).

Satisfaction of (4) at each time-step i on the infinite horizon $i \geq N$ can be ensured through a finite number of constraints by introducing constraints on the N step-ahead state $\hat{x}_N$. This approach uses a fixed feedback law, $u_T(x)$, to define a postulated input sequence after the initial N-step horizon via $\hat{u}_i = u_T(\hat{x}_i)$ for all $i \geq N$. The constraints of (4) are necessarily satisfied for all $i \geq N$ if a constraint

$$\hat{x}_N \in X_T$$

is imposed, where X_T is robustly positively invariant with probability 1 under $u_T(x)$, i.e.

$$f(x, u_T(x), w) \in X_T,\ \forall x \in X_T,\ \forall w \in \mathcal{W}, \tag{5}$$

and furthermore the constraint $\Pr_x(z \leq 1) \geq p$ is satisfied at each point in X_T under $u_T(x)$, i.e.

$$\Pr_x(g(x, u_T(x), v) \leq 1) \geq p,\ \forall x \in X_T.$$

Although the recursively feasible constraints (4) account robustly for the future realizations of the unknown parameter w in (1), the key difference between SMPC and RMPC is that the conditions in (4) depend on the probability distribution of the parameter v in (2). It also follows from the necessity of hard constraints for feasibility that the distribution of w must in general have finite support in order that feasibility can be guaranteed recursively. On the other hand the support of v in the definition of z may be unbounded (an important exception being the case of state constraints in which $v = w$).

4 Stability and convergence

This section outlines the stability properties of SMPC strategies based on cost indices (a)-(c) of Section 2 and related variants. We use $V_N^*(x) = V_N(x, \hat{\mathbf{u}}^*(x))$ to denote the optimal value of the SMPC cost index, and X_T denotes a subset of state space satisfying the robust invariance condition (5). We also denote the solution at time i of the system (1) with initial state $x_0 = x$ and under a given feedback control law $u = \kappa(x)$ and disturbance sequence $\mathbf{w} = \{w_0, w_1, \dots\}$ as $x_i(x, \kappa, \mathbf{w})$.

The expected value cost index in (a) results in mean-square stability of the closed loop system provided the terminal term $V_f(x)$ in (3) satisfies

$$\mathbb{E}_x V_f\big(f(x, u_T(x), w)\big) \leq V_f(x) - \|x\|_Q^2 - \|u_T(x)\|_R^2$$

for all x in the terminal set X_T. The optimal cost is then a stochastic Lyapunov function satisfying

$$\mathbb{E}_x V_N^*\big(f(x, \hat{u}_0^*(x), w)\big) \leq V_N^*(x) - \|x\|_Q^2 - \|\hat{u}_0^*(x)\|_R^2.$$

For positive definite Q this implies the closed loop system under the SMPC law is mean-square stable, so that $x_i(x, \hat{u}_0^*, \mathbf{w}) \to 0$ as $i \to \infty$ with probability 1 for any feasible initial condition x. For the case of systems (1) subject to additive disturbances, the modified cost

$$V_N(x, \hat{\mathbf{u}}) := \mathbb{E}_x\left[\sum_{i=0}^{N-1} (\|\hat{x}_i\|_Q^2 + \|\hat{u}_i\|_R^2 - l_{ss}) + V_f(\hat{x}_N)\right]$$

where $l_{ss} := \lim_{i\to\infty} \mathbb{E}_x(\|x_i(x, u_T, \mathbf{w})\|_Q^2 + \|u_i\|_R^2)$ under $u_i = u_T(x_i)$, results in the asymptotic bound

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}_x(\|x_i(x, \hat{u}_0^*, \mathbf{w})\|_Q^2 + \|u_i\|_R^2) \leq l_{ss}$$

along the closed loop trajectories of (1) under the SMPC law $u_i = \hat{u}_0^*(x_i)$, for any feasible initial condition x.

For the worst-case cost (b), if $V_f(x)$ is designed as a control Lyapunov function for (1), with

$$V_f\big(f(x, u_T(x), w)\big) \leq V_f(x) - \|x\|_Q^2 - \|u_T(x)\|_R^2$$

for all $w \in \mathcal{W}$ and all $x \in X_T$, then $V_N^*(x)$ is a Lyapunov function satisfying

$$V_N^*(f(x, \hat{u}_0^*(x), w) \leq V_N^*(x) - \|x\|_Q^2 - \|\hat{u}_0^*(x)\|_R^2$$

for all $w \in \mathcal{W}$, implying $x = 0$ is an asymptotically stable equilibrium of (1) under the SMPC law $u = \hat{u}_0^*(x)$. Clearly the system model (1) cannot be subject to unknown additive disturbances in this case. However, for the case in which the system (1) is subject to additive disturbances, a variant of this approach uses a modified cost which is equal to zero inside some set of states, leading to asymptotic stability of this set rather than an equilibrium point. Also in the context of additive disturbances, an alternative approach uses an $\mathcal{H}_\infty$-type cost,

$$V_N(x, \hat{\mathbf{u}}) := \max_{\hat{\mathbf{w}} \in \mathcal{W}^N} \Bigg[\sum_{i=0}^{N-1} (\|\hat{x}_i\|_Q^2 + \|\hat{u}_i\|_R^2 - \gamma^2 \|\hat{w}_i\|^2) + V_f(\hat{x}_N) \Bigg]$$

for which the closed loop trajectories of (1) under the associated SMPC law $u_i = \hat{u}_0^*(x_i)$ satisfy

$$\sum_{i=0}^{\infty} (\|x_i(x, \hat{u}_0^*, \mathbf{w})\|_Q^2 + \|u_i\|_R^2) \leq \gamma^2 \sum_{i=0}^{\infty} \|w_i\|^2 + V_N^*(x_0)$$

provided $V_f\big(f(x, u_T(x), w)\big) \leq V_f(x) - (\|x\|_Q^2 + \|u_T(x)\|_R^2 - \gamma^2 \|w\|^2)$ for all $w \in \mathcal{W}$ and $x \in X_T$.

Algorithms employing the nominal cost (c) typically rely on the existence of a feedback law $u_T(x)$ such that the system (1) satisfies, in the absence of constraints and under $u_i = u_T(x_i)$, an input-to-state stability (ISS) condition of the form

$$\sum_{i=0}^{\infty} (\|x_i(x, u_T, \mathbf{w})\|_Q^2 + \|u_i\|_R^2) \leq \gamma^2 \sum_{i=0}^{\infty} \|w_i\|^2 + \beta \tag{6}$$

for some γ and $\beta > 0$. If $V_f(x)$ satisfies

$$V_f\big(f(x, u_T(x), 0)\big) \leq V_f(x) - (\|x\|_Q^2 + \|u_T(x)\|_R^2)$$

for all $x \in X_T$, then the closed loop system under SMPC with the nominal cost (c) satisfies an ISS condition with the same gain γ as the unconstrained case (6) but a different constant β.

5 Implementation issues

In general stochastic MPC algorithms require more computation than their robust counterparts because of the need to determine the probability distributions of future states. An important exception is the case of linear dynamics and purely additive disturbances, for which the model (1)-(2) becomes

$$x^+ = Ax + Bu + w \tag{7}$$

$$z = Cx + Du + v \tag{8}$$

where A, B, C, D are known matrices. In this case the expected value constraints (A) and probabilistic constraints (B), as well as hard constraints that ensure future feasibility of the SMPC optimization in each case can be invoked non-conservatively through tightened constraints on the expectations of future states. Furthermore the required degree of tightening can be computed offline using numerical integration of probability distributions or using random sampling techniques, and the online computational load is similar to MPC with no model uncertainty.

The case in which the matrices A, B, C, D in the model (7)-(8) depend on unknown stochastic parameters is more difficult because the predicted states then involve products of random variables. An effective approach to this problem uses a sequence of sets (known as a tube) to recursively bound the sequence of predicted states via one step-ahead set inclusion conditions. By using polytopic bounding sets that are defined as the intersection of a fixed number of half-spaces, the complexity of these tubes can be controlled by the designer, albeit at the expense of conservative inclusion conditions. Furthermore an application of Farkas' Lemma allows these sets to be computed online through linear conditions on optimization variables.

Random sampling techniques developed for general stochastic programming problems provide effective means of handling the soft constraints arising in SMPC. These techniques use finite sets of discrete samples to represent the probability distributions of model states and parameters. Furthermore bounds are available on the number of samples that are needed in order to meet specified confidence levels on the satisfaction of constraints. Probabilistic and expected value constraints can be imposed using random sampling, and this approach has also been applied to the case of probabilistic constraints over a horizon (C) through a scenario-based optimization approach.

6 Summary and Future directions

This article describes how the ideas of MPC and RMPC can be extended to the case of stochastic model uncertainty. Crucial in this development is the assumption that the uncertainty has bounded support, which allows the assertion of recursive feasibility of the SMPC optimization problem. For simplicity of presentation we have considered the case of full state feedback. However stochastic MPC can also be applied to the output feedback case using a state estimator if the probability distributions of measurement and estimation noise are known.

An area of future development is optimization over sequences of feedback policies. Although an observer at initial time cannot know the future realizations of random uncertainty, information on $\hat{x}_i$ will be available to the controller i-steps ahead, and, as mentioned in Section 2 in the context of feasible initial condition sets, $\hat{u}_i$ must therefore depend on $\hat{x}_i$. In general the optimal control decision is of the form $\hat{u}_i = \mu_i(\hat{x}_i)$ where $\mu_i(\cdot)$ is a feedback policy. This implies optimization over arbitrary feedback policies, which is generally considered to be intractable since the required online computation grows exponentially with the horizon N. However approximate approaches to this problem have been suggested which optimize over restricted classes of feedback laws, and further developments in this respect are expected in the future.

7 Cross-references

$\longrightarrow$ Nominal MPC
$\longrightarrow$ Robust MPC
$\longrightarrow$ Distributed MPC
$\longrightarrow$ Tracking MPC
$\longrightarrow$ Economic MPC

8 Recommended Reading

A historical perspective on SMPC is provided by Åström and Wittenmark (1973), Charnes and Cooper (1963) and Schwarm and Nikolaou (1999). A treatment of constraints stated in terms of expected values can be found for example in Primbs and Sung (2009). Probabilistic constraints and the conditions for recursive feasibility can be found in Kouvaritakis et al. (2010) for the additive case whereas the general case of multiplicative and additive uncertainty is described in Evans et al. (2012), which uses random sampling techniques. Random sampling techniques were developed for random convex programming (Calafiore and Campi, 2005), and were used in a scenario based approach to predictive control in Calafiore and Fagiano (2013). An output feedback SMPC strategy incorporating state estimation is described in Cannon et al. (2012).

The use of the expectation of a quadratic cost and associated mean square stability results are discussed in Lee and Cooley (1998). Robust stability results for MPC based on worst-case costs are given by Lee and Yu (1997) and Mayne et al. (2005). Input-to-state stability of MPC based on a nominal cost is discussed in Marruedo et al. (2002)

Descriptions of SMPC based on closed loop optimization can be found in Lee and Yu (1997) and Stoorvogel et al. (2007). These algorithms are computationally intensive and approximate solutions can be found by restricting the class of closed loop predictions as discussed for example in van Hessem and Bosgra (2002) and Primbs and Sung (2009).

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