Simple Homothetic Tube Model Predictive Control

Saša V. Raković, Basil Kouvaritakis, Rolf Findeisen and Mark Cannon

Abstract—This paper considers the robust model predictive control synthesis problem for constrained linear discrete time systems. The manuscript introduces a simple homothetic tube model predictive control synthesis method. The proposed method employs several novel features including: a more general parameterization of the state and control tubes based on homothety and invariance; a more flexible form of the terminal constraint set; and a relaxation of the dynamics of the sets that define the state and control tubes. Under rather mild assumptions it is demonstrated that the proposed method is computationally efficient while it induces strong system theoretic properties.

I. INTRODUCTION

Both classical and contemporary research have recognized the need for a mathematical framework that deals with control synthesis under constraints and uncertainty. The computational complexity normally associated with dynamic programming [1]–[3] is certainly an issue with the control synthesis problem in the presence of constraints and uncertainty. Tube model predictive control (TMPC) [4]–[14] constitutes a sensible approximate solution methodology. The deployment of tubes in robust model predictive control is made possible through a parameterization of the control policy that allows for the direct handling of uncertainty and its interaction with the system dynamics, constraints and performance. In particular, in the case of linear systems with additive bounded uncertainty and convex constraint sets it is possible to use separable control policies. These, in turn, allow for the separation of the evolution of the nominal system (uncertainty free system) and the evolution of the local uncertain system. Then, the effect of the uncertainty can be accounted for by considering the exact reachable tubes centered around the trajectories of the nominal system and invoking suitably modified constraints on the nominal variables; at the same time it is possible to consider simpler and computationally more tractable performance objectives. Early proposals employing this construction include [15]–[17] and were followed with research investigations along similar lines [18]–[21] which adopt the same paradigm for the case of the parametric (multiplicative) uncertainty and for local robustification of nonlinear MPC. The approaches reported in [15]–[22] fall into the class of TMPC synthesis methods, even though the tube terminology was not invoked. These approaches resulted in a reduction of the computational complexity and guaranteed desirable system theoretic properties.

In this manuscript, we utilize the basic concepts of the general homothetic TMPC (HTMPC), recently developed in [23], and offer a simplified HTMPC synthesis for a more structured setting. As in [23], we make use of the homothetic state and control tubes and the corresponding parameterized, separable, control policy. The homothetic state and control tubes are sequences of sets \( \{ X_k = z_k + \alpha_k S \}_{k \in \mathbb{N}} \) and \( \{ U_k = v_k + \alpha_k R \}_{k \in \mathbb{N}} \) which are parameterized via fixed, but suitably chosen, basic shape sets \( S \) and \( R \) and the sequences of the centers and diameters \( \{ z_k \}_{k \in \mathbb{N}}, \{ \alpha_k \}_{k \in \mathbb{N}} \) and \( \{ v_k \}_{k \in \mathbb{N}} \). The associated control policy \( \{ \pi_k(\cdot, X_k, U_k) \}_{k \in \mathbb{N}} \) is a sequence of control laws parameterized via a local control function \( \nu(\cdot) \) (which is a continuous and positively homogeneous function of the first degree) and a sequence of the centers of the homothetic state and control tubes \( \{ z_k \}_{k \in \mathbb{N}} \) and \( \{ v_k \}_{k \in \mathbb{N}} \) (so that \( \pi_k(y, X_k, U_k) = v_k + \nu(y - z_k) \) for all \( y \in X_k \)). The tube basic shape sets \( S \) and \( R \) and the local control function \( \nu(\cdot) \) are designed off-line and are required to satisfy a reasonable assumption, while the sequences of the centers and diameters \( \{ z_k \}_{k \in \mathbb{N}}, \{ \alpha_k \}_{k \in \mathbb{N}} \) and \( \{ v_k \}_{k \in \mathbb{N}} \) are decision variables in the on-line optimization. We employ globally relaxed set-dynamics of the underlying state and control tubes which allows for less conservative on-line constraint handling, performed only locally with respect to the actual state-control tube process. An additional novelty of our proposal is the construction of a suitable terminal constraint set obtained by analyzing the local homothetic state and control tube dynamics. The exact local set-dynamics of the homothetic state and control tubes is simplified by employing vector–valued dynamics describing the evolution of the centers and the diameters of the outer–bounding homothetic state and control tubes. For computational simplicity we employ a linear/affine approximation of the dynamics of the centers and diameters allowing, in turn, for the utilization of the classical set invariance concepts albeit in a suitably lifted space.

From the computational point of view, our proposal carries a modest increase in complexity compared to the existing methods [5], [7], [8], [11] but makes use of relaxed assumptions. Our method guarantees, under mild assumptions, strong system theoretic properties of the controlled uncertain dynamics.

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II. PRELIMINARIES & PROBLEM FORMULATION

We consider linear, time–invariant, discrete time systems, given by:

\[ x^+ = Ax + Bu + w, \]  

where \( x \in \mathbb{R}^n \) is the current state, \( u \in \mathbb{R}^m \) is the current control, \( x^+ \) is the successor state, \( w \in \mathbb{R}^p \) is the disturbance taking values in the set \( \mathbb{W} \subseteq \mathbb{R}^p \) and matrices \( A \) and \( B \) are of compatible dimensions, i.e. \( (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \). Thus, if at any time \( k \in \mathbb{N} \) the state is \( x_k \), the applied control is \( u_k \) and the disturbance is \( w_k \), the state at time \( k+1 \) satisfies

\[ x_{k+1} = Ax_k + Bu_k + w_k. \]

The system variables \( x, u \) and \( w \) are subject to hard constraints:

\[ x \in \mathcal{X}, \ u \in \mathcal{U} \text{ and } w \in \mathcal{W}. \]

We work, throughout this note, under the following standing assumptions and clarifying interpretation:

\begin{itemize}
    \item \textbf{Assumption 1:} The matrix pair \( (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \) is stabilizable.
    \item \textbf{Assumption 2:} The state and control constraint sets \( \mathcal{X} \) and \( \mathcal{U} \) are \( PC \)-polytopic sets in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively, and the disturbance constraint set \( \mathcal{W} \) is a non–trivial \( C \)-polytopic set in \( \mathbb{R}^p \).
\end{itemize}

\begin{itemize}
    \item \textbf{Interpretation 1:} At any time instance \( k \in \mathbb{N} \), the state \( x_k \) is known when the current control action \( u_k \) is determined, while the current disturbance \( w_k \) and future disturbances \( w_{k+i}, i \in \mathbb{N}_+ \) are not known and can take any arbitrary values \( w_{k+i} \in \mathcal{W}, i \in \mathbb{N} \).
\end{itemize}

Central focus of this note is the efficient computation of homothetic state and control tubes as well as associated control policies. For any non–empty sets \( S \subseteq \mathbb{R}^n \) and \( R \subseteq \mathbb{R}^m \), any function \( \nu (\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) and any positive integer \( N \in \mathbb{N}_+ \), the homothetic state tube is a sequence of sets \( \mathcal{X}_N := \{X_k\}_{k \in \mathbb{N}_N} \) where sets \( X_k \) are given, for all \( k \in \mathbb{N}_N \), by:

\[ X_k := z_k + \alpha_k S \text{ with } (z_k, \alpha_k) \in \mathbb{R}^n \times \mathbb{R}^+. \]

Likewise, the homothetic control tube is a sequence of sets \( \mathcal{U}_{N-1} := \{U_k\}_{k \in \mathbb{N}_{N-1}} \) where sets \( U_k \) are given, for all \( k \in \mathbb{N}_{N-1} \), by:

\[ U_k := v_k + \alpha_k R \text{ with } (v_k, \alpha_k) \in \mathbb{R}^m \times \mathbb{R}^+, \]

and the corresponding \( \nu \)-parameterized control policy is a sequence of control laws \( \Pi_{N-1} := \{\pi_k(\cdot, X_k, U_k)\}_{k \in \mathbb{N}_{N-1}} \), where control laws \( \pi_k(\cdot, X_k, U_k) \) are such that, \( \forall k \in \mathbb{N}_{N-1} \):

\[ \forall y \in X_k, \pi_k(y, X_k, U_k) := v_k + \nu(y - z_k). \]
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\section{Homothetic Tubes: Constituting Components}

The first mild but simplifying assumption is concerned with the desirable properties of the local control function \( \nu(\cdot) \), state and control tube cross-section shape sets \( S \) and \( R \) and an additional simplifying component, namely the nominal successor state set \( S^+ \):

\begin{itemize}
  \item [(i)] The control function \( \nu(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous, positively homogeneous function of the first degree.
  \item [(ii)] The state tube cross-section shape set \( S \) is a non-trivial \( C \)-polytopic set in \( \mathbb{R}^n \) such that
    \[
    \{ As + Bv(s) : s \in S \} \oplus \mathbb{W} \subseteq S. \tag{III.1}
    \]
  \item [(iii)] The control tube cross-section shape set \( R \) is a \( C \)-set in \( \mathbb{R}^m \) given by:
    \[
    R := \text{convh}(\{\nu(s) : s \in S\}). \tag{III.2}
    \]
  \item [(iv)] The nominal successor state set \( S^+ \) is a \( C \)-set in \( \mathbb{R}^n \) given by:
    \[
    S^+ := \text{convh}(\{ As + Bv(s) : s \in S \}). \tag{III.3}
    \]
\end{itemize}

We observe that, under Assumption 3, for any given \( \alpha \in \mathbb{R}_+ \) and all \( s \in \alpha S \) it holds that \( \nu(s) \in \alpha R \) and \( As + Bv(s) \in \alpha S^+ \) due to homogeneity of \( \nu(\cdot) \) and the \( C \)-property of the relevant sets. Clearly \( Ay + B(v + \nu(y - z)) = A\hat{z} + Bv + A(y - z) + Bv(y - z) \) and consequently, under Assumption 3, the set of constraints in (II.6e) is equivalently expressed, for all \( k \in \mathbb{N}_{N-1} \), as:

\[
A_{z} + Bv_{k} + \alpha_{k}S^+ \oplus \mathbb{W} \subseteq z_{k+1} + \alpha_{k+1}S. \tag{III.4}
\]

while, in addition, the set of constraints in (II.6f) is satisfied by construction. We point out that the constraints (III.4) specify the global homothetic state tube set–dynamics and do not necessarily require that \( \forall k \in \mathbb{N}_{N-1}, z_{k+1} = A_{z} + Bv_{k} + \alpha_{k+1} = \alpha_{k} = 1 \) as in earlier proposals [7].

A further simplification is obtained by considering strictly convex and quadratic stage and terminal cost functions \( \ell(\cdot, \cdot, \cdot) \) and \( V_{f}(\cdot, \cdot) \) given, for all \((z, \alpha, v) \in \mathbb{R}^{n+1} \), by:

\[
\ell(z, \alpha, v) = z'Q_{z}z + q_{\alpha}(\alpha - \bar{\alpha})^2 + v'Q_{v}v, \tag{III.5}
\]

and, for all \((z, \alpha) \in \mathbb{R}^{n+1} \),

\[
V_{f}(z, \alpha) = z'P_{z}z + p_{\alpha}(\alpha - \bar{\alpha})^2, \tag{III.6}
\]

where \( Q_{z} \in \mathbb{R}^{n \times n}, Q_{\alpha} = Q_{z}^2 > 0, Q_{v} \in \mathbb{R}^{m \times m}, Q_{v} = Q_{z}^2 > 0, P_{z} \in \mathbb{R}^{n \times n}, P_{z} = P_{z}^2 > 0, q_{\alpha} \in \mathbb{R}_+ \) and \( p_{\alpha} \in \mathbb{R}_+ \). The scalar \( \bar{\alpha} \in \mathbb{R}_+ \) is specified below in Section III-B.

\section{Homothetic Tubes: Simplified Local Tube Dynamics}

Our second step is to provide a suitable choice of the terminal constraint set \( G_f(\cdot) \) and invoke additional assumptions on stage and terminal cost functions \( \ell(\cdot, \cdot, \cdot) \) and \( V_{f}(\cdot, \cdot) \) guaranteeing desirable invariance and stabilizing properties. Before proceeding, let for any \((z, v, \alpha) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}_+ \),

\[
X(z, \alpha) := z \oplus \alpha S \text{ and } U(v, \alpha) := v \oplus \alpha R. \tag{III.7}
\]

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We consider the local control function, \( \pi_f(x, z) \), given, for all \( x \in X(z) := z + aS \) with \((z, a) \in \mathbb{R}_+^{m+n} \times \mathbb{R}_+ \), by:
\[
\pi_f(x, z) = Kz + \nu(x-z),
\]
where \( K \in \mathbb{R}^{n \times n} \). We note that, for any given \((z, a) \in \mathbb{R}_+^{n \times n} \times \mathbb{R}_+ \) and all \( x \in X(z) := z + aS \), it holds that
\[
\pi_f(x, z) = Kz + \nu(x-z) \in U(Kz, a).
\]
Our primary aim, in this subsection, is to determine the terminal set \( G_f \) such that for all \((z, a) \in G_f \) it holds that:
\[
\begin{align*}
x(z, a) &= z + aS \subseteq X, \\
U(Kz, a) &= Kz + aR \subseteq U, \quad \text{and}, \\
(A + BK)z + aS + \mathbb{W} &\subseteq z + aS
\end{align*}
\]
for some \((z^+, a^+) \in G_f \). Motivated by simplicity and computational considerations, we enforce the condition (III.9c) by setting:
\[
z^+ = (A + BK)z + a^+ = \lambda a + \mu,
\]
where:
\[
\begin{align*}
\mu &:= \min_{\eta} \{\eta : \mathbb{W} \subseteq \eta S, \eta \geq 0\}, \quad \text{and}, \\
\lambda &:= \min_{\eta} \{\eta : S^+ \subseteq \eta S, \eta \geq 0\}.
\end{align*}
\]
Note that, under Assumption 3, it is guaranteed that \( \mu \in [0, 1] \) and \( \lambda \in [0, 1] \). In order to ensure the satisfaction of the conditions (III.9a) and (III.9b) we introduce the set of constraints on the \((z, a)\) variable:
\[
(z, a) \in G \quad \text{where} \quad (z, a) \in G := \{(z, a) : z + aS \subseteq X, \ Kz + aR \subseteq U \text{ and } a \geq 0\}.
\]
It is important to note that, under Assumptions 2 and 3, the set \( G \) is a \( C^\infty \)-polypolytopic set in \( \mathbb{R}_+^{n \times n} \times \mathbb{R}_+ \). We underline that the ordinary vector-valued dynamics specified in (III.10) define completely the local, outer-bounding, homothetic set--dynamics (see (III.9c)) while the conditions (III.9a) and (III.9b) impose constraints on the \((z, a)\) dynamics via (III.12). Consequently, the determination of an appropriate terminal constraint set \( G_f \) ensuring that the relations (III.9) are satisfied, is reduced to the problem of the determination of a positively invariant set for the dynamics (III.10) subject to constraints (III.12).

We are now in position to invoke our second mild and simplifying assumption:

**Assumption 4:** (i) The matrix \( K \in \mathbb{R}^{m \times n} \) and scalar \( \lambda \in (0, 1) \) in (III.10) and are such that \( \rho(A + BK) < 1 \) and \( \lambda \in [0, 1] \).

(ii) The scalar \( a \in \mathbb{R}_+ \) is given by:
\[
a = (1 - \lambda)^{-1} \mu,
\]
where \( \lambda \) and \( \mu \) are given as in (III.10), and is such that \( (0, a) \in \text{interior}(G) \), where the set \( G \) is specified in (III.12b).

(iii) The terminal constraint set \( G_f \) is a non–trivial \( C^\infty \)-polypolytopic set in \( \mathbb{R}_+^{n \times n} \times \mathbb{R}_+ \) and is a positively invariant set for the dynamics (III.10) subject to constraints (III.12), i.e. it is such that \( G_f \subseteq G \) and:
\[
\forall (z, a) \in G_f, \ ((A + BK)z, \lambda a + \mu) \in G_f. \quad \text{(III.14)}
\]
(iv) The stage and terminal cost functions \( \ell(t, \cdot, \cdot) \) and \( V_f(t, \cdot) \) are given as in (III.5) and (III.6) and:
\[
(A + BK)P(A + BK) - P_z \leq -Q_z + K'Q_kK,
\]
and \( p_a \geq (1 - \lambda^2)^{-1}q_{a}. \quad \text{(III.15)}
\]
Now, we comment briefly on the consequences of the invoked assumptions. As already indicated, under Assumptions 2 and 3, the set \( G \) in (III.12b) is a \( C^\infty \)-polypolytopic set in \( \mathbb{R}_+^{n \times n} \times \mathbb{R}_+ \). The conditions (i) and (ii) in Assumption 4 guarantee then the existence of the terminal constraint set \( G_f \) postulated in the condition (iii) in Assumption 4. In fact, under above mentioned assumptions, a direct use of results in [24], [25] implies that the maximal positively invariant set for the dynamics (III.10) subject to constraints (III.12) is finitely determined and is a non–trivial \( C^\infty \)-polypolytopic set in \( \mathbb{R}_+^{n \times n} \times \mathbb{R}_+ \). The conditions (i) and (ii) in Assumption 4 also imply that:
\[
\begin{align*}
\bar{S} \subseteq X, \quad \bar{R} \subseteq U \text{ and } \bar{S}^+ \subseteq \mathbb{W} \subseteq \bar{S}, \text{ where}, \\
\bar{S} := \bar{a}S, \quad \bar{R} := \bar{a}R, \text{ and,} \\
\bar{S}^+ := \bar{a}S^+.
\end{align*}
\]
Note that sets \( \bar{S} \) and \( \bar{R} \) play a role of the equilibrium set pair for the homothetic state and control tubes in analogy with the equilibrium pair \((0, 0) \in \mathbb{R}_+^{n \times n} \times \mathbb{R}_+^{m \times n} \) relevant for the deterministic case (i.e. in the absence of the uncertainty). The condition (iv) in Assumption 4 implies that there exist scalars \( c_1 \in (0, \infty), \ c_2 \in (0, \infty), \ c_3 \in (0, \infty) \) and \( c_4 \in (0, \infty) \) such that, for all \((z, a) \in G_f \) it holds that
\[
|c_1(z, a) - (0, a)| \leq \ell(z, a, Kz) \leq c_2(z, a) - (0, a),
\]
\[
|c_3(z, a) - (0, a)| \leq V_f(z, a) \leq c_4|z, a) - (0, a)| \leq V_f((A + BK)z, \lambda a + \mu) - V_f(z, a) \leq -\ell(z, a, Kz).
\]
In other words, the terminal cost function \( V_f(\cdot, \cdot) \) is a local Lyapunov function for the dynamics (III.10) relative to the equilibrium point \((0, a)\) and with the basin of attraction being equal to the positively invariant set \( G_f \). Summarizing the discussion above we have:

**Proposition 1:** Suppose Assumptions 1–4 hold and consider any sequence \( \{(z_k, \alpha_k)\}_{k \in \mathbb{N}} \) generated, for all \( k \in \mathbb{N} \), by \( z_{k+1} = (A + BK)z_k \) and \( \alpha_{k+1} = \lambda \alpha_k + \mu \) for any arbitrary \((z_0, \alpha_0) \in G_f \). Then:
(i) For all \( k \in \mathbb{N} \), \((z_k, \alpha_k) \in G_f \).
(ii) For all \( k \in \mathbb{N} \), \(|(z_k, \alpha_k) - (0, a)| \leq a^k b |(z_0, \alpha_0) - (0, a)| \) for some \((a, b) \in [0, 1] \times [0, \infty) \), and
(iii) The pair \((0, a) \in \mathbb{R}_+^{n \times n} \times \mathbb{R}_+ \) is an exponentially stable attractor for dynamics (III.10) with the basin of attraction being equal to the set \( G_f \).

We close this section by summarizing relevant consequences of our construction and Proposition 1:

**Remark 1:** Under the conditions of Proposition 1, the set sequences \( \{X(z_k, \alpha_k) = z_k + \alpha_k S\}_{k \in \mathbb{N}} \) and \( \{U(Kz_k, \alpha_k) = \}

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Furthermore, the set sequences \( \{X(z_k, \alpha_k)\}_{k \in \mathbb{N}} \) and \( \{U(Kz_k, \alpha_k)\}_{k \in \mathbb{N}} \) converge exponentionally fast, with respect to the Hausdorff distance, to the sets \( \bar{S} = \tilde{a}S \) and \( \bar{R} = \tilde{a}R \) respectively. The corresponding Hausdorff distances between sets \( X(z_k, \alpha_k) \) and \( S \) as well as between sets \( U(Kz_k, \alpha_k) \) and \( R \) are smaller or equal to \( a^k \bar{b}(\alpha, \alpha) = (0, \alpha) \) and \( a^k \bar{c}(\alpha) = (0, \alpha) \), respectively, for some scalars \((a, b, c) \in [0, 1) \times [0, \infty) \times [0, \infty)\).

Remark 2: The set of states \( x \in X \) for which there exists at least one pair of the set sequences \( \{X(z_k, \alpha_k)\}_{k \in \mathbb{N}} \) and \( \{U(Kz_k, \alpha_k)\}_{k \in \mathbb{N}} \) enjoying the properties discussed in Remark 1 and satisfying, in addition, that \( x_0 \in X(z_0, \alpha_0) \) is implicitly characterized via the conditions \( x \in z \oplus \alpha S \) and \((z, \alpha) \in GF\). More precisely, this is the set \( \mathcal{A}_0 \) referred to as the 0-step homothetic tubes controllability set and given by:

\[
\mathcal{A}_0 = \text{Projection}_{\mathbb{R}^n} \{(x, z, \alpha) : x \in z \oplus \alpha S, \quad (z, \alpha) \in GF\}. \tag{III.17}
\]

Under our assumptions, the set \( \{(x, z, \alpha) : x \in z \oplus \alpha S, \quad (z, \alpha) \in GF\} \) is a non–trivial \( C \)-polytopic set in \( \mathbb{R}^{2n} \times \mathbb{R}_+ \) and, consequently, the 0-step homothetic tubes controllability set \( \mathcal{A}_0 \) is itself a non–trivial \( C \)-polytopic set in \( \mathbb{R}^n \) such that \( \bar{S} \subseteq \mathcal{A}_0 \). We note that, by construction, for all \( x_0 \in \mathcal{A}_0 \) there exists at least one pair \((z_0, \alpha_0) \) such that \( x_0 \in \bar{S} \). Let, for any positive integer \( n \in \mathbb{N} \), the \( n \)-step homothetic tubes controllability set and given by:

\[
\mathcal{X}_n = \text{Projection}_{\mathbb{R}^n} \{(x, z, \alpha) : x \in z \oplus \alpha S, \quad (z, \alpha) \in GF\}. \tag{IV.19}
\]

IV. HOMOTHETIC TUBE OPTIMAL CONTROL

We utilize the analysis of Section III and deploy corresponding simplifications in order to discuss the relevant properties of the finite horizon HTOC problem \( \mathcal{P}_N(x), x \in X \) specified in (II.9). Our simplifying assumptions allow us to establish structural properties of the graph of the set–valued map \( \mathcal{D}_N(\cdot) \) and the cost function \( \mathcal{V}_N(\cdot) \) specified in (II.7) and (II.8) respectively and, in turn, to ascertain relevant topological properties of the value function \( \mathcal{V}_N(\cdot) \) and its optimizer \( \mathcal{d}_N(\cdot) \) given in (II.9). For any positive integer \( N \in \mathbb{N} \), the set \( \mathcal{X}_N \) given by:

\[
\mathcal{X}_N = \{x : \mathcal{D}_N(x) \neq \emptyset\} \tag{IV.1}
\]

is referred to as the \( N \)-step homothetic tubes controllability set and is, in fact, the effective domain of the value function \( \mathcal{V}_N(\cdot) \) and its optimizer \( \mathcal{d}_N(\cdot) \). Let, for any integer \( N \in \mathbb{N} \),

\[
\mathcal{d}_N \subseteq X \subseteq \mathcal{X}_N \subseteq X.
\]

Lemma 1: Suppose Assumption 4 (iv) holds. Then the cost function \( \mathcal{V}_N(\cdot) : \mathbb{R}^{(n+m+1)+n+1} \rightarrow \mathbb{R}_+ \) given by (II.8), (III.5) and (III.6) is strictly convex and quadratic function such that for all \( \mathcal{d}_N \in \mathbb{R}^{(n+m+1)+n+1} \) it holds that:

\[
c_5 |\mathcal{d}_N - \mathcal{d}_N| \leq \mathcal{V}_N(\mathcal{d}_N) \leq c_6 |\mathcal{d}_N - \mathcal{d}_N| \tag{IV.2}
\]

for some scalars \( c_5 \in (0, \infty) \) and \( c_6 \in (0, \infty) \).

Let, for any positive integer \( N \in \mathbb{N} \), the set \( \mathcal{D}_N \) be given by:

\[
\mathcal{D}_N = \{x : \mathcal{d}_N \} \tag{IV.6 \text{ holds}}
\]

We recall that, under Assumption 3, the constraints in (II.6f) are satisfied by construction while the constraints in (II.6e) are equivalently expressed via the constraints in (III.4). With this in mind, under Assumptions 2, 3 and 4, a direct use of basic algebra of support functions [26], [27] yields the fact that the set \( \mathcal{D}_N \) is a non–trivial \( C \)-polytopic set in \( \mathbb{R}^{n+N(\alpha+1)+n+1} \) (i.e. in \( (x, \mathcal{d}_N) \)–space). A concrete algebraic details are easily obtained by utilizing analysis provided in [23, Section 5].

Lemma 2: Suppose Assumptions 1–4 hold. Then the set \( \mathcal{D}_N \) given by (IV.3) is a non–trivial \( C \)-polytopic set in \( \mathbb{R}^{n+N(\alpha+1)+n+1} \). Since, by construction, it holds that \( \mathcal{X}_N = \text{Projection}_{\mathbb{R}^n} \mathcal{D}_N \) it follows that the set \( \mathcal{X}_N \) is a non–trivial \( C \)-polytopic set in \( \mathbb{R}^n \) which, under our assumptions, is such that \( \bar{S} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}_N \subseteq X \). Furthermore, we also have that for all \( x \in \mathcal{X}_N \), \( \mathcal{D}_N(x) = \{\mathcal{X}_N : (x, \mathcal{d}_N) \in \mathcal{D}_N \} \) and consequently for any fixed \( x \in \mathcal{X}_N \) the set \( \mathcal{D}_N(x) \) is a polytopic set. Hence, under our assumptions, the finite horizon HTOC problem \( \mathcal{P}_N(x) \) given in (II.9) is, for any fixed \( x \in \mathcal{X}_N \) a strictly convex quadratic programming problem. The basic results in parametric mathematical programming [28] allows us to establish relevant topological properties of the value function \( \mathcal{V}_N(\cdot) \) and its optimizer \( \mathcal{d}_N(\cdot) \) given in (II.9):

Proposition 2: Suppose Assumptions 1–4 hold. Then:

(i) The \( N \)-step homothetic tubes controllability set \( \mathcal{X}_N \) is a non–trivial \( C \)-polytopic set in \( \mathbb{R}^n \) such that \( \bar{S} \subseteq \mathcal{X}_N \subseteq X \).
(ii) The value function $V^0_N(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is a convex, piecewise quadratic and continuous function such that \( \forall x \in S, \quad V^0_N(x) = 0 \), and
(iii) The function \( d^N_k(\cdot) : \mathcal{X}_N \to \mathbb{R}^{N(n+m+1)+n+1} \) is a single-valued, piecewise affine and continuous function such that \( \forall x \in S, \quad d^N_k(x) = d_N \).

The solution of the finite horizon HTOC problem $P_N(x)$, $x \in \mathcal{X}_N$ allows for the construction of the optimal simple homothetic state and control tubes $X^0_N(x)$ and $U^0_{N-1}(x)$ as well as the optimal $\nu$-parametrized control policy $\Pi^0_{N-1}$. Namely, the corresponding optimizer $d^N_k(\cdot)$ induces the optimal simple homothetic optimal state and control tubes $X^0_N(x) := \{X^0_N(x) := (z^0_x, x^0, \alpha^0_x, v^0_x, \ldots, v^0_{N-1}(x)) \}$ and $U^0_{N-1}(x) := \{U^0_N(x) = v^0_x + \alpha^0_x R \}$ as well as the optimal $\nu$-parametrized control policy $\Pi^0_{N-1} = \{ \pi^N_k(X^0_N(x), U^0_N(x)) \}_{k \in \mathbb{N}}$ such that, for all $k \in \mathbb{N}$ and for all $y \in X^0_N(x)$,

\[
\pi^N_k(y, X^0_N(x), U^0_N(x)) = v^0_x + \nu(y - z^0_x).
\]

The optimal simple homothetic state and control tubes $X^0_N(x)$ and $U^0_{N-1}(x)$ and the corresponding optimal $\nu$-parametrized control policy $\Pi^0_{N-1}$ satisfy the constraints specified in (II.6). Proposition 1 and Remark 1 imply directly that the corresponding optimizer $d^N_k(\cdot)$ can be employed to construct a feasible decision variable $d_N(y) := (z^0_x, \ldots, z^0_N, \alpha^0_x, \ldots, \alpha^0_N, v^0_x, \ldots, v^0_{N-1}(y))$ for any $y \in X^0_N(x)$ by setting for all $k \in \mathbb{N}$:

\[
\hat{z}_k(y) = z^0_{k+1} + \hat{y}_N(y) = (A + BK)^{y^0_x + \nu(y - z^0_x)} \frac{d_N(y)}{\lambda^0_N(x)} + \mu,
\]

and, for all $k \in \mathbb{N}$,

\[
\hat{v}_k(y) = v^0_{k+1} + \hat{y}_{N-1}(y) = K z^{0_N}_N(x).
\] (IV.4)

Indeed, the following result guarantees robust recursive feasibility of the finite horizon HTOC problem $P_N(x)$, $x \in \mathcal{X}_N$:

**Proposition 3:** Suppose Assumptions 1–4 hold. Then, for all $x \in \mathcal{X}_N$ and all $y \in X^0_N(x) := z^0_x(\cdot) \oplus \alpha^0_x(\cdot) S$, it holds that

\[
d_N(y) \in D_N(y),
\] (IV.5)

where $d_N(y)$ is specified in (IV.4), and, consequently, for all $x \in \mathcal{X}_N$ it holds that $X^0_N(x) \subseteq \mathcal{X}_N$.

V. HOMOTHETIC TUBE MPC

We now examine the repetitive on-line application of the solution of the finite horizon HTOC problem $P_N(x)$, $x \in \mathcal{X}_N$. We consider the simple homothetic tube model predictive controller $\kappa^0_N(\cdot) : \mathcal{X}_N \to \mathbb{U}$ given by:

\[
k^0_N(x) = \pi^0_N(x, X^0_N(x), U^0_N(x)) = v^0_x + \nu(x - z^0_x).
\] (V.1)

Under Assumptions 1–4, Proposition 3 establishes that the optimizer $d^N_k(\cdot)$ is single-valued and continuous implying, in view of (V.1) and Assumption 3, that the control law $k^0_N(\cdot) : \mathcal{X}_N \to \mathbb{U}$ is also a single-valued and continuous function. The simple homothetic tube model predictive controller $\kappa^0_N(\cdot) : \mathcal{X}_N \to \mathbb{U}$ induces the controlled, uncertain, dynamics given, for all $x \in \mathcal{X}_N$, by:

\[
x^+ \in F(x) := \{Ax + Bv^0_N(x) + w : w \in \mathbb{W}\},
\] (V.2)

and it ensures, by construction, that for all $x \in \mathcal{X}_N$:

\[
F(x) \subseteq X^0_N(x) = z^0_x(\cdot) \oplus \alpha^0_x(\cdot) S \subseteq \mathcal{X}_N,
\] (V.3)

as evident from (II.6) and Proposition 3.

Any state sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by (V.1) with $x_0 \in \mathcal{X}_N$ and the corresponding control actions sequence $\{u_k\}_{k \in \mathbb{N}}$ with $u_k = k^0_N(x_k)$ for each $k$, for any admissible disturbance sequence $\{w_k\}_{k \in \mathbb{N}}$ with $w_k \in \mathbb{W}$ for each $k$, however, lie in the simple HTMPC state and control tubes $\{X^0_N(x_k)\}_{k \in \mathbb{N}}$ and $\{U^0_N(x_k)\}_{k \in \mathbb{N}}$, respectively. Therefore, for all $k \in \mathbb{N}$ it holds that:

\[
x_k \in X^0_N(x_k) = z^0_x(\cdot) \oplus \alpha^0_x(\cdot) S \subseteq \mathcal{X}_N \subseteq \mathbb{X}
\] (V.4a)

\[
k^0_N(x_k) \in U^0_N(x_k) = v^0_x(\cdot) \oplus \alpha^0_x(\cdot) R \subseteq \mathbb{U}.
\] (V.4b)

The limiting behavior of the simple HTMPC state and control tubes $\{X^0_N(x_k)\}_{k \in \mathbb{N}}$ and $\{U^0_N(x_k)\}_{k \in \mathbb{N}}$, respectively, is, therefore, completely induced from the limiting behavior of the sequences $\{x^0_k\}_{k \in \mathbb{N}}$, $\{\alpha^0_k(x_k)\}_{k \in \mathbb{N}}$ and $\{v^0_k(x_k)\}_{k \in \mathbb{N}}$. We examine the latter by utilizing the value function $V^0_N(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ as a Lyapunov function to establish the exponential convergence of the sequences $\{x^0_k\}_{k \in \mathbb{N}}$, $\{\alpha^0_k(x_k)\}_{k \in \mathbb{N}}$ and $\{v^0_k(x_k)\}_{k \in \mathbb{N}}$ to the triplet $(0, \bar{\alpha}, 0)$ in a stable manner, i.e., that $x^0_k(0) \to 0$, $\alpha^0_k(x_k) \to 1$ and $v^0_k(x_k) \to 0$, as $k \to \infty$, exponentially fast and in a stable fashion for any realized state sequence $\{x_k\}_{k \in \mathbb{N}}$ arising due to an admissible disturbance sequence $\{w_k\}_{k \in \mathbb{N}}$. Proposition 2 establishes that the value function $V^0_N(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is a convex and continuous function, which is, in addition, such that for all $x \in \mathcal{S}$ it holds that $V^0_N(x) = 0$ or equivalently, that for all $x \in \mathcal{S}$ it holds that $(\alpha^0(x), \alpha^0(x), v^0(x)) = (0, \bar{\alpha}, 0)$. Before proceeding, let:

\[
y^0_0(x) := (z^0_x, \alpha^0(x), v^0(x)) \quad \text{and} \quad \bar{y} := (0, \bar{\alpha}, 0).
\] (V.5)

A further relevant property of the value function $V^0_N(\cdot)$ is stated next:

**Proposition 4:** Suppose Assumptions 1–4 hold. Then there exists a scalar pair $(\epsilon_\gamma, \epsilon_\alpha) \in (0, \infty) \times (0, \infty)$ such that for all $x \in \mathcal{X}_N$ it holds that:

\[
\epsilon_\gamma |y^*_0(x) - \bar{y}| \leq V^0_N(x) \leq \epsilon_\alpha |y^*_0(x) - \bar{y}|
\] (V.6a)

\[
\forall x^+ \in F(x), \quad V^0_N(x^+) - V^0_N(x) \leq -\epsilon_\gamma |y^*_0(x) - \bar{y}|.
\] (V.6b)

where $F(\cdot)$, $y^*_0(\cdot)$ and $\bar{y}$ are given by (V.2) and (V.5), respectively.

A direct, but relevant, consequence of Proposition 4 is the following fact:
Corollary 1: Suppose Assumptions 1–4 hold. Then there exists a scalar pair \((a_N, b_N) \in (0,1) \times (0,\infty)\) such that the inequalities
\[
\forall k \in \mathbb{N}, \quad V_0^k(x_k) \leq a_N^k V_0^0(x_0) \quad \text{and} \quad (V.7a)
\]
\[
\forall k \in \mathbb{N}, \quad |y_0^k(x_k) - \bar{y}| \leq b_N^k |y_0^0(x_0) - \bar{y}| \quad \text{for any} \quad x_0 \in X_N \text{ and any corresponding state sequence} \{x_k\}_{k \in \mathbb{N}} \text{ generated by} (V.2). \quad (V.7b)
\]
hold true for any \(x_0 \in X_N\) and any corresponding state sequence \(\{x_k\}_{k \in \mathbb{N}}\) generated by \((V.2)\).

We also need the following technical lemma:

Lemma 3: Suppose Assumptions 1–4 hold. Then there exists a scalar pair \((c_0, c_{10}) \in (0,\infty) \times (0,\infty)\) such that for all \(x \in X_N\) it holds that:
\[
c_0 |y_0^0(x) - \bar{y}| \leq H(B_0^2, X_0^0(x), \bar{S}) \leq c_{10} |y_0^0(x) - \bar{y}| \quad (V.8)
\]
The sequence of our preliminary results now easily yields our main result:

Theorem 1: Suppose Assumptions 1–4 hold. Then:
(i) The \(N\)-step homothetic tubes controllability set \(X_N\) is a robust positively invariant set for the system \((V.2)\) and constraint set \((\bar{X}, \bar{U}, \bar{W})\), i.e. for all \(x \in X_N\) it holds that\(\kappa_N^k(x) \in \bar{U}\) and \(F(x) \subseteq X_N\) where \(\kappa_N^k(\cdot)\) and \(F(\cdot)\) are given by \((V.1)\) and \((V.2)\), respectively.
(ii) There exists a scalar pair \((\hat{a}_N, \hat{b}_N) \in [0,1) \times \mathbb{R}_+\) such that for all \(k \in \mathbb{N},\)
\[
H(B_0^2, X_0^0(x_k), \bar{S}) \leq \hat{a}_N^k \hat{b}_N H(B_0^2, X_0^0(x_0), \bar{S}), \quad \text{and} \quad (VI.3b)
\]
holds true for all state sequence \(\{x_k\}_{k \in \mathbb{N}}\) generated by \((V.2)\) with arbitrary \(x_0 \in X_N\).
(iii) The set \(\bar{S}\) is robustly exponentially stable for the controlled uncertain system \((V.2)\) with the basin of attraction being the \(N\)-step homothetic tubes controllability set \(X_N\).

Remark 3: Theorem 1 establishes the stable and exponential convergence of the simple HTMPC state tubes \(\{X_0^0(x_k)\}_{k \in \mathbb{N}}\) to the set \(\bar{S}\) as well as the exponential convergence of the simple HTMPC control tubes \(\{U_0^0(x_k)\}_{k \in \mathbb{N}}\) to the set \(\bar{R}\). The established property is equivalent to the stability of the set \(\bar{S}\) and the exponential convergence of the possible state trajectories \(\{x_k\}_{k \in \mathbb{N}}\) to the set \(\bar{R}\). The corresponding control actions sequence \(\{u_k\}_{k \in \mathbb{N}}\) converges exponentially fast to \(0\) in a stable manner. Moreover, the corresponding control actions sequence \(\{u_k\}_{k \in \mathbb{N}}\) converges exponentially fast to \(\bar{R}\) in the sense that the sequence \(\{h(B_0^2, \{x_k\}, \bar{S})\}_{k \in \mathbb{N}}\) converges exponentially fast to \(0\).

Remark 4: As far as the implementation of the simple HTMPC is concerned, the main computational burden is the off–line determination of the sets \(\bar{S}, R, S^+\) and \(G_f\). However, a number of methods exists in the literature that can be utilized to execute the necessary computations \([3, 24, 25, 29–37]\). Once the off–line computations are performed, the on–line implementation of the simple HTMPC can be performed by utilizing the standard convex optimization software (since the finite horizon (HTOC) problem \(P_N(x)\) given in (II.9) is, for any fixed \(x \in X_N\) a strictly convex quadratic programming problem). Finally, since \(\forall x \in \bar{S}, V_0^0(x) = 0, d_0^0(x) = d_N\) and, consequently, \(\forall x \in \bar{S}, k_N^k(x) = \nu(x)\) we note that on–line optimization can be terminated once the state \(x_k\) enters the set \(\bar{S}\).

VI. YET SIMPLER HTMPC

We outline a further plausible simplification of the simple HTMPC. Namely, the set of constraints in (II.6e) or its equivalent reformulation (III.4) (under Assumption 3) can be enforced as described in Section III. Indeed, the set of constraints in (III.4) can be enforced by setting for all \(k \in \mathbb{N}_{N-1}\):
\[
z_{k+1} = A z_k + B v_k \quad \text{and} \quad \alpha_{k+1} = \lambda \alpha_k + \mu, \quad (VI.1)
\]
where \(\lambda\) and \(\mu\) are given as in (III.10). In this case, we introduce the set:
\[
G_S = \{(z, \alpha, v) : \quad z + \alpha s \subseteq X, \quad v + \alpha R \subseteq U \quad \text{and} \quad \alpha \geq 0\}, \quad (VI.2)
\]
We note that, under Assumptions 2 and 3, the set \(G_S\) is a \(C\)-polytopic set in \(\mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v}\).

In order to simplify further the simple HTMPC we utilize (VI.1) and (VI.2) and invoke the following set of constraints on the decision variable \(d_N\):
\[
x \in z_0 + \alpha_0 S, \quad (VI.3a)
\]
\[
\forall k \in \mathbb{N}_{N-1}, \quad z_{k+1} = A z_k + B v_k, \quad (VI.3b)
\]
\[
\forall k \in \mathbb{N}_{N-1}, \quad \alpha_{k+1} = \lambda \alpha_k + \mu, \quad (VI.3c)
\]
\[
\forall k \in \mathbb{N}_{N-1}, \quad (z_k, \alpha_k, v_k) \in G_S, \quad \text{and}, \quad (VI.3d)
\]
\[
(z_N, \alpha_N) \in G_f. \quad (VI.3e)
\]

Let, similarly as in Section IV, the set \(D_{SN}\) be given by:
\[
D_{SN} := \{(x, d_N) : \quad (VI.3) \text{ holds}\} \quad (VI.4)
\]
Under Assumptions 1–4, similarly as in Lemma 2, the set \(D_{SN}\) is a non–trivial polytopic set in \(\mathbb{R}^{n_x+n_y+n_z+1}\).

The simplified HTOC problem \(P_{SN}(x)\) is given, for all \(x \in X\), by:
\[
V_{SN}^0(x) := \inf_{d_N} \{V_N(d_N) : \quad (x, d_N) \in D_{SN}\}, \quad \text{and}, \quad (VI.5)
\]
\[
d_{SN}^0(x) := \arg \inf_{d_N} \{V_N(d_N) : \quad (x, d_N) \in D_{SN}\}. \quad (VI.5)
\]
Under Assumptions 1–4, the effective domain of the value function \(V_{SN}^0(\cdot)\) and its optimizer \(d_{SN}^0(\cdot)\) is the set:
\[
\bar{X}_{SN} := \text{Projection}_{d_N}^x D_{SN}, \quad (VI.6)
\]
which is a non–trivial \(C\)-polytopic set in \(\mathbb{R}^n\) such that \(\bar{S} \subseteq X_0 \subseteq \bar{X}_{SN} \subseteq X\).

Remark 5: Under Assumptions 1–4, all the results established in Section IV apply directly to the simplified HTOC problem \(P_{SN}(x)\) with minor notational changes. In particular, Propositions 2 and 3 hold true with obvious notational changes.
Remark 6: The simplified homothetic tube model predictive controller \( \kappa^0_{SN}(\cdot) : \mathcal{X}_{SN} \to \mathbb{U} \) given as in (V.1) (i.e. for all \( x \in \mathcal{X}_{SN}, \kappa^0_{SN}(x) = v^0_0(x) + \nu(x - z^0_0(x)) \)) ensures that the controlled uncertain dynamics given as in (V.2) (i.e. \( x^+ \in F(x) := \{Az + B\kappa^0_{SN}(x) + w : w \in \mathcal{W} \} \)) are well behaved. Indeed, as in Remark 5, under Assumptions 1–4, Propositions 4, Corollary 1, Lemma 3 and Theorem 1 hold true with obvious notational changes. However, since the set of constraints in (V.1) is more restrictive than the set of constraints (III.4), the computationally beneficial simplifications outlined here come at the expense of the potential degradation of optimality, the speed of convergence and the reduction of the size of the domain of the attraction compared to the HTMPC method discussed in Section V.

VII. CONCLUDING REMARKS

We considered the robust model predictive control synthesis problem for constrained linear discrete time systems and proposed two simple homothetic tube model predictive control synthesis methods. Under rather mild assumptions it is demonstrated that the proposed methods are computationally efficient and induce strong system theoretic properties.

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