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by

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Abstract

Robust model predictive control (RMPC) is an area of significant practical importance that has received a lot of research attention in recent years. Despite this, there still remains a considerable challenge, that of reaching a reasonable compromise between computational tractability and degree of optimality. Early results [1] were based on an open–loop worst case optimization, but were only practicable for low dimension systems and could be considerably suboptimal. Tube MPC [2–12] provides an effective and efficient alternative which nevertheless is still based on a semi closed–loop optimization. Optimality can be achieved through the use of closed–loop optimization [13, 14], but computational complexity grows exponentially with respect to the prediction horizon. The so–called disturbance affine control policy [15–18] provides a reasonable compromise and it is the aim of this paper to propose a new methodology which supersedes the disturbance affine control policy in that it provides a more general nonlinear framework with which to achieve more optimal results at the same computational cost. The work is based on a suitable parameterization of state and input tubes for systems which are subject to additive polytopic uncertainty and is underpinned by guarantees of strong system theoretic properties for the controlled uncertain dynamics.

Keywords: robust model predictive control, tube model predictive control, set invariance, set–dynamics.

1 Introduction

Tube model predictive control (TMPC) [2–12] forms a sensible approximate solution methodology for the control synthesis problem in the presence of constraints and uncertainty, and offers a computationally tractable alternative to dynamic programming [19–21]. The development of TMPC is made possible by making use of a particular parameterization of the control policy that allows for the direct handling of uncertainty and its interaction with the system dynamics, constraints and performance. More precisely, in the case of linear systems with additive, bounded, uncertainty and convex constraint sets, the deployment of separable control policies allows for the separation of the evolution of the nominal system (uncertainty free system) and the evolution of the local uncertain system. In this construction, the propagation of the uncertainty can be accounted for by considering the exact reachable tubes centered around the trajectories of the nominal system and invoking suitably modified constraints on the nominal variables. Early proposals employing this construction include [22–30]. These approaches employ the state and control tubes with the time varying cross–sections and result in a reduction of the computational complexity and guaranteed desirable system theoretic properties. More recent proposals [3, 5, 6, 9, 31] offer several novel and distinct features such as use of tubes with constant cross–sections, the optimization of the initial condition of the nominal system and robust stability and attractivity of the corresponding minimal invariant set. Additional investigations on the topic [4, 7–12] provided generalizations of tube MPC handling the output feedback case [7, 12] and some classes of nonlinear systems [4, 8, 10, 11, 31]. Further advances of TMPC are recently reported in [32, 33] where the homothetic state and control tubes and a more general control policy is employed. The homothetic tube model predictive control (HTMPC) [32, 33] employed several novel features including: a more general parameterization of the state and control tubes based on homothety and invariance; a more flexible form of the terminal constraint set; and a relaxation of the dynamics of the sets that define the state and control tubes. As its predecessors, HTMPC [32, 33] is computationally efficient and it induces strong system theoretic properties.

In this paper, we propose a novel TMPC allowing uniquely for simultaneous online optimization of the state and control tubes as well as the corresponding control policy. The proposed parameterization of the state and control tubes and the corresponding control policy is more general than existing methods while the online optimization reduces to a single tractable linear programming problem. The main idea behind our proposal is to employ the state decomposition in the sense that at time k within the prediction horizon $\mathbb{N}_N := \{0, 1, \ldots, k, \ldots, N\}$, the possible state x_k is decomposed into $k+1$ components $x_{(0,k)}, x_{(1,k)}, \ldots, x_{(k,k)}$ satisfying $x_k = \sum_{j=0}^{k} x_{(j,k)}$.

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The state decomposition motivates the corresponding control action $u_k(x_k)$ decomposition into $k+1$ components $u_{(0,k)}(x_{(0,k)}), u_{(1,k)}(x_{(1,k)}), \ldots, u_{(k,k)}(x_{(k,k)})$ satisfying $u_k(x_k) = \sum_{j=0}^{k} u_{(j,k)}(x_{(j,k)})$. The state and control decompositions allow, in turn, for the utilization of the controlled but nominal dynamics of the state components via the recursion $x_{(j,k+1)} = Ax_{(j,k)} + Bu_{(j,k)}(x_{(j,k)})$ leading to the overall state at time $k+1$ given by the sum of $k+2$ components $x_{(0,k+1)}, x_{(1,k+1)}, \ldots, x_{(k,k+1)}, x_{(k+1,k+1)}$ where the last component $x_{(k+1,k+1)}$ accounts for the effect of the uncertainty w_k (i.e. $x_{k+1} = \sum_{j=0}^{k+1} x_{(j,k+1)}$ and $x_{(k+1,k+1)} = w_k$).

As is customary in TMPC, the presence of the uncertainty is accounted for by utilizing the state and control tubes that are sequences of sets $\mathbf{X}_N := \{X_k\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{N-1} := \{U_k\}_{k\in\mathbb{N}_{N-1}}$. The state and control action decompositions naturally suggest the parameterization of the state and control tubes by utilizing the partial state and control tubes sets. Namely, at time k within the prediction horizon $\mathbb{N}_N$ the state tube cross–section (i.e. X_k) is obtained from a collection of partial state tube cross–sections $X_{(0,k)}, X_{(1,k)}, \ldots, X_{(k,k)}$. Similarly, the control tube cross–section at time k (i.e. U_k) is obtained from a collection of partial control tube cross–sections $U_{(0,k)}, U_{(1,k)}, \ldots, U_{(k,k)}$. More precisely, the state tube cross–sections at time k are parameterized via $X_k = \bigoplus_{j=0}^{k} X_{(j,k)}$ and, likewise, the control tubes are parameterized via $U_k = \bigoplus_{j=0}^{k} U_{(j,k)}$. Aside the parameterized state and control tubes $\mathbf{X}_N = \{X_k\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{N-1} = \{U_k\}_{k\in\mathbb{N}_{N-1}}$ we also utilize the separable control policy $\Pi_{N-1} := \{\pi_k(\cdot, X_k, U_k)\}_{k\in\mathbb{N}_{N-1}}$. Namely, for each k the control laws $\pi_k(\cdot, X_k, U_k) : X_k \to U_k$ are such that for all $y \in X_k$, $\pi_k(y, X_k, U_k) = \sum_{j=0}^{k} \pi_{(j,k)}(y_j, X_{(j,k)}, U_{(j,k)})$ where $y_0, y_1, \ldots, y_k$ represents the decomposition of y (so that $y = \sum_{j=0}^{k} y_j$ with $y_j \in X_{(j,k)}$) and, where for each j and k, $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)}) : X_{(j,k)} \to U_{(j,k)}$ are the partial control laws at time k.

The state and control tube parameterization $\mathbf{X}_N = \{X_k\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{N-1} = \{U_k\}_{k\in\mathbb{N}_{N-1}}$ deployed in this paper is exact in that explicit use is made of additive uncertainty set (assumed to be polytopic and described by its extreme points) in such a way that the employed state tube $\mathbf{X}_N$ contains only all possible state trajectories when the input trajectories are chosen to lie in the control tube $\mathbf{U}_{N-1}$ and, at each prediction time, the additive disturbance is allowed to take any value within the polytopic uncertainty set.

In this manuscript, we focus on the linear systems with polytopic constraint sets and develop a TMPC method, termed throughout as the parameterized TMPC (PTMPC), which allows for the efficient computation of the parameterized state and control tubes as well as the corresponding control policy via optimization. We introduce a parameterized tube optimal control (PTOC) problem, discuss its relevant topological properties and provide its reformulation as a single tractable linear programming problem. We also study the application of PTOC to the synthesis of robustly stabilizing PTMPC laws guaranteeing strong system theoretic properties of the controlled constrained but uncertain dynamics. Under mild conditions, we establish that the proposed PTMPC control law guarantees relevant set invariance and robust stability properties. We also discuss necessary computational issues and offer three illustrative examples. We demonstrate the advantages of our method and, in particular, we show that it is more general than recent method for robust model predictive control synthesis utilizing the so–called disturbance affine control policies [15–18].

Paper Structure

Sections 2 and 3 provide preliminaries and discuss the main idea of parameterized state and control tubes. Section 4 discusses the relevant tube model predictive control terminal constraint set and examines the local behavior of the parameterized state tubes within that set (and also indicates relevant properties for the corresponding control tubes). Section 5 introduces the parameterized tube optimal control problem and comments on the corresponding topological properties. Section 6 discusses the parameterized tube model predictive control and establishes the relevant system theoretic properties. Section 7 comments on computational issues and provides illustrative examples while Section 8 closes the paper with a few concluding remarks.

Basic Nomenclature and Definitions

The sets of non–negative, positive integers and non–negative reals are denoted by $\mathbb{N}$, $\mathbb{N}_+$, and $\mathbb{R}_+$, respectively, i.e. $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ := \{1, 2, \ldots\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. Given non–negative integers $a \in \mathbb{N}$ and $b \in \mathbb{N}$ such that $a < b$ we denote $\mathbb{N}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$; We write $\mathbb{N}_b$ for $\mathbb{N}_{[0:b]}$. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ denotes the largest absolute value of its eigenvalues.

Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y : x \in X, y \in Y\}$, we write $x \oplus X$ instead of $\{x\} \oplus X$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) the image of X under M is denoted by $MX := \{Mx : x \in X\}$. Given a set $Z \subset \mathbb{R}^{n+m}$ its projection onto $\mathbb{R}^n$ is denoted by $\text{Projection}_{\mathbb{R}^n}(Z) = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ such that } (x, y) \in Z\}$. If $f(\cdot)$ is a set–valued function from, say, X into U, namely, its values are subsets of U, then its graph is the set $\text{graph}(f) := \{(x, y) : x \in X, y \in f(x)\}$.

A set $X \subset \mathbb{R}^n$ is said to be a non–trivial set if it is a proper, non–empty, subset of $\mathbb{R}^n$ and it is not a singleton set. A set $X \subset \mathbb{R}^n$ is *a C–set* if it is compact, convex, and contains the origin. A set $X \subset \mathbb{R}^n$ is *a proper C–set, or just a PC–set,* if it is a C–set and contains the origin in its (non–empty) interior. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half–spaces and a *polytope* is the closed and bounded polyhedron. The interior of a set X is denoted by interior(X). Given a set $X \subset \mathbb{R}^n$, convh(X) denotes its convex hull. Given a non–empty closed convex set $X \subset \mathbb{R}^n$ the support function $\mathcal{S}(X, \cdot)$ is given by:

$$\mathcal{S}(X, y) := \sup_x \{y'x \ : \ x \in X\} \text{ for } y \in \mathbb{R}^n.$$

Given a PC–set L in $\mathbb{R}^n$, the function $\mathcal{G}(L, \cdot)$ defined by:

$$\mathcal{G}(L, x) := \min_\mu \{\mu \ : \ x \in \mu L, \ \mu \in \mathbb{R}_+\} \text{ for } x \in \mathbb{R}^n$$

is called the gauge (Minkowski) function of L. If L is a symmetric PC–set in $\mathbb{R}^n$, then the gauge (Minkowski) function of L induces the vector norm $|x|_L := \mathcal{G}(L, x)$. Given a symmetric PC–set L in $\mathbb{R}^n$ and a non–empty closed set $X \subset \mathbb{R}^n$, the function dist$(L, \cdot, X)$ given by:

$$\text{dist}(L, y, X) := \inf_x \{|x - y|_L \ : \ x \in X\} \text{ for } y \in \mathbb{R}^n,$$

is called the distance function associated with the set X (often abbreviated to the distance function for the typographical reasons).

For typographical convenience, we distinguish row vectors from column vectors only when needed and employ the same symbol for a variable x and its vectorized form in the algebraic expressions. For clarity, proofs of less obvious statements are given in the appendices.

2 Preliminaries & Motivation for Tube Patameterization

2.1 Setting and Basic Definitions

We consider linear, time–invariant, discrete time systems, given by:

$$x^+ = Ax + Bu + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control, x^+ is the successor state, $w \in \mathbb{R}^n$ is the disturbance taking values in the set $\mathbb{W} \subset \mathbb{R}^n$ and matrices A and B are of compatible dimensions, i.e. $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. Thus, if at any time $k \in \mathbb{N}$ the state is x_k, the applied control is u_k and the disturbance is w_k, the state at time $k+1$ satisfies $x_{k+1} = Ax_k + Bu_k + w_k$. The system variables x, u and w are subject to hard constraints:

$$x \in \mathbb{X}, \ u \in \mathbb{U} \text{ and } \ w \in \mathbb{W}. \tag{2.2}$$

The following standing assumptions and clarifying interpretation will be employed throughout the paper:

Assumption 1 *The matrix pair $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is stabilizable.*

Assumption 2 *The state and control constraint sets $\mathbb{X}$ and $\mathbb{U}$ are PC–polytopic sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, given by irreducible representations:*

$$\mathbb{X} := \{x \in \mathbb{R}^n \ : \ \forall i \in \mathbb{N}_{[1:p]}, \ F_i^T x \le 1\}, \ \textit{and}, \tag{2.3a}$$

$$\mathbb{U} := \{u \in \mathbb{R}^m \ : \ \forall i \in \mathbb{N}_{[1:r]}, \ G_i^T u \le 1\}. \tag{2.3b}$$

The disturbance constraint set $\mathbb{W}$ is a non–trivial C–polytopic set in $\mathbb{R}^n$ given by:

$$\mathbb{W} := \text{convh}(\{\tilde{w}_i \in \mathbb{R}^n \ : \ i \in \mathbb{N}_{[1:q]}\}), \tag{2.4}$$

where $\tilde{w}_i \in \mathbb{R}^n, \ i \in \mathbb{N}_{[1:q]}$ are extreme points of the disturbance set $\mathbb{W}$ and are known.

Interpretation 1 *At any time instance $k \in \mathbb{N}$, the state x_k is known when the current control action u_k is determined, while the current disturbance w_k and future disturbances $w_{k+i}, \ i \in \mathbb{N}_+$ are not known and can take any arbitrary values $w_{k+i} \in \mathbb{W}, \ i \in \mathbb{N}$.*

For a given control function $\kappa(\cdot) \ : \ \mathbb{R}^n \to \mathbb{R}^m$, the controlled uncertain dynamics takes the form:

$$x^+ = Ax + B\kappa(x) + w, \tag{2.5}$$

whose variables are, in view of (2.2), subject to constraints:

$$x \in \mathbb{X}_\kappa := \{x \in \mathbb{X} \ : \ \kappa(x) \in \mathbb{U}\} \text{ and } w \in \mathbb{W}. \tag{2.6}$$

We can now recall basic set invariance and stability related notions utilized in this manuscript.

Definition 1 *A set $\Omega \subseteq \mathbb{R}^n$ is said to be a robust positively invariant (RPI) set for the system $x^+ = Ax + B\kappa(x) + w$ given by (2.5) and the constraint set $(\mathbb{X}_\kappa, \mathbb{W})$ given by (2.6) if and only if $\Omega \subseteq \mathbb{X}_\kappa$ and for all $x \in \Omega$ and all $w \in \mathbb{W}$ it holds that $Ax + B\kappa(x) + w \in \Omega$.*

Definition 2 *A set $\underline{\Omega} \subseteq \mathbb{R}^n$ is robustly exponentially stable for the system $x^+ = Ax + B\kappa(x) + w$ given by (2.5) and the constraint set $(\mathbb{X}_\kappa, \mathbb{W})$ given by (2.6) with the basin of attraction being equal to the set $\bar{\Omega} \subseteq \mathbb{R}^n$ if and only if $\underline{\Omega} \subseteq \bar{\Omega} \subseteq \mathbb{X}_\kappa$, and, for any sequence $\{x_k\}_{k \in \mathbb{N}}$, generated by (2.5) with any arbitrary $x_0 \in \bar{\Omega}$ and any arbitrary disturbance sequence $\{w_k\}_{k \in \mathbb{N}}$, it holds that, for all $k \in \mathbb{N}$, $\operatorname{dist}(L, x_k, \bar{\Omega}) = 0$ and $\operatorname{dist}(L, x_k, \underline{\Omega}) \leq a^k b \operatorname{dist}(L, x_0, \underline{\Omega})$ for some scalars $a \in [0, 1)$ and $b \in [0, \infty)$ and where L is a symmetric, PC–set in $\mathbb{R}^n$.*

2.2 Separable Prediction Scheme

Let $x = x_0$ denote the current state, x_k for $k \in \mathbb{N}_{[1,N]}$ denote the state at prediction time k and w_k for $k \in \mathbb{N}_{N-1}$ the relevant value of the additive disturbance at prediction time k. Linearity of the system (2.1), in conjunction with Interpretation 1, suggests that if at any prediction time $k \in \mathbb{N}$ it was known that the state x_k satisfied $x_k = \sum_{j=0}^{k} x_{(j,k)}$ and the applied control u_k was given by $u_k = \sum_{j=0}^{k} u_{(j,k)}$ then the state at time $k+1 \in \mathbb{N}$ would satisfy $x_{k+1} = \sum_{j=0}^{k+1} x_{(j,k+1)}$ with, for all $j \in \mathbb{N}_k$, $x_{(j,k+1)} = Ax_{(j,k)} + Bu_{(j,k)}$ and $x_{(k+1,k+1)} = w_k$. The benefit of treating each of w_0, w_1, ... separately is (as will become apparent in the sequel) that it avoids the exponential growth of the number of extreme points necessary to describe the effect of the propagation of the polytopic uncertainty over the prediction horizon. These observations suggest that it would be beneficial within the context of model predictive control under uncertainty to perform both the prediction and corresponding optimization by utilizing the separable prediction scheme shown in Table 1.

$\mathbf{x}_{(0,N)}$	$x_{(0,0)} = x$	$x_{(0,1)} = Ax_{(0,0)} + Bu_{(0,0)}$	$x_{(0,2)} = Ax_{(0,1)} + Bu_{(0,1)}$	$\dots$	$x_{(0,N-1)} = Ax_{(0,N-2)} + Bu_{(0,N-2)}$	$x_{(0,N)} = Ax_{(0,N-1)} + Bu_{(0,N-1)}$
$\mathbf{u}_{(0,N-1)}$	$u_{(0,0)}$	$u_{(0,1)}$	$u_{(0,2)}$	$\dots$	$u_{(0,N-1)}$	
$\mathbf{x}_{(1,N)}$		$x_{(1,1)} = w_0$	$x_{(1,2)} = Ax_{(1,1)} + Bu_{(1,1)}$	$\dots$	$x_{(1,N-1)} = Ax_{(1,N-2)} + Bu_{(1,N-2)}$	$x_{(1,N)} = Ax_{(1,N-1)} + Bu_{(1,N-1)}$
$\mathbf{u}_{(1,N-1)}$		$u_{(1,1)}$	$u_{(1,2)}$	$\dots$	$u_{(1,N-1)}$	
$\mathbf{x}_{(2,N)}$			$x_{(2,2)} = w_1$	$\dots$	$x_{(2,N-1)} = Ax_{(2,N-2)} + Bu_{(2,N-2)}$	$x_{(2,N)} = Ax_{(2,N-1)} + Bu_{(2,N-1)}$
$\mathbf{u}_{(2,N-1)}$			$u_{(2,2)}$	$\dots$	$u_{(2,N-1)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{x}_{(N-1,N)}$					$x_{(N-1,N-1)} = w_{N-2}$	$x_{(N-1,N)} = Ax_{(N-1,N-1)} + Bu_{(N-1,N-1)}$
$\mathbf{u}_{(N-1,N-1)}$				$\dots$	$u_{(N-1,N-1)}$	
$\mathbf{x}_{(N,N)}$						$x_{(N,N)} = w_{N-1}$
$\mathbf{x}_N$	$x = x_{(0,0)}$	$x_1 = \sum_{j=0}^{1} x_{(j,1)}$	$x_2 = \sum_{j=0}^{2} x_{(j,2)}$	$\dots$	$x_{N-1} = \sum_{j=0}^{N-1} x_{(j,N-1)}$	$x_N = \sum_{j=0}^{N} x_{(j,N)}$
$\mathbf{u}_{N-1}$	$u_0 = u_{(0,0)}$	$u_1 = \sum_{j=0}^{1} u_{(j,1)}$	$u_2 = \sum_{j=0}^{2} u_{(j,2)}$	$\dots$	$u_{N-1} = \sum_{j=0}^{N-1} u_{(j,N-1)}$	

Table 1: The Separable Prediction Scheme.

Within the context of robust model predictive control, the separable prediction scheme illustrated in Table 1 motivates the prediction and optimization over the sets of state and control sequences $\mathbf{x}_{(k,N)}$ and $\mathbf{u}_{(k,N-1)}$ specified as follows:

$$\forall k \in \mathbb{N}_N,\ \mathbf{x}_{(k,N)} := \{x_{(k,k)}, x_{(k,k+1)}, \dots, x_{(k,N-1)}, x_{(k,N)}\},\ \text{and,} \tag{2.7a}$$

$$\forall k \in \mathbb{N}_{N-1},\ \mathbf{u}_{(k,N-1)} := \{u_{(k,k)}, u_{(k,k+1)}, \dots, u_{(k,N-1)}\}, \tag{2.7b}$$

and satisfying, for all $k \in \mathbb{N}_{N-1}$,

$$\forall l \in \mathbb{N}_{[k:N-1]},\ x_{(k,l+1)} = Ax_{(k,l)} + Bu_{(k,l)}, \tag{2.8}$$

with the additional conditions that $x_{(0,0)} = x$ and, for all $k \in \mathbb{N}_{[1:N]}$, $x_{(k,k)} = w_{k-1}$. The sets of state sequences $\{\mathbf{x}_{(k,N)}\}_{k \in \mathbb{N}_N}$ and corresponding control sequences $\{\mathbf{u}_{(k,N-1)}\}_{k \in \mathbb{N}_{N-1}}$ satisfying (2.7)–(2.8) are shown in the rows of Table 1 and are referred to as the partial state and control sequences. Clearly, the partial state and control sequences $\mathbf{x}_{(0,N)}$ and $\mathbf{u}_{(0,N-1)}$ are uncertainty free from the time instant $j = 0$. Likewise, for any $k \in \mathbb{N}_{[1:N-1]}$, the partial state and control sequences $\mathbf{x}_{(j,N)}$ and $\mathbf{u}_{(j,N-1)}$ are uncertainty free from time instants $j \in \mathbb{N}_{[1:k]}$ and the singleton partial state sequence $\mathbf{x}_{(N,N)} = \{x_{(N,N)}\}$ is uncertainty free at the time instant $j = N$.

As the prediction has to be made at time $j = 0$, the knowledge of the actual sets of partial state sequences $\{\mathbf{x}_{(k,N)}\}_{k \in \mathbb{N}_N}$ and corresponding control sequences $\{\mathbf{u}_{(k,N-1)}\}_{k \in \mathbb{N}_{N-1}}$ is incomplete due to the uncertainty and this has to be taken into account appropriately. More precisely, at time instant $j = 0$, only partial state and control sequences $\mathbf{x}_{(0,N)}$ and $\mathbf{u}_{(0,N-1)}$ are uncertainty free, while the possible knowledge of the sets of partial state sequences $\{\mathbf{x}_{(k,N)}\}_{k \in \mathbb{N}_{[1:N]}}$ and corresponding control sequences $\{\mathbf{u}_{(k,N-1)}\}_{k \in \mathbb{N}_{[1:N-1]}}$ is limited due to dependence of the partial initial conditions $x_{(k,k)}$ on the actual values of the uncertainty w_{k-1} (the knowledge of which is not available at the time instant $j = 0$ but becomes available at the time instant $j = k$). This crucial difficulty can be avoided, under Assumption 2, by considering assumed sets of extreme control sequences $\{\mathbf{u}_{(i,k,N-1)}\}$ and the corresponding sets of extreme partial state sequences $\{\mathbf{x}_{(i,k,N)}\}$ specified as follows:

$$\forall k \in \mathbb{N}_{[1:N]},\ \forall i \in \mathbb{N}_{[1:q]},\ \mathbf{x}_{(i,k,N)} := \{x_{(i,k,k)}, x_{(i,k,k+1)}, \dots, x_{(i,k,N-1)}, x_{(i,k,N)}\},\ \text{and,} \tag{2.9a}$$

$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall i \in \mathbb{N}_{[1:q]},\ \mathbf{u}_{(i,k,N-1)} := \{u_{(i,k,k)}, u_{(i,k,k+1)}, \dots, u_{(i,k,N-1)}\}, \tag{2.9b}$$

and satisfying, for all $k \in \mathbb{N}_{[1:N-1]}$ and all $i \in \mathbb{N}_{[1:q]}$,

$$\forall l \in \mathbb{N}_{[k:N-1]},\ x_{(i,k,l+1)} = Ax_{(i,k,l)} + Bu_{(i,k,l)}, \tag{2.10}$$

with the additional conditions that, for all $k \in \mathbb{N}_{[1:N]}$ and all $i \in \mathbb{N}_{[1:q]}$, $x_{(i,k,k)} = \tilde{w}_i$ where $\tilde{w}_i$ is the i^{th} extreme point of the disturbance set $\mathbb{W}$. In accordance with usual MPC practice it would be customary to consider the extreme control sequences $\{\mathbf{u}_{(i,k,N-1)}\}$ only as degrees of freedom, but here for notational convenience both $\{\mathbf{x}_{(i,k,N)}\}$ and $\{\mathbf{u}_{(i,k,N-1)}\}$ are treated as degrees of freedom which of course need to obey the equality constraint (2.10).

In this paper we formalize the utilization of the separable prediction scheme within the context of model predictive control under uncertainty. We make use of the prediction and optimization over the sets of extreme partial state and corresponding control sequences (i.e. the sequences $\mathbf{x}_{(0,N)}$ and $\mathbf{u}_{(0,N-1)}$ as well as the sequences $\{\mathbf{x}_{(i,k,N)}\}$ and $\{\mathbf{u}_{(i,k,N-1)}\}$) and develop a computationally efficient tube model predictive control synthesis method. The prediction and optimization over the sets of extreme partial state and corresponding control sequences (i.e. the sequences $\mathbf{x}_{(0,N)}$ and $\mathbf{u}_{(0,N-1)}$ as well as the sequences $\{\mathbf{x}_{(i,k,N)}\}$ and $\{\mathbf{u}_{(i,k,N-1)}\}$) leads naturally to the parameterized state and control tubes discussed next.

3 Parameterized State and Control Tubes

Our first step is to introduce the notions of parameterized state and control tubes utilizing our preliminary observations outlined in Section 2. This is done in two stages; first, the partial state and control tubes parameterization is introduced, and, second, the state and control tubes parameterization is provided.

3.1 Partial State and Control Tubes : Parameterization & Controlled Dynamics

As already indicated in Section 2, we consider a set of extreme partial state and corresponding control sequences which lead to the notions of partial state and control tubes. Namely, for any time instant $k \in \mathbb{N}_N$, the partial state tube is a sequence of sets $\mathbf{X}_{(k,N)} := \{X_{(k,l)}\}_{l \in \mathbb{N}_{[k:N]}}$ formed from the corresponding sets of extreme partial state sequences $\{\mathbf{x}_{(i,k,N)}\}$ specified in (2.9a). More precisely, for any time instant $k \in \mathbb{N}_N$, the partial state tube is formed from the sets $X_{(k,l)}$ referred to as the partial state tube cross–sections and specified by:

$$\forall l \in \mathbb{N}_N,\ X_{(0,l)} := \{\tilde{x}_{(0,l)}\},\ \text{and,} \tag{3.1a}$$

$$\forall k \in \mathbb{N}_{[1:N]},\ \forall l \in \mathbb{N}_{[k:N]},\ X_{(k,l)} := \text{convh}(\{\tilde{x}_{(i,k,l)} \in \mathbb{R}^n\ :\ i \in \mathbb{N}_{[1:q]}\}). \tag{3.1b}$$

Similarly, for any time instant $k \in \mathbb{N}_{N-1}$, the partial control tube is a sequence of sets $\mathbf{U}_{(k,N-1)} := \{U_{(k,l)}\}_{l \in \mathbb{N}_{[k:N-1]}}$ formed from the corresponding sets of extreme partial control sequences $\{\mathbf{u}_{(i,k,N-1)}\}$ specified in (2.9b). As above, for any time instant $k \in \mathbb{N}_{[1:N-1]}$, the partial control tube is formed from the sets $U_{(k,l)}$ referred to as the partial control tube cross–sections and given by:

$$\forall l \in \mathbb{N}_{N-1},\ U_{(0,l)} := \{\tilde{u}_{(0,l)}\},\ \text{and,} \tag{3.2a}$$

$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ U_{(k,l)} := \text{convh}(\{\tilde{u}_{(i,k,l)} \in \mathbb{R}^m\ :\ i \in \mathbb{N}_{[1:q]}\}). \tag{3.2b}$$

It is noted that $\tilde{x}_{(0,l)}$ and $\tilde{u}_{(0,l)}$ are going to be taken (respectively) for the elements $x_{(0,l)}$ and $u_{(0,l)}$ of the nominal sequences $\mathbf{x}_{(0,N)}$ and $\mathbf{u}_{(0,N-1)}$; the tilde symbol is used here in order to keep the notation uniform.

Due to the presence of the uncertainty it is necessary to consider a generalized form of the separable prediction scheme given in Table 1. Namely, it is necessary to employ the tube separable prediction scheme given in Table 2, wherein the partial state and control tubes $\mathbf{X}_{(k,N)}$ and $\mathbf{U}_{(k,N-1)}$ as well as their corresponding cross–sections $X_{(j,k)}$ and $U_{(j,k)}$ are shown in the rows of Table 2.

As in (2.10), the partial state tubes controlled dynamics is specified via following recursions:

$$\forall l \in \mathbb{N}_{N-1},\ \tilde{x}_{(0,l+1)} = A\tilde{x}_{(0,l)} + B\tilde{u}_{(0,l)}, \tag{3.3a}$$

$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ \tilde{x}_{(i,k,l+1)} = A\tilde{x}_{(i,k,l)} + B\tilde{u}_{(i,k,l)}, \tag{3.3b}$$

with, in addition, conditions that:

$$\tilde{x}_{(0,0)} = x,\ \text{and,} \tag{3.4a}$$

$$\forall k \in \mathbb{N}_{[1:N]},\ \forall i \in \mathbb{N}_{[1:q]},\ \tilde{x}_{(i,k,k)} = \tilde{w}_i, \tag{3.4b}$$

$\mathbf{X}_{(0,N)}$	$X_{(0,0)}$	$X_{(0,1)}$	$X_{(0,2)}$	$\dots$	$X_{(0,N-1)}$	$X_{(0,N)}$
$\mathbf{U}_{(0,N-1)}$	$U_{(0,0)}$	$U_{(0,1)}$	$U_{(0,2)}$	$\dots$	$U_{(0,N-1)}$	
$\mathbf{X}_{(1,N)}$		$X_{(1,1)}$	$X_{(1,2)}$	$\dots$	$X_{(1,N-1)}$	$X_{(1,N)}$
$\mathbf{U}_{(1,N-1)}$		$U_{(1,1)}$	$U_{(1,2)}$	$\dots$	$U_{(1,N-1)}$	
$\mathbf{X}_{(2,N)}$			$X_{(2,2)}$		$X_{(2,N-1)}$	$X_{(2,N)}$
$\mathbf{U}_{(2,N-1)}$			$U_{(2,2)}$	$\dots$	$U_{(2,N-1)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{X}_{(N-1,N)}$					$X_{(N-1,N-1)}$	$X_{(N-1,N)}$
$\mathbf{U}_{(N-1,N-1)}$				$\dots$	$U_{(N-1,N-1)}$	
$\mathbf{X}_{(N,N)}$						$X_{(N,N)}$
$\mathbf{X}_N$	$X_0 = X_{(0,0)}$	$X_1 = \bigoplus_{j=0}^{1} X_{(j,1)}$	$X_2 = \bigoplus_{j=0}^{2} X_{(j,2)}$	$\dots$	$X_{N-1} = \bigoplus_{j=0}^{N-1} X_{(j,N-1)}$	$X_N = \bigoplus_{j=0}^{N} X_{(j,N)}$
$\mathbf{U}_{N-1}$	$U_0 = U_{(0,0)}$	$U_1 = \bigoplus_{j=0}^{1} U_{(j,1)}$	$U_2 = \bigoplus_{j=0}^{2} U_{(j,2)}$	$\dots$	$U_{N-1} = \bigoplus_{j=0}^{N-1} U_{(j,N-1)}$	

Table 2: The Tube Separable Prediction Scheme.

where $\tilde{w}_i,\ i \in \mathbb{N}_{[1:q]}$ are extreme points of the disturbance set $\mathbb{W}$ (so that for all $k \in \mathbb{N}_{[1:N]}$ we have $X_{(k,k)} = \mathbb{W}$).

A closer inspection of relations (3.1)–(3.4) reveals that the controlled dynamics of the partial state tubes is deterministic and, in fact, the controlled partial state tubes $\mathbf{X}_{(k,N)}$ are induced from the initial partial state tube cross–section $X_{(k,k)}$ and the corresponding partial control tube $\mathbf{U}_{(k,N-1)}$ via (3.3) and (3.4). A more relevant property of partial state and control tubes stems from the fact that once the uncertain state component $x_{(k,k)} = w_{k-1} \in \mathbb{W}$ becomes known at time instant $k \in \mathbb{N}_{[1:N-1]}$ it is possible to select at least one particular control sequence $\mathbf{u}_{(k,N-1)} = \{u_{(k,k)}, u_{(k,k+1)}, \dots, u_{(k,N-1)}\}$ whose terms belong to the corresponding partial control tube $\mathbf{U}_{(k,N-1)}$ and which ensures that the corresponding controlled partial state sequence $\mathbf{x}_{(k,N)} = \{x_{(k,k)}, x_{(k,k+1)}, \dots, x_{(k,N)}\}$ (where $\forall l \in \mathbb{N}_{[k:N-1]},\ x_{(k,l+1)} = Ax_{(k,l)} + Bu_{(k,l)}$) is maintained within the partial state tube $\mathbf{X}_{(k,N)}$. We now proceed to demonstrate this desirable property.

Let for any $k \in \mathbb{N}_{[1:N-1]}$, $\lambda_{(k,k)} := (\lambda_{(1,k,k)}, \lambda_{(2,k,k)}, \dots, \lambda_{(q,k,k)}) \in \mathbb{R}^q$ and let $\Lambda := \{\lambda \in \mathbb{R}^q\ :\ \forall i \in \mathbb{N}_{[1:q]},\ \lambda_{(i,k,k)} \geq 0,\ \text{and},\ \sum_{i=1}^q \lambda_{(i,k,k)} = 1\}$. We define, for any $k \in \mathbb{N}_{[1:N-1]}$ with $N \geq 2$,

$$\forall x_{(k,k)} \in X_{(k,k)},\ \Lambda_{(k,k)}(x_{(k,k)}) := \{\lambda_{(k,k)} \in \Lambda\ :\ x_{(k,k)} = \sum_{i=1}^{q} \lambda_{(i,k,k)} \tilde{x}_{(i,k,k)}\}, \tag{3.5a}$$

$$\forall x_{(k,k)} \in X_{(k,k)},\ \lambda^*_{(k,k)}(x_{(k,k)}) := \arg\min_{\lambda_{(k,k)}} \{\ \lambda^T_{(k,k)}\lambda_{(k,k)}\ :\ \lambda_{(k,k)} \in \Lambda_{(k,k)}(x_{(k,k)})\},\ \text{and}, \tag{3.5b}$$

$$\forall x_{(k,k)} \in X_{(k,k)},\ \mathbf{u}^*_{(k,N-1)}(x_{(k,k)}) := \{u^*_{(k,k)}(x_{(k,k)}), u^*_{(k,k+1)}(x_{(k,k)}), \dots, u^*_{(k,N-1)}(x_{(k,k)})\},\ \text{with}, \tag{3.5c}$$

$$\forall l \in \mathbb{N}_{[k:N-1]},\ u^*_{(k,l)}(x_{(k,k)}) := \sum_{i=1}^{q} \lambda^*_{(i,k,k)}(x_{(k,k)}) \tilde{u}_{(i,k,l)}. \tag{3.5d}$$

We note that for any arbitrary fixed $N \in \mathbb{N}_+$ with $N \geq 2$, $k \in \mathbb{N}_{[1:N-1]}$ and $x \in \mathbb{R}^n$, and any fixed partial state and control tubes $\mathbf{X}_{(k,N)}$ and $\mathbf{U}_{(k,N-1)}$ satisfying relations (3.1)–(3.4), the set $\Lambda_{(k,k)}(x_{(k,k)})$ is a polytope in $\mathbb{R}^q$ for each fixed $x_{(k,k)} \in X_{(k,k)}$. Furthermore, the graph of the set–valued map $\Lambda_{(k,k)}(\cdot)$, namely the set $\{(x_{(k,k)}, \lambda_{(k,k)})\ :\ x_{(k,k)} \in X_{(k,k)},\ \lambda_{(k,k)} \in \Lambda_{(k,k)}(x_{(k,k)})\}$ is a polytopic set in $\mathbb{R}^{n+q}$. Consequently, standard results [34] yield the fact that the function $\lambda^*_{(k,k)}(\cdot)\ :\ X_{(k,k)} \to \Lambda$, defined in (3.5b), is also, in general, a single–valued, piecewise affine and continuous function of $x_{(k,k)} \in X_{(k,k)}$. With this in mind, it follows that the function $\mathbf{u}^*_{(k,N-1)}(\cdot)$ defined in (3.5c) is, in general, a single–valued, piecewise affine and continuous function of $x_{(k,k)} \in X_{(k,k)}$. By construction, the function $\mathbf{u}^*_{(k,N-1)}(\cdot)$ satisfies, for all $x_{(k,k)} \in X_{(k,k)}$ and all $l \in \mathbb{N}_{[k:N-1]}$,:

$$u^*_{(k,l)}(x_{(k,k)}) = \sum_{i=1}^{q} \lambda^*_{(i,k,k)}(x_{(k,k)}) \tilde{u}_{(i,k,l)} \in \text{convh}(\{\tilde{u}_{(i,k,l)} \in \mathbb{R}^m\ :\ i \in \mathbb{N}_{[1:q]}\}) = U_{(k,l)}$$

and ensures that, for all $x_{(k,k)} \in X_{(k,k)}$ and all $l \in \mathbb{N}_{[k:N-1]}$, $x_{(k,l+1)} = Ax_{(k,l)} + Bu^*_{(k,l)}(x_{(k,k)}) \in X_{(k,l+1)}$. This follows by a direct mathematical induction argument given that $x_{(k,k)} = \sum_{i=1}^q \lambda^*_{(i,k,k)}(x_{(k,k)}) \tilde{x}_{(i,k,k)}$ and:

$$\begin{aligned} x_{(k,l+1)} = Ax_{(k,l)} + Bu^*_{(k,l)}(x_{(k,k)}) &= A\sum_{i=1}^{q} \lambda^*_{(i,k,k)}(x_{(k,k)}) \tilde{x}_{(i,k,l)} + B\sum_{i=1}^{q} \lambda^*_{(i,k,k)}(x_{(k,k)}) \tilde{u}_{(i,k,l)} \\ &= \sum_{i=1}^{q} \lambda^*_{(i,k,k)}(x_{(k,k)}) \left(A\tilde{x}_{(i,k,l)} + B\tilde{u}_{(i,k,l)}\right) = \sum_{i=1}^{q} \lambda^*_{(i,k,k)}(x_{(k,k)}) \tilde{x}_{(i,k,l+1)} \\ &\in \text{convh}(\{\tilde{x}_{(i,k,l+1)} \in \mathbb{R}^m\ :\ i \in \mathbb{N}_{[1:q]}\}) = X_{(k,l+1)}. \end{aligned}$$

These relevant observations are summarized formally by:

Proposition 1 *Pick arbitrary $N \in \mathbb{N}_+$ with $N \geq 2$, $k \in \mathbb{N}_{[1:N-1]}$ and $x \in \mathbb{R}^n$, and fix an arbitrary pair of the partial state and control tubes $\mathbf{X}_{(k,N)} = \{X_{(k,l)}\}_{l \in \mathbb{N}_{[k:N]}}$ and $\mathbf{U}_{(k,N-1)} = \{U_{(k,l)}\}_{l \in \mathbb{N}_{[k:N-1]}}$ satisfying* (3.1)–(3.4)*. Then: (i) the function $\lambda^*_{(k,k)}(\cdot) : X_{(k,k)} \to \Lambda$ is, in general, single–valued and continuous piecewise affine function, and, (ii) the function $\mathbf{u}^*_{(k,N-1)}(\cdot) : X_{(k,k)} \to U_{(k,k)} \times U_{(k,k+1)} \times \ldots \times U_{(k,N-1)}$ is, in general, single–valued and continuous piecewise affine function such that*

$$\forall x_{(k,k)} \in X_{(k,k)},\ \forall l \in \mathbb{N}_{[k:N-1]},\ x_{(k,l+1)} = Ax_{(k,l)} + Bu^*_{(k,l)}(x_{(k,k)}) \in X_{(k,l+1)}.$$

We note that for $N = 1$ the partial state and control tubes are composed from singletons, i.e. $\mathbf{X}_{(0,1)} = \{\tilde{x}_{(0,0)}, \tilde{x}_{(0,1)}\}$ and $\mathbf{U}_{(0,0)} = \{\tilde{u}_{(0,0)}\}$, satisfying $\tilde{x}_{(0,1)} = A\tilde{x}_{(0,0)} + B\tilde{u}_{(0,0)}$. Likewise, if $N \geq 2$, then for $k = 0$ the partial state and control tubes are also composed from singletons , i.e. $\mathbf{X}_{(0,N)} = \{\tilde{x}_{(0,0)}, \tilde{x}_{(0,1)}, \ldots, \tilde{x}_{(0,N)}\}$ and $\mathbf{U}_{(0,N-1)} = \{\tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}, \ldots, \tilde{u}_{(0,N)}\}$, satisfying by construction, for all $l \in \mathbb{N}_{[0:N-1]}$, $\tilde{x}_{(0,l+1)} = A\tilde{x}_{(0,l)} + B\tilde{u}_{(0,l)}$. Note that Proposition 1 establishes the relevant properties of the functions $\lambda^*_{(k,k)}(\cdot)$ and $\mathbf{u}^*_{(k,N-1)}(\cdot)$ in the general case; in particular special cases (e.g. when the disturbance set $\mathbb{W}$ is a simplex) the functions $\lambda^*_{(k,k)}(\cdot)$ and $\mathbf{u}^*_{(k,N-1)}(\cdot)$ can take the linear/affine form rather than piecewise affine.

Remark 1 *Notice that for any arbitrary $N \in \mathbb{N}_+$ with $N \geq 2$, $k \in \mathbb{N}_{[1:N-1]}$ and $x \in \mathbb{R}^n$, and any arbitrary but fixed pair of the partial state and control tubes $\mathbf{X}_{(k,N)}$ and $\mathbf{U}_{(k,N-1)}$ satisfying* (3.1)–(3.4)*, the functions $\lambda^*_{(k,k)}(\cdot) : X_{(k,k)} \to \Lambda$ and $\mathbf{u}^*_{(k,N-1)}(\cdot) : X_{(k,k)} \to U_{(k,k)} \times U_{(k,k+1)} \times \ldots \times U_{(k,N-1)}$ can be constructed and/or evaluated without the need to explicitly compute the convex hull operation appearing in* (3.1b) *and* (3.2b)*. As evident from* (3.5) *what is required is the knowledge of the sets of points $\{\tilde{x}_{(i,k,k)}\}_{i=1}^q$ and $\{\tilde{u}_{(i,j,k)}\}_{i=1}^q$ forming the sets of extreme points of the initial partial state tube cross–section $X_{(k,k)} = \mathbb{W}$ and the corresponding partial control tubes $\mathbf{U}_{(k,N-1)} = \{U_{(k,k)}, U_{(k,k+1)}, \ldots, U_{(k,N-1)}\}$. Indeed, for any $k \in \mathbb{N}_{[1:N-1]}$, the value of the function $\lambda^*_{(k,k)}(x_{(k,k)})$ can be evaluated for any fixed $x_{(k,k)} \in X_{(k,k)}$ by solving a qudratic programming problem in* (3.5b)*, while the value of the function $\mathbf{u}^*_{(k,N-1)}(x_{(k,k)})$ is then easily calculated by performing algebraic operations in* (3.5d)*. Furthermore, we can, by utilizing our construction, induce actual partial state sequences $\{\mathbf{x}_{(k,N)}\}_{k \in \mathbb{N}_{[0:N]}}$ via the following recursions:*

$$\forall l \in \mathbb{N}_{[0:N-1]},\ \tilde{x}_{(0,l+1)} = A\tilde{x}_{(0,l)} + B\tilde{u}_{(0,l)},\ \text{with } \tilde{x}_{(0,0)} = x, \tag{3.6a}$$

$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ x_{(k,l+1)} = Ax_{(k,l)} + Bu^*_{(k,l)}(x_{(k,k)}),\ \text{with } x_{(k,k)} = w_{k-1}, \tag{3.6b}$$

where $\mathbf{u}_{(0,N-1)}$ is the partial control sequence associated with the partial state sequence $\mathbf{x}_{(0,N)}$ and the partial control sequences $\{\mathbf{u}_{(k,N-1)}\}_{k \in \mathbb{N}_{[0:N-1]}}$ are obtained by utilizing (3.5)*. We note that if we set:*

$$u_0 = \tilde{u}_{(0,0)},\ \text{and, } \forall k \in \mathbb{N}_{[1:N-1]},\ u_k = \tilde{u}_{(0,k)} + \sum_{j=1}^{k} u^*_{(j,k)}(x_{(j,j)}). \tag{3.7}$$

Then, for any admissible disturbance sequence $\mathbf{w}_{N-1} := \{w_k \in \mathbb{W}\}_{k \in \mathbb{N}_{N-1}}$, the corresponding state sequence $\{x_k\}_{k \in \mathbb{N}_N}$ generated by:

$$\forall k \in \mathbb{N}_{[0:N-1]},\ x_{k+1} = Ax_k + Bu_k + w_k, \tag{3.8}$$

for $x = \tilde{x}_{(0,0)}$, satisfies:

$$\forall k \in \mathbb{N}_{[1:N]},\ x_k = \tilde{x}_{(0,k)} + \sum_{j=1}^{k} x_{(j,k)}, \tag{3.9}$$

where partial state components $x_{(j,k)}$ are given as in (3.6b)*. We note that (for any $k \in \mathbb{N}_{[0:N-1]}$ and any $j \in \mathbb{N}_{[0:k]}$) our construction and Propositions 1 yield the fact that if $x_{(j,k)} \in X_{(j,k)}$ then $x_{(j,k+1)} = Ax_{(j,k)} + Bu^*_{(j,k)}(x_{(j,j)}) \in X_{(j,k+1)}$. Consequently, we can guarantee at the time instant $j = 0$ that any actual state and control sequences generated via* (3.7)–(3.9)*, for any $x \in \mathbb{R}^n$ (with, of course, $\tilde{x}_{(0,0)} = x$) and for any admissible disturbance sequence $\mathbf{w}_{N-1} := \{w_k \in \mathbb{W}\}_{k \in \mathbb{N}_{N-1}}$, satisfy:*

$$\forall k \in \mathbb{N}_N,\ x_k \in \bigoplus_{j=0}^{k} X_{(j,k)} \text{ and } \forall k \in \mathbb{N}_{N-1},\ u_k \in \bigoplus_{j=0}^{k} U_{(j,k)}. \tag{3.10}$$

Hence, the partial state and control tubes provide a feedback mechanism to account for and counteract the uncertainty and its effect on the system evolution within the prediction horizon in a feedback fashion.

3.2 State and Control Tubes : Parameterization, Controlled Dynamics & Induced Control Policy

Following the discussion of Remark 1, relations (3.6)–(3.10) suggest that, as already indicated in Table 2, the partial state and control tubes can be employed to obtain the parameterized state and control tubes. More precisely, the parameterized state tube is a sequence of sets $\mathbf{X}_N := \{X_k\}_{k\in\mathbb{N}_N}$ where sets X_k are given, for all $k \in \mathbb{N}_N$, by:

$$X_k := \bigoplus_{j=0}^{k} X_{(j,k)}, \tag{3.11}$$

and the sets $X_{(0,k)}, X_{(1,k)}, \ldots, X_{(k,k)}$ are the partial state tubes cross–sections at time k given as in (3.1). Likewise, the parameterized control tube is a sequence of sets $\mathbf{U}_{N-1} := \{U_k\}_{k\in\mathbb{N}_{N-1}}$ where sets U_k are given, for all $k \in \mathbb{N}_{N-1}$, by:

$$U_k := \bigoplus_{j=0}^{k} U_{(j,k)}, \tag{3.12}$$

and the sets $U_{(0,k)}, U_{(1,k)}, \ldots, U_{(k,k)}$ are the partial control tubes cross–sections at time k given as in (3.2).

Remark 2 *Before proceeding, we wish to stress that the Minkowski sum and convex hull operations employed for the parameterization of the partial and overall state and control tubes are merely utilized for the purpose of necessary analysis. We will demonstrate that the actual implementation of our method does not require any explicit computation of the relevant set theoretic operations and is computationally tractable.*

We now demonstrate that our construction and Proposition 1 allows us to induce a separable control policy Π_{N-1} alluded to in the introduction and associated with the parameterized state and control tubes $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ satisfying relations (3.1)–(3.4) and (3.11)–(3.12). Namely, Proposition 1 and discussion of Remark 1 imply that, once a pair of the parameterized state and control tubes $\mathbf{X}_N = \{X_0, X_1, \ldots, X_N\}$ and $\mathbf{U}_{N-1} = \{U_0, U_1, \ldots, U_{N-1}\}$ satisfying relations (3.1)–(3.4) and (3.11)–(3.12) is chosen, we can ensure the controlled transition from the parameterized state tube cross–section X_k at the time instant $k \in \mathbb{N}_{[0:N-1]}$ to the parameterized state tube cross–section X_{k+1} at the time instant $k+1 \in \mathbb{N}_{[1:N]}$ regardless the presence of the uncertainty. Furthermore, the appropriate control actions u_k applied at the possible states x_k of the the parameterized state tube cross–section X_k at the time instant $k \in \mathbb{N}_{[0:N-1]}$ can be generated in such a way that they belong to the corresponding parameterized control tube cross–section U_k at the time instant k. Suppose that $N \in \mathbb{N}_+$ is such that $N \geq 2$ as otherwise there is nothing to discuss. Let, for any $k \in \mathbb{N}_{[1:N-1]}$ and any $j \in \mathbb{N}_{[1:k]}$, $\lambda_{(j,k)} := (\lambda_{(1,j,k)}, \lambda_{(2,j,k)}, \ldots, \lambda_{(q,j,k)}) \in \mathbb{R}^q$. Let also $\Lambda := \{\lambda \in \mathbb{R}^q \ : \ \forall i \in \mathbb{N}_{[1:q]},\ \lambda_{(i,j,k)} \geq 0, \text{ and, } \sum_{i=1}^{q} \lambda_{(i,j,k)} = 1\}$. Similarly as in (3.5), we define, for any $k \in \mathbb{N}_{[1:N-1]}$ and any $j \in \mathbb{N}_{[1:k]}$,

$$\forall x_{(j,k)} \in X_{(j,k)},\ \Lambda_{(j,k)}(x_{(j,k)}) := \{\lambda_{(j,k)} \in \Lambda \ : \ x_{(j,k)} = \sum_{i=1}^{q} \lambda_{(i,j,k)} \tilde{x}_{(i,j,k)}\}, \tag{3.13a}$$

$$\forall x_{(j,k)} \in X_{(j,k)},\ \lambda^*_{(j,k)}(x_{(j,k)}) := \arg\min_{\lambda_{(j,k)}} \{\ \lambda^T_{(j,k)} \lambda_{(j,k)} \ : \ \lambda_{(j,k)} \in \Lambda_{(j,k)}(x_{(j,k)})\}, \text{ and,} \tag{3.13b}$$

$$\forall x_{(j,k)} \in X_{(j,k)},\ \pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)}) := \sum_{i=1}^{q} \lambda^*_{(i,j,k)}(x_{(j,k)}) \tilde{u}_{(i,j,k)}. \tag{3.13c}$$

In view of the discussion succeeding relations (3.5), we stress that for any given $x \in \mathbb{R}^n$ and any fixed parameterized state and control tubes $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ satisfying relations (3.1)–(3.4) and (3.11)–(3.12) and for any $k \in \mathbb{N}_{[1:N-1]}$ and any $j \in \mathbb{N}_{[1:k]}$, the set $\Lambda_{(j,k)}(x_{(j,k)})$ is a polytope in $\mathbb{R}^q$ for each fixed $x_{(j,k)} \in X_{(j,k)}$ and, furthermore, the set $\{(x_{(j,k)}, \lambda_{(j,k)}) \ : \ x_{(j,k)} \in X_{(j,k)},\ \lambda_{(j,k)} \in \Lambda_{(j,k)}(x_{(j,k)})\}$ is a polytopic set in $\mathbb{R}^{n+q}$. Consequently, the functions $\lambda^*_{(j,k)}(\cdot) \ : \ X_{(j,k)} \to \Lambda$ and $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)}) \ : \ X_{(j,k)} \to U_{(j,k)}$, defined in (3.13b) and (3.13c), are, in general, single–valued, piecewise affine and continuous function of $x_{(j,k)} \in X_{(j,k)}$. Furthermore the function $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)})$ ensures, by construction, that for all $x_{(j,k)} \in X_{(j,k)}$ it holds that:

$$\begin{aligned} x_{(j,k+1)} &= Ax_{(j,k)} + B\pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)}) = A\sum_{i=1}^{q} \lambda^*_{(i,j,k)}(x_{(j,k)})\tilde{x}_{(i,j,k)} + B\sum_{i=1}^{q} \lambda^*_{(i,j,k)}(x_{(j,k)})\tilde{u}_{(i,j,k)} \\ &= \sum_{i=1}^{q} \lambda^*_{(i,j,k)}(x_{(j,k)}) \left(A\tilde{x}_{(i,j,k)} + B\tilde{u}_{(i,j,k)}\right) = \sum_{i=1}^{q} \lambda^*_{(i,j,k)}(x_{(j,k)})\tilde{x}_{(i,j,k+1)} \\ &\in \text{convh}(\{\tilde{x}_{(i,j,k+1)} \in \mathbb{R}^m \ : \ i \in \mathbb{N}_{[1:q]}\}) = X_{(j,k+1)}. \end{aligned}$$

These observations are summarized formally by:

Proposition 2 *Pick an arbitrary $N \in \mathbb{N}_+$ with $N \geq 2$ and $x \in \mathbb{R}^n$, and fix an arbitrary pair of the parameterized state and control tubes $\mathbf{X}_N = \{X_0, X_1, \ldots, X_N\}$ and $\mathbf{U}_{N-1} = \{U_0, U_1, \ldots, U_{N-1}\}$ satisfying relations* (3.1)–(3.4) *and* (3.11)–(3.12)*. Then, for any $k \in \mathbb{N}_{[1:N-1]}$ and any $j \in \mathbb{N}_{[1:k]}$,: (i) the function $\lambda^*_{(j,k)}(\cdot) \ : \ X_{(j,k)} \to \Lambda$ is, in general, single–valued and continuous piecewise affine function, and, (ii) the function $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)}) \ : \ X_{(j,k)} \to U_{(j,k)}$ is, in general, single–valued and continuous piecewise affine function such that:*

$$\forall x_{(j,k)} \in X_{(j,k)},\ \ Ax_{(j,k)} + B\pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)}) \in X_{(j,k+1)}.$$

Remark 3 *Similarly as discussed in Remark 1, the functions* $\lambda^*_{(j,k)}(\cdot) : X_{(j,k)} \to \Lambda$ *and* $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)}) : X_{(j,k)} \to U_{(j,k)}$ *can be constructed and/or evaluated without the need to compute explicitly the convex hull operation appearing in* (3.1b) *and* (3.2b). *The knowledge of the sets of points* $\{\tilde{x}_{(i,j,k)}\}_{i=1}^q$ *and* $\{\tilde{u}_{(i,j,k)}\}_{i=1}^q$ *forming the set of extreme points of the corresponding partial state and control tubes cross–sections* $X_{(j,k)}$ *and* $U_{(j,k)}$ *suffices for the necessary computations. Indeed, for any* $k \in \mathbb{N}_{[1:N-1]}$ *and any* $j \in \mathbb{N}_{[1:k]}$*, the value of the function* $\lambda^*_{(j,k)}(x_{(j,k)})$ *can be evaluated for any fixed* $x_{(j,k)} \in X_{(j,k)}$ *by solving a quadratic programming problem in* (3.13b)*, so that the value of the function* $\pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)})$ *is then easily calculated by performing algebraic operations in* (3.13c).

Our next observation shows that once the parameterized state and control tubes $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ satisfying relations (3.1)–(3.4) and (3.11)–(3.12) are fixed we can construct a separable control policy Π_{N-1} ensuring the controlled transition from the parameterized state tube cross–section X_k at the time instant $k \in \mathbb{N}_{[0:N-1]}$, regardless the presence of the uncertainty, to the parameterized state tube cross–section X_{k+1} at the time instant $k+1 \in \mathbb{N}_{[1:N]}$:

Proposition 3 *Pick an arbitrary* $N \in \mathbb{N}_+$ *with* $N \geq 2$ *and* $x \in \mathbb{R}^n$*, and fix an arbitrary pair of the parameterized state and control tubes* $\mathbf{X}_N = \{X_0, X_1, \ldots, X_N\}$ *and* $\mathbf{U}_{N-1} = \{U_0, U_1, \ldots, U_{N-1}\}$ *satisfying relations* (3.1)–(3.4) *and* (3.11)–(3.12). *Then (i) for all* $k \in \mathbb{N}_{[0:N]}$ *and all* $j \in \mathbb{N}_{[0:k]}$*,*

$$\forall x_k \in X_k,\ \exists \{x_{(j,k)} \in X_{(j,k)}\}_{j=0}^k \text{ such that } x_k = \sum_{j=0}^k x_{(j,k)},$$

and, (ii) for all $k \in \mathbb{N}_{[0:N-1]}$ *and all* $j \in N_{[0:k]}$*,*

$$\forall \{x_{(j,k)} \in X_{(j,k)}\}_{j=0}^k,\ x_k = \sum_{j=0}^k x_{(j,k)} \in X_k, \tag{3.14a}$$

$$\pi_k(x_k, X_k, U_k) := \tilde{u}_{(0,k)} + \sum_{j=1}^k \pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)}) \in U_k \text{ and } Ax_k + B\pi_k(x_k, X_k, U_k) \oplus \mathbb{W} \subseteq X_{k+1}, \tag{3.14b}$$

where the functions $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)}) : X_{(j,k)} \to U_{(j,k)}$ *are defined as in* (3.13).

The main consequence of Propositions 2 and 3 is summarized next:

Corollary 1 *Pick an arbitrary* $N \in \mathbb{N}_+$ *with* $N \geq 2$ *and* $x \in \mathbb{R}^n$*, and fix an arbitrary pair of the parameterized state and control tubes* $\mathbf{X}_N = \{X_0, X_1, \ldots, X_N\}$ *and* $\mathbf{U}_{N-1} = \{U_0, U_1, \ldots, U_{N-1}\}$ *satisfying relations* (3.1)–(3.4) *and* (3.11)–(3.12). *Let* $\mathbf{w}_{N-1} := \{w_k \in \mathbb{W}\}_{k=0}^{N-1}$ *be an arbitrary uncertainty sequence. Let also, for all* $k \in \mathbb{N}_{[1:N]}$*,* $x_{(k,k)} = w_{k-1}$*. Finally let:*

$$\forall k \in \mathbb{N}_{[0:N-1]},\ x_{k+1} = Ax_k + B\pi_k(x_k, X_k, U_k) + w_k,\ \text{and,}$$
$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ x_{(k,l+1)} = Ax_{(k,l)} + B\pi_{(k,l)}(x_{(k,l)}, X_{(k,l)}, U_{(k,l)}),$$

where functions $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)})$ *and* $\pi_k(\cdot, X_k, U_k)$ *are defined as in* (3.7) *and* (3.8). *Then:*

$$\forall k \in \mathbb{N}_{[0:N]},\ x_k = \tilde{x}_{(0,k)} + \sum_{j=1}^k x_{(j,k)} \in X_k,\ \text{and,}$$

$$\forall k \in \mathbb{N}_{[0:N-1]},\ \pi_k(x_k, X_k, U_k) = \tilde{u}_{(0,k)} + \sum_{j=1}^k \pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)}) \in U_k.$$

Remark 4 *At this stage we are ready to make a few remarks indicating advantages of the introduced parameterization of the state and control tubes over some of existing proposals in the literature. As evident, the implicit representation of parameterized state tubes requires the knowledge of the sets of points* $\{\tilde{x}_{(0,k)}\}_{k \in \mathbb{N}_N}$ *and* $\{\tilde{x}_{(i,j,k)}\}$ *where* $i \in \mathbb{N}_{[1:q]}$*,* $k \in \mathbb{N}_{[1:N]}$ *and* $j \in \mathbb{N}_{[k:N]}$*. Consequently, the parameterized state tube is induced from* $N_{\mathbf{X}_N}$ n*–dimensional variables (where* $N_{\mathbf{X}_N} := N+1+q\frac{N(N+1)}{2}$*) or, equivalently, from* $n(N+1)+qn\frac{N(N+1)}{2}$ *real numbers. Likewise, the parameterized control tube is induced from* $N_{\mathbf{U}_{N-1}}$ m*–dimensional variables (where* $N_{\mathbf{U}_{N-1}} := N + q\frac{(N-1)N}{2}$*) or, equivalently, from* $mN + qm\frac{(N-1)N}{2}$ *real numbers. Therefore, the total number of real variables characterizing a pair of parameterized state and control tubes is given by* $n(N+1) + qn\frac{N(N+1)}{2} + mN + qm\frac{(N-1)N}{2}$ *and is, clearly, a quadratic function of the horizon length* N*. We also note that the sets of points* $\{\tilde{x}_{(0,k)}\}_{k \in \mathbb{N}_N}$ *and* $\{\tilde{x}_{(i,j,k)}\}$ *where* $i \in \mathbb{N}_{[1:q]}$*,* $k \in \mathbb{N}_{[1:N]}$ *and* $j \in \mathbb{N}_{[k:N]}$ *can be eliminated by utilizing the equality constraints* (2.10) *which, in turn, reduces the free variables to those belonging to the sets of points* $\{\tilde{u}_{(0,k)}\}_{k \in \mathbb{N}_{N-1}}$ *and* $\{\tilde{u}_{(i,j,k)}\}$ *where* $i \in \mathbb{N}_{[1:q]}$*,* $k \in \mathbb{N}_{[1:N-1]}$ *and* $j \in \mathbb{N}_{[k:N-1]}$ *and which are clearly induced from* $mN + qm\frac{(N-1)N}{2}$ *real numbers. This is in strong contrast with* *proposals [13,14] which suffer from the fact that the total number of decision variables employed in methods of [13,14] is an exponential function of the horizon length N (i.e. it grows proportionally to q^N). In addition, as established in Propositions 2, the functions $\pi_{(j,k)}(\cdot, X_{(j,k)}, U_{(j,k)})$, $k \in \mathbb{N}_{[1:N-1]}$, $j \in \mathbb{N}_{[1:k]}$ are, in general, single–valued, piecewise affine and continuous functions of $x_{(j,k)} \in X_{(j,k)}$. Hence,the control policy specified via* (3.14b) *and* (3.13) *is more general than the so–called disturbance affine control policy utilized in [15–18]. The benefits of this extra degree of generality are illustrated by means of a numerical example in Section 7.*

As the parameterized state and control tubes allow for the prediction under uncertainty in a feedback fashion, our next step is to study the local behavior of the parameterized state and control tubes.

4 Local Behavior of Parameterized State and Control Tubes in Terminal Constraint Set

As it is customary in robust model predictive control, an appropriate terminal constraint set is introduced in order to guarantee the relevant invariance and stability properties. The terminal constraint set is, as usual, obtained by considering simpler local state and control tube dynamics which are guaranteed to satisfy constraints. More precisely, we consider the simpler form of local dynamics under uncertainty induced by utilizing a linear state feedback control law $u = Kx$:

$$x^+ = (A + BK)x + w. \tag{4.1}$$

We invoke the standard assumptions on the terminal constraint set and the corresponding matrix $K \in \mathbb{R}^{m\times n}$:

Assumption 3 *(i) The matrix $K \in \mathbb{R}^{m\times n}$ is such that $A+BK$ is strictly stable, i.e. $\rho(A+BK) < 1$, and, (ii) The terminal constraint set $\mathbb{X}_f$ is the maximal robust positively invariant set for the system* (4.1) *and the constraint set $(\mathbb{X}_K, \mathbb{W})$ where $\mathbb{X}_K := \{x \in \mathbb{X} \ : \ Kx \in \mathbb{U}\}$, i.e. $\mathbb{X}_f$ is the maximal set (with respect to the set inclusion) such that*

$$(A + BK)\mathbb{X}_f \oplus \mathbb{W} \subseteq \mathbb{X}_f,\ \mathbb{X}_f \subseteq \mathbb{X},\ \textit{and},\ K\mathbb{X}_f \subseteq \mathbb{U}, \tag{4.2}$$

and, in addition, is a PC–polytopic set in $\mathbb{R}^n$ with its irreducible representation given by:

$$\mathbb{X}_f := \{x \in \mathbb{R}^n \ : \ \forall i \in \mathbb{N}_{[1:t]},\ H_i^T x \leq 1\}. \tag{4.3}$$

Remark 5 *We note that, as is well known (e.g. [35, 36]), Assumption 3 is easily satisfied under rather mild conditions. In particular, Assumption 3(ii) is satisfied under Assumptions 1, 2 and 3(i) and an additional requirement that the minimal robust positively invariant set (e.g. [35,37,38]), which is given by $X_\infty = \bigoplus_{i=0}^{\infty}(A+BK)^i\mathbb{W}$, satisfies $X_\infty \subseteq \mathrm{interior}(\mathbb{X}_K)$, $\mathbb{X}_K = \{x \in \mathbb{X} \ : \ Kx \in \mathbb{U}\}$. In fact, it is also well known that [38, 39], the minimal robust positively invariant set X_∞ is an exponentially stable attractor for the induced set–dynamics $X^+ = (A+BK)X \oplus \mathbb{W}$ with the basin of attraction being the space of compact subsets of the maximal robust positively invariant set $\mathbb{X}_f$. Consequently, any set sequence $\{Y_k\}_{k=0}^{\infty}$ generated, for all $k \in \mathbb{N}$, by $Y_{k+1} = (A + BK)Y_k \oplus \mathbb{W}$, with Y_0 being an arbitrary compact subset of $\mathbb{X}_f$, converges exponentially fast, with respect to the Hausdorff distance, to the minimal robust positively invariant set X_∞ as $k \to \infty$ while satisfying, for all $k \in \mathbb{N}$, $Y_k \subseteq \mathbb{X}_f$.*

The first observation of interest is concerned with the local state and control tubes $\mathbf{Z}_N := \{Z_0, Z_1, \ldots, Z_N\}$ and $\mathbf{V}_{N-1} := \{V_0, V_1, \ldots, V_{N-1}\}$ given, for any $N \in \mathbb{N}_+$ and $z \in \mathbb{X}_f$, by:

$$Z_0 := z \text{ and } \forall k \in \mathbb{N}_{[1:N]},\ Z_k := (A + BK)^k z \oplus \bigoplus_{j=1}^{k}(A + BK)^{k-j}\mathbb{W},\ \text{and}, \tag{4.4a}$$

$$V_0 := Kz \text{ and } \forall k \in \mathbb{N}_{[1:N-1]},\ V_k := K(A + BK)^k z \oplus \bigoplus_{j=1}^{k} K(A + BK)^{k-j}\mathbb{W}. \tag{4.4b}$$

The relevant properties of the local state and control tubes given above in (4.4) are summarized by:

Proposition 4 *Suppose Assumptions 1, 2 and 3 hold and take any arbitrary $N \in \mathbb{N}_+$. Then, for all $z \in \mathbb{X}_f$, the local state and control tubes $\mathbf{Z}_N := \{Z_0, Z_1, \ldots, Z_N\}$ and $\mathbf{V}_{N-1} := \{V_0, V_1, \ldots, V_{N-1}\}$ given by* (4.4) *satisfy:*

$$\forall k \in \mathbb{N}_N,\ Z_k \subseteq \mathbb{X}_f \subseteq \mathbb{X}, \tag{4.5a}$$

$$\forall k \in \mathbb{N}_{N-1},\ V_k \subseteq K\mathbb{X}_f \subseteq \mathbb{U},\ \textit{and}, \tag{4.5b}$$

$$\forall k \in \mathbb{N}_{N-1},\ Z_{k+1} = (A + BK)Z_k \oplus \mathbb{W}. \tag{4.5c}$$

Clearly, Proposition 4 establishes that the local state and control tubes as specified in (4.4) form a feasible parameterized state and control tubes for any $x \in \mathbb{X}_f$. This observation is helpful in selecting a sensible cost function leading to appropriate stabilizing properties of PTMPC discussed in the sequel of this manuscript. We proceed to discuss a more important consequence of Assumption 3 (*ii*), which allows us to indicate appropriate invariance properties relevant for PTMPC. To this end, let, for any $N \in \mathbb{N}_+$,

$$
\begin{aligned}
&Y_N := \tilde{y}_{(0,N)} \oplus \bigoplus_{j=1}^{N} Y_{(j,N)} \text{ with} \\
&\tilde{y}_{(0,N)} \in \mathbb{R}^n \text{ and } \forall j \in \mathbb{N}_{[1:N]},\ Y_{(j,N)} := \operatorname{convh}(\{\tilde{y}_{(i,j,N)} \in \mathbb{R}^n \ :\ i \in \mathbb{N}_{[1:q]}\}).
\end{aligned} \tag{4.6}
$$

The set Y_N taking the form specified in (4.6) represents a parameterized state tube cross–section X_N (see (3.11)). Our next objective is to show that for any set Y_N satisfying $Y_N \subseteq \mathbb{X}_f$ it is possible to construct sets $\hat{Y}_N(y_{(1,N)})$ and $\hat{V}_{N-1}(y_{(1,N)})$, for any $y_{(1,N)} \in Y_{(1,N)}$, such that $\hat{Y}_N(y_{(1,N)}) = (A+BK)\left((\tilde{y}_{(0,N)} + y_{(1,N)}) \oplus \bigoplus_{j=2}^{N} Y_{(j,N)}\right) \oplus \mathbb{W}$, $\hat{Y}_N(y_{(1,N)}) \subseteq \mathbb{X}_f \subseteq \mathbb{X}$ and $\hat{V}_{N-1}(y_{(1,N)}) \subseteq \mathbb{U}$ and taking structural form of parameterized state and control tube cross–sections X_N and U_{N-1} as specified in (3.11) and (3.12). This property plays an important role in establishing relevant invariance properties of PTMPC analyzed in the sequel of the manuscript. In particular, it allows the assertion of recursive feasibility of the PTMPC according to which if the online optimization of PTMPC is feasible at any particular time instant, then it is guaranteed to be feasible at the next (and by recursion) at all subsequent time instants.

Proposition 5 *Suppose Assumptions 1, 2 and 3 hold, consider any arbitrary $N \in \mathbb{N}_+$ and any arbitrary set Y_N given as in* (4.6) *satisfying that $Y_N \subseteq \mathbb{X}_f$. Then for all $y_{(1,N)} \in Y_{(1,N)}$ it holds that:*

$$
\begin{aligned}
&\hat{Y}_N(y_{(1,N)}) \subseteq \mathbb{X}_f \subseteq \mathbb{X}, \\
&\hat{V}_{N-1}(y_{(1,N)}) \subseteq K\mathbb{X}_f \subseteq \mathbb{U}, \ \textit{and,} \\
&\hat{Y}_N(y_{(1,N)}) = (A+BK)\left((\tilde{y}_{(0,N)} + y_{(1,N)}) \oplus \bigoplus_{j=2}^{N} Y_{(j,N)}\right) \oplus \mathbb{W},
\end{aligned} \tag{4.7}
$$

where

$$
\begin{aligned}
\hat{Y}_N(y_{(1,N)}) &:= \hat{y}_{(0,N)} \oplus \bigoplus_{j=1}^{N} \hat{Y}_{(j,N)}, \ \textit{with,} \\
&\hat{y}_{(0,N)} := (A+BK)(\tilde{y}_{(0,N)} + y_{(1,N)}),\ \forall j \in \mathbb{N}_{[1:N-1]},\ \hat{Y}_{(j,N)} := (A+BK)Y_{(j+1,N)},\ \textit{and, } \hat{Y}_{(N,N)} := \mathbb{W}, \\
\hat{V}_{N-1}(y_{(1,N)}) &:= \hat{v}_{(0,N-1)} \oplus \bigoplus_{j=0}^{N-1} \hat{V}_{(j,N-1)}, \ \textit{with,} \\
&\hat{v}_{(0,N-1)} := K(\tilde{y}_{(0,N)} + y_{(1,N)}),\ \textit{and, } \forall j \in \mathbb{N}_{[1:N-1]},\ \hat{V}_{(j,N-1)} := KY_{(j+1,N)}.
\end{aligned} \tag{4.8}
$$

We are now ready to utilize our preliminary analysis of Sections 3 and 4 in order to formulate an appropriate parameterized tube optimal control problem and discuss its utilization for the synthesis of parameterized tube model predictive control.

5 Parameterized Tube Optimal Control

In the absence of information on the values of future uncertainty, the local linear feedback control law discussed in Section 4 provides an obvious control strategy for steering the current state to the minimal robust invariant set X_∞. However this is only feasible within the terminal constraint set $\mathbb{X}_f$ while it may not be feasible outside of it on account of the state and input constraints and a sensible solution, in this case, is to minimize a cost that measures the distance away from the dynamical behavior induced by the local linear feedback control law discussed in Section 4. This feature has obvious advantages in terms of performance but also allows the assertion of a guarantee of closed–loop robust stability and will be utilized in this paper. Clearly, in the PTMPC context the distance away from the local linear feedback control law has to be measured in terms of the distance of the tubes structure of Section 3 away from the terminal control tube structures of Section 3 as discussed next.

5.1 Simplifying State Substitution

We first introduce a simplifying state substitution which is motivated by stability considerations. We express equivalently the partial state and control tubes $\mathbf{X}_{(0,N)}$ and $\mathbf{U}_{(0,N-1)}$ by employing, for any horizon length $N \in \mathbb{N}_+$ and any $x \in \mathbb{X}$, the following substitutions:

$$\forall k \in \mathbb{N}_N,\ \tilde{x}_{(0,k)} := (A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{e}_{(0,k)},\ \text{and,} \tag{5.1a}$$

$$\forall k \in \mathbb{N}_{N-1},\ \tilde{u}_{(0,k)} := K(A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)}. \tag{5.1b}$$

The sequences $\mathbf{e}_0 := \{\tilde{e}_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{v}_0 := \{\tilde{v}_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ are required to satisfy, in view of (3.3a), the following dynamic constraint:

$$\forall k \in \mathbb{N}_{N-1},\ \tilde{e}_{(0,k+1)} = A\tilde{e}_{(0,k)} + B\tilde{v}_{(0,k)}, \tag{5.2}$$

which ensures that the sequences $\{\tilde{x}_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\{\tilde{u}_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ satisfy (3.3a) and, in addition, that $\tilde{x}_{(0,0)} = x$ as required in (3.4a). With this simplifying substitution, the partial state and control tubes $\mathbf{X}_{(0,N)}$ and $\mathbf{U}_{(0,N-1)}$ satisfy $\mathbf{X}_{(0,N)} = \{(A+BK)^k(x-\tilde{e}_{(0,0)}) + \tilde{e}_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{(0,N-1)} = \{K(A+BK)^k(x-\tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ and are, for a given $x \in \mathbb{X}$, completely characterized by the sequences $\mathbf{e}_0 = \{\tilde{e}_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{v}_0 = \{\tilde{v}_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$. We also note that the initial term $\tilde{e}_{(0,0)}$ of the sequence $\mathbf{e}_0 = \{\tilde{e}_{(0,k)}\}_{k\in\mathbb{N}_N}$ can be freely chosen subject to adequate stabilizing constraint specified below in (5.3c).

5.2 Admissible Set of Parameterized State and Control Tubes

For any horizon length $N \in \mathbb{N}_+$ let $\theta_N := \{\mathbf{X}_N, \mathbf{U}_{N-1}\}$ denote a pair of parameterized state and control tubes. In order to ensure that the state, control and terminal constraints are robustly satisfied as well as to allow for the minimization of the distance of the tubes structure of Section 3 away from the terminal control tube structures of Section 3, for a given horizon length $N \in \mathbb{N}_+$ and initial state $x \in \mathbb{X}$, we require the parameterized state and control tubes $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ to satisfy the following set of constraints:

$$\forall k \in \mathbb{N}_{N-1},\ X_k \subseteq \mathbb{X},\ X_N \subseteq \mathbb{X}_f, \tag{5.3a}$$

$$\forall k \in \mathbb{N}_{N-1},\ U_k \subseteq \mathbb{U},\ \text{and,} \tag{5.3b}$$

$$x - \tilde{e}_{(0,0)} \in \mathbb{X}_f. \tag{5.3c}$$

The set of admissible pairs of parametrized state and control tubes, for a given $x \in \mathbb{X}$, is specified as a value of the set–valued map $\Theta_N(\cdot)$ evaluated at x:

$$\Theta_N(x) := \{\theta_N\ :\ (3.1),\ (3.2),\ (3.3\text{b}),\ (3.4\text{b}),\ (3.11),\ (3.12),\ (5.1),\ (5.2)\ \text{and}\ (5.3)\ \text{hold}\} \tag{5.4}$$

First, we note that the stabilizing constraint (5.3c) does not affect any of other constraints specified in (5.4) since it can be trivially satisfied for any $x \in \mathbb{X}$. We also remark that the graph of the set–valued map $\Theta_N(\cdot)$ admits an equivalent tractable polyhedral reformulation of the set–valued map $\Theta_N(\cdot)$ provided in Section 7 where we address the corresponding computational issues.

5.3 Parameterized State and Control Tubes: Sensible Cost Function

For any horizon length $N \in \mathbb{N}_+$ and with any pair of parameterized state and control tubes $\theta_N = \{\mathbf{X}_N, \mathbf{U}_{N-1}\}$ we associate a cost function $V_N(\cdot)$ given by:

$$V_N(\theta_N) := V_{(0,N)}(\mathbf{e}_0, \mathbf{v}_0) + \sum_{k=1}^{N-1} V_{(k,N)}(\mathbf{X}_{(k,N)}, \mathbf{U}_{(k,N-1)}) + V_{(N,N)}(\mathbf{X}_{(N,N)}), \tag{5.5}$$

which is composed from the partial cost functions $V_{(k,N)}(\cdot),\ k \in \mathbb{N}_N$ associated with the partial state and control tubes $\mathbf{X}_{(k,N)}$ and $\mathbf{U}_{(k,N)}$. The partial cost functions $V_{(k,N)}(\cdot),\ k \in \mathbb{N}_N$ are specified as discussed next. Since, as already noticed, the partial state and control tubes $\mathbf{X}_{(0,N)} = \{(A+BK)^k(x-\tilde{e}_{(0,0)}) + \tilde{e}_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{(0,N-1)} = \{K(A+BK)^k(x-\tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ are, for a given $x \in \mathbb{X}$, completely characterized by the sequences $\mathbf{e}_0 = \{\tilde{e}_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{v}_0 = \{\tilde{v}_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ we specify the partial cost function $V_{(0,N)}(\cdot)\ :\ \mathbb{R}^{(N+1)n+Nm} \to \mathbb{R}_+$ by:

$$V_{(0,N)}(\mathbf{e}_0, \mathbf{v}_0) := \sum_{l=0}^{N-1} \left(\mathcal{G}(\mathcal{Q}, \tilde{e}_{(0,l)}) + \mathcal{G}(\mathcal{R}, \tilde{v}_{(0,l)})\right) + \mathcal{G}(\mathcal{P}, \tilde{e}_{(0,N)}). \tag{5.6}$$

We recall that, for any $k \in \mathbb{N}_{[1:N-1]}$, as specified in (3.1) and (3.2) the partial state and control tubes $\mathbf{X}_{(k,N)}$ and $\mathbf{U}_{(k,N)}$ are completely characterized by the sequences of the corresponding sets of extreme partial state and control sequences $\{\mathbf{x}_{(i,k,N)}\}$ and $\{\mathbf{u}_{(i,k,N-1)}\}$ specified in (2.9). Consequently, we specify, for any $k \in \mathbb{N}_{[1:N-1]}$, the partial cost functions $V_{(k,N)}(\cdot)\ :\ \mathbb{R}^{q(N+1-k)n+q(N=k)m} \to \mathbb{R}_+$ by:

$$V_{(k,N)}(\mathbf{X}_{(k,N)}, \mathbf{U}_{(k,N-1)}) := \sum_{i=1}^{q}\left(\sum_{l=k}^{N-1}\left(\mathcal{G}(\mathcal{Q}, \tilde{x}_{(i,k,l)} - \bar{x}_{(i,k,l)}) + \mathcal{G}(\mathcal{R}, \tilde{u}_{(i,k,l)} - \bar{u}_{(i,k,l)})\right) + \mathcal{G}(\mathcal{P}, \tilde{x}_{(i,k,N)} - \bar{x}_{(i,k,N)})\right), \tag{5.7}$$

where, for all $i \in \mathbb{N}_{[1:q]}$, all $k \in \mathbb{N}_{[1:N-1]}$ and all $l \in \mathbb{N}_{[k:N-1]}$,

$$\bar{x}_{(i,k,l)} := (A + BK)^{l-k}\tilde{w}_i,\ \bar{u}_{(i,k,l)} := K(A + BK)^{l-k}\tilde{w}_i \text{ and } \bar{x}_{(i,k,N)} := (A + BK)^{N-k}\tilde{w}_i. \tag{5.8}$$

The partial cost function $V_{(N,N)}(\cdot) : \mathbb{R}^{qn} \to \mathbb{R}_+$ is specified by:

$$V_{(N,N)}(\mathbf{X}_{(N,N)}) := \sum_{i=1}^{q} \mathcal{G}(\mathcal{P}, \tilde{x}_{(i,N,N)} - \bar{x}_{(i,N,N)}), \tag{5.9}$$

where, as above for all $i \in \mathbb{N}_{[1:q]}$, $\bar{x}_{(i,N,N)} := \tilde{w}_i$.

Our final standard assumption is concerned with the relevant properties of the sets $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P}$ utilized via the gauge (Minkowski) function to specify the parameterized state and control tubes $\theta_N = \{\mathbf{X}_N, \mathbf{U}_{N-1}\}$ cost function $V_N(\cdot)$:

Assumption 4 *The sets $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P}$ are symmetric PC–polytopic sets in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^n$, respectively, given by irreducible representations:*

$$\mathcal{Q} := \{x \in \mathbb{R}^n \ :\ \forall i \in \mathbb{N}_{[1:s_1]},\ Q_i^T x \leq 1\} \tag{5.10a}$$
$$\mathcal{R} := \{u \in \mathbb{R}^m \ :\ \forall i \in \mathbb{N}_{[1:s_2]},\ R_i^T u \leq 1\},\ \textit{and}, \tag{5.10b}$$
$$\mathcal{P} := \{x \in \mathbb{R}^n \ :\ \forall i \in \mathbb{N}_{[1:s_3]},\ P_i^T x \leq 1\}. \tag{5.10c}$$

Furthermore, for all $x \in \mathbb{R}^n$ it holds that:

$$\mathcal{G}(\mathcal{P}, (A + BK)x) - \mathcal{G}(\mathcal{P}, x) \leq -\left(\mathcal{G}(\mathcal{Q}, x) + \mathcal{G}(\mathcal{R}, Kx)\right). \tag{5.11}$$

5.4 Parameterized Tube Optimal Control: Formulation and Basic Properties

The parameterized tube optimal control (PTOC) problem $\mathbb{P}_N(x)$ is specified, for all $x \in \mathbb{X}$, by:

$$V_N^0(x) := \min_{\theta_N}\{V_N(\theta_N) \ :\ \theta_N \in \Theta_N(x)\},\ \text{and}, \tag{5.12a}$$
$$\theta_N^0(x) := \arg\min_{\theta_N}\{V_N(\theta_N) \ :\ \theta_N \in \Theta_N(x)\}. \tag{5.12b}$$

The effective domain of the value function $V_N^0(\cdot)$ is referred to as N–step parameterized tubes controllability set and is, clearly, given by, for any $N \in \mathbb{N}_+$,:

$$\mathcal{X}_N := \{x \ :\ \Theta_N(x) \neq \emptyset\},\ \text{and},\ \mathcal{X}_0 := \mathbb{X}_f. \tag{5.13}$$

At this stage, we wish to stress that, under Assumptions 1–4, the PTOC problem $\mathbb{P}_N(x), x \in \mathbb{X}$ admits an equivalent reformulation as a single and tractable linear programming problem. The corresponding algebraic details will be provided in in Section 7. In fact, under Assumptions 1–4, the following facts can be asserted:

Proposition 6 *Suppose Assumptions 1–4 hold. Then:*

(i) The N–step parameterized tubes controllability set $\mathcal{X}_N$ is a PC–polytopic set in $\mathbb{R}^n$ such that $\mathbb{X}_f \subseteq \mathcal{X}_N \subseteq \mathbb{X}$,

(ii) The value function $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is convex, piecewise affine and continuous function such that $\forall x \in \mathbb{X}_f$, $V_N^0(x) = 0$ and $\forall x \in \mathcal{X}_N \setminus \mathbb{X}_f$, $V_N^0(x) > 0$, and

(iii) There exist piecewise affine and continuous functions $\mathbf{e}_0^*(\cdot) : \mathcal{X}_N \to \mathbb{R}^{(N+1)n}$, $\mathbf{v}_0^*(\cdot) : \mathcal{X}_N \to \mathbb{R}^{Nm}$, $\mathbf{x}^*_{(i,k,N)}(\cdot) : \mathcal{X}_N \to \mathbb{R}^{(N+1-k)n}$, $i \in \mathbb{N}_{[1:q]}$, $k \in \mathbb{N}_{[1:N]}$ and $\mathbf{u}^*_{(i,k,N)}(\cdot) : \mathcal{X}_N \to \mathbb{R}^{(N-k)m}$, $i \in \mathbb{N}_{[1:q]}$, $k \in \mathbb{N}_{[1:N-1]}$ inducing the parameterized state and control tube pair $\theta_N^*(x) = \{\mathbf{X}_N^*(x), \mathbf{U}_{N-1}^*(x)\}$ such that for all $x \in \mathcal{X}_N$ it holds that $\theta_N^*(x) \in \theta_N^0(x)$ (here $\theta_N^*(\cdot)$ denotes a selection of $\theta_N^0(\cdot)$). Furthermore, for all $x \in \mathbb{X}_f$ it holds that:

$$\forall k \in \mathbb{N}_N,\ \tilde{e}^*_{(0,k)}(x) = 0,\ \forall k \in \mathbb{N}_{N-1},\ \tilde{v}^*_{(0,k)}(x) = 0,$$
$$\forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N]},\ \forall l \in \mathbb{N}_{[k:N]},\ \tilde{x}^*_{(i,k,l)}(x) = (A + BK)^{l-k}\tilde{w}_i,$$
$$\forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ \tilde{u}^*_{(i,k,l)}(x) = K(A + BK)^{l-k}\tilde{w}_i.$$

In other words, for all $x \in \mathbb{X}_f$, $\theta_N^0(x)$ is a singleton and it takes the form of the local state and control tubes pair given as in (4.4) *(with $x = z$).*

For any $x \in \mathcal{X}_N$, the selection of the solution of the (PTOC) problem $\mathbb{P}_N(x)$, namely $\theta_N^*(x)$, is well–defined and allows for the construction of optimal parameterized state and control tubes $\mathbf{X}_N^*(x) = \{X_k^*(x) = \bigoplus_{j=0}^k X_{(j,k)}^*(x)\}_{k \in \mathbb{N}_N}$ and $\mathbf{U}_{N-1}^*(x) = \{U_k^*(x) = \bigoplus_{j=0}^k U_{(j,k)}^*(x)\}_{k \in \mathbb{N}_N}$ given via:

$$\forall l \in \mathbb{N}_N,\ X_{(0,l)}^*(x) := \{(A+BK)^l(x - \tilde{e}_{(0,0)}^*(x)) + \tilde{e}_{(0,l)}^*(x)\},\ \text{and,} \tag{5.14a}$$

$$\forall k \in \mathbb{N}_{[1:N]},\ \forall l \in \mathbb{N}_{[k:N]},\ X_{(k,l)}^*(x) := \text{convh}(\{\tilde{x}_{(i,k,l)}^*(x) \in \mathbb{R}^n\ :\ i \in \mathbb{N}_{[1:q]}\}), \tag{5.14b}$$

and

$$\forall l \in \mathbb{N}_{N-1},\ U_{(0,l)}^*(x) := \{K(A+BK)^l(x - \tilde{e}_{(0,0)}^*(x)) + \tilde{v}_{(0,l)}^*(x)\},\ \text{and,} \tag{5.15a}$$

$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ U_{(k,l)}^*(x) := \text{convh}(\{\tilde{u}_{(i,k,l)}^*(x) \in \mathbb{R}^m\ :\ i \in \mathbb{N}_{[1:q]}\}), \tag{5.15b}$$

The optimal pair of the parameterized state and control tubes $\mathbf{X}_N^*(x)$ and $\mathbf{U}_{N-1}^*(x)$, given above in (5.14) and (5.15), satisfies, by construction, all relationships specified in (3.3b), (3.4b), (3.11), (3.12), (5.1), (5.2) and (5.3). In addition, Propositions 1 and 2 allow for the construction of the associated separable control policy $\Pi_{N-1}^*(x)$ according to either (3.5) or (3.13), which is, in view of Propositions 1, 2 and 6, composed, in general, from the sequence of continuous piecewise affine control laws.

An appropriate utilization of Proposition 5 allows us to utilize the optimal pair of the parameterized state and control tubes $\mathbf{X}_N^*(x)$ and $\mathbf{U}_{N-1}^*(x)$, given in (5.14) and (5.15), to construct a feasible and cost decreasing parameterized state and control tubes pair $\hat{\theta}_N(y(x, x_{(1,1)})) := \{\hat{\mathbf{X}}_N(y(x, x_{(1,1)})), \hat{\mathbf{U}}_{N-1}(y(x, x_{(1,1)}))\}$ for any $y(x, x_{(1,1)}) := Ax + B(K(x - \tilde{e}_{(0,0)}^*(x)) + \tilde{v}_{(0,0)}^*(x)) + x_{(1,1)},\ x_{(1,1)} \in X_{(1,1)}^*(x) = \mathbb{W}$ as discussed next. With any $x_{(1,1)} \in X_{(1,1)}^*(x) = \mathbb{W}$ we associate the sequences $\{x_{(1,k)}^*(x_{(1,1)})\}_{k \in \mathbb{N}_{[1:N]}}$ and $\{u_{(1,k)}^*(x_{(1,1)})\}_{k \in \mathbb{N}_{[1:N-1]}}$ by setting:

$$\forall l \in \mathbb{N}_{[1:N-1]},\ u_{(1,l)}^*(x_{(1,1)}) := \sum_{i=1}^{q} \lambda_{(i,1,1)}^*(x_{(1,1)}) u_{(i,1,l)}^*(x_{(1,1)})\ \text{and} \tag{5.16a}$$

$$\forall l \in \mathbb{N}_{[1:N-1]},\ x_{(1,l+1)}^*(x_{(1,1)}) := Ax_{(1,l)}^*(x_{(1,1)}) + Bu_{(1,l)}^*(x_{(1,1)})\ \text{with}\ x_{(1,1)}^*(x_{(1,1)}) := x_{(1,1)}, \tag{5.16b}$$

where the function $\lambda_{(1,1)}^*(\cdot)$ is given by (3.5). We recall that by Proposition 1 it holds that, for all $x_{(1,1)} \in X_{(1,1)}^*(x) = \mathbb{W}$ and all $l \in \mathbb{N}_{[1:N-1]}$, $u_{(1,l)}^*(x_{(1,1)}) \in U_{(1,l)}^*(x)$ as well as $x_{(1,l)}^*(x_{(1,1)}) \in X_{(1,l)}^*(x)$ and $x_{(1,N)}^*(x_{(1,1)}) \in X_{(1,N)}^*(x)$. We set, for all $k \in \mathbb{N}_{[0:N]}$, $\hat{X}_k(y(x, x_{(1,1)})) := \bigoplus_{j=0}^k \hat{X}_{(j,k)}(y(x, x_{(1,1)}))$ and, for all $k \in \mathbb{N}_{[0:N-1]}$, $\hat{U}_k(y(x, x_{(1,1)})) := \bigoplus_{j=0}^k \hat{U}_{(j,k)}(y(x, x_{(1,1)}))$ with:

$$\begin{aligned}
&\forall l \in \mathbb{N}_N,\ \hat{X}_{(0,l)}(y(x, x_{(1,1)})) := \{(A+BK)^l(y(x, x_{(1,1)}) - \hat{e}_{(0,0)}(y(x, x_{(1,1)}))) + \hat{e}_{(0,l)}(y(x, x_{(1,1)}))\},\ \text{with,}\\
&\forall l \in \mathbb{N}_{N-1},\ \hat{e}_{(0,l)}(y(x, x_{(1,1)})) := \tilde{e}_{(0,l+1)}^*(x) + x_{(1,l+1)}^*(x_{(1,1)}) - (A+BK)^l x_{(1,1)},\ \text{and,}\\
&\hat{e}_{(0,N)}(y(x, x_{(1,1)})) := (A+BK)\left(\tilde{e}_{(0,N)}^*(x) + x_{(1,N)}^*(x_{(1,1)}) - (A+BK)^{N-1} x_{(1,1)}\right),\ \text{and,}
\end{aligned} \tag{5.17a}$$

$$\begin{aligned}
&\forall k \in \mathbb{N}_{[1:N]},\ \forall l \in \mathbb{N}_{[k:N]},\ \hat{X}_{(k,l)}(y(x, x_{(1,1)})) := \text{convh}(\{\hat{x}_{(i,k,l)}(y(x, x_{(1,1)})) \in \mathbb{R}^n\ :\ i \in \mathbb{N}_{[1:q]}\}),\ \text{with,}\\
&\forall k \in \mathbb{N}_{[1:N-1]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ \hat{x}_{(i,k,l)}(y(x, x_{(1,1)})) := \tilde{x}_{(i,k+1,l+1)}^*(x),\ \text{and,}\\
&\forall i \in \mathbb{N}_{[1:q]},\ \hat{x}_{(i,k,N)}(y(x, x_{(1,1)})) := (A+BK)\tilde{x}_{(i,k+1,N)}^*(x),\ \text{and,}\ \hat{x}_{(i,N,N)}(y(x, x_{(1,1)})) := \tilde{w}_i,
\end{aligned} \tag{5.17b}$$

and

$$\begin{aligned}
&\forall l \in \mathbb{N}_{N-1},\ \hat{U}_{(0,l)}(y(x, x_{(1,1)})) := \{K(A+BK)^l(y(x, x_{(1,1)})) - \hat{e}_{(0,0)}(y(x, x_{(1,1)}))) + \hat{v}_{(0,l)}(y(x, x_{(1,1)}))\},\ \text{with,}\\
&\forall l \in \mathbb{N}_{N-2},\ \hat{v}_{(0,l)}(y(x, x_{(1,1)})) := \tilde{v}_{(0,l+1)}^*(x) + u_{(1,l+1)}^*(x_{(1,1)}) - K(A+BK)^l x_{(1,1)},\ \text{and,}\\
&\hat{v}_{(0,N-1)}(y(x, x_{(1,1)})) := K\left(\tilde{e}_{(0,N)}^*(x) + x_{(1,N)}^*(x_{(1,1)}) - (A+BK)^{N-1} x_{(1,1)}\right),\ \text{and,}
\end{aligned} \tag{5.18a}$$

$$\begin{aligned}
&\forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ \hat{U}_{(k,l)}(y(x, x_{(1,1)})) := \text{convh}(\{\hat{u}_{(i,k,l)}(y(x, x_{(1,1)})) \in \mathbb{R}^m\ :\ i \in \mathbb{N}_{[1:q]}\}),\ \text{with,}\\
&\forall k \in \mathbb{N}_{[1:N-2]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall l \in \mathbb{N}_{[k:N-2]},\ \hat{u}_{(i,k,l)}(y(x, x_{(1,1)})) := \tilde{u}_{(i,k+1,l+1)}^*(x),\ \text{and,}\\
&\forall i \in \mathbb{N}_{[1:q]},\ \hat{u}_{(i,k,N-1)}(y(x, x_{(1,1)})) := K\tilde{x}_{(i,k+1,N)}^*(x),\ \text{and,}\ \hat{u}_{(i,N-1,N-1)}(y(x, x_{(1,1)})) := K\tilde{w}_i.
\end{aligned} \tag{5.18b}$$

Indeed, the following result guarantees robust recursive feasibility of the PTOC problem $\mathbb{P}_N(x),\ x \in \mathcal{X}_N$:

Proposition 7 *Suppose Assumptions 1–4 hold. Then for all $x \in \mathcal{X}_N$ and all $y(x, x_{(1,1)}) := Ax + B(K(x - \tilde{e}_{(0,0)}^*(x)) + \tilde{v}_{(0,0)}^*(x)) + x_{(1,1)},\ x_{(1,1)} \in X_{(1,1)}^*(x) = \mathbb{W}$ it holds that:*

$$y(x, x_{(1,1)}) \in X_1^*(x), \tag{5.19a}$$

$$\hat{\theta}_N(y(x, x_{(1,1)})) \in \Theta_N(y(x, x_{(1,1)})),\ \textit{and, consequently,}\ X_1^*(x) \subseteq \mathcal{X}_N,\ \textit{and, in addition,} \tag{5.19b}$$

$$V_N^0(y(x, x_{(1,1)})) - V_N^0(x) \leq V_N(\hat{\theta}_N(y(x, x_{(1,1)}))) - V_N^0(x) \leq -\left(\mathcal{G}(\mathcal{Q}, \tilde{e}_{(0,0)}^*(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}_{(0,0)}^*(x))\right), \tag{5.19c}$$

where $\hat{\theta}_N(y(x, x_{(1,1)}))$ is specified via (5.16)–(5.18).

Remark 6 *A direct consequence of Propositions 5 and 7 is the fact that the N–step parameterized tubes controllability sets $\mathcal{X}_N$ given by* (5.13)*, in addition to the PC–polytopic property asserted in Proposition 6* (*i*)*, satisfy that for all $N \in \mathbb{N}$, $\mathcal{X}_N \subseteq \mathcal{X}_{N+1}$.*

6 Parameterized Tube MPC

We now examine the repetitive online application of the solution of the PTOC problem $\mathbb{P}_N(x),\ x \in \mathbb{X}$ in the context of the PTMPC. Namely, we consider the parameterized tube model predictive controller $\kappa_N^0(\cdot)\ :\ \mathcal{X}_N \to \mathbb{U}$ given by:

$$\kappa_N^*(x) = \tilde{u}_{(0,0)}^*(x) = \tilde{v}_{(0,0)}^*(x) + K(x - \tilde{e}_{(0,0)}^*(x)). \tag{6.1}$$

Under Assumptions 1–4, Proposition 6 establishes that the functions $\tilde{v}_{(0,0)}^*(\cdot)$ and $\tilde{e}_{(0,0)}^*(\cdot)$ are single–valued piecewise affine and continuous implying, in view of (6.1), that the control law $\kappa_N^*(\cdot)\ :\ \mathcal{X}_N \to \mathbb{U}$ is also a single–valued piecewise affine and continuous function. The parameterized tube model predictive controller $\kappa_N^*(\cdot)\ :\ \mathcal{X}_N \to \mathbb{U}$ induces the controlled, uncertain, dynamics given, for all $x \in \mathcal{X}_N$, by:

$$x^+ \in \mathcal{F}(x) := \{Ax + B\kappa_N^*(x) + w\ :\ w \in \mathbb{W}\}, \tag{6.2}$$

and it ensures, by construction, that for all $x \in \mathcal{X}_N$:

$$\begin{aligned}\mathcal{F}(x) &= X_1^*(x) = X_{(0,1)}^*(x) \oplus X_{(1,1)}^*(x) = \tilde{x}_{(0,1)}^*(x) \oplus X_{(1,1)}^*(x)\\ &= ((A+BK)(x - \tilde{e}_{(0,0)}^*(x)) + \tilde{e}_{(0,1)}^*(x)) \oplus \mathbb{W} \subseteq \mathcal{X}_N,\end{aligned} \tag{6.3}$$

as evident from (3.4), (3.10), (5.1) and Proposition 7.

The dynamical behavior of the controlled, uncertain, dynamics given in (6.2) is examined by utilizing the value function $V_N^0(\cdot)\ :\ \mathcal{X}_N \to \mathbb{R}_+$ as a Lyapunov function relative to the terminal constraint set $\mathbb{X}_f$. Namely, we proceed to show that any state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2) with $x_0 \in \mathcal{X}_N$ and the corresponding control actions sequence $\{u_k\}_{k\in\mathbb{N}}$ with $u_k = \kappa_N^*(x_k)$ for each k, for any admissible disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$ with $w_k \in \mathbb{W}$ for each k, converge respectively to the sets $\mathbb{X}_f$ and $K\mathbb{X}_f$, as $k \to \infty$, exponentially fast and in a stable fashion for any realized state sequence $\{x_k\}_{k\in\mathbb{N}}$ arising due to an admissible disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$.

Before proceeding, we note that Proposition 6 establishes that the value function $V_N^0(\cdot)\ :\ \mathcal{X}_N \to \mathbb{R}_+$ is a convex piecewise affine and continuous function, and, in addition, that for all $x \in \mathbb{X}_f$ it holds that $V_N^0(x) = 0$, $\tilde{e}_{(0,0)}^*(x) = 0$ and $\tilde{v}_{(0,0)}^*(x) = 0$. In turn, in view of (6.1) and (6.2), it follows that for all $x \in \mathbb{X}_f$ we have:

$$\kappa_N^*(x) = Kx \text{ and } \mathcal{F}(x) = (A+BK)x \oplus \mathbb{W}. \tag{6.4}$$

We now establish the relevant technical and preliminary results, which lead to our main result summarizing the properties of the introduced PTMPC. Our first technical lemma is:

Lemma 1 *Suppose Assumptions 1–4 hold. Then:*

(i) for all $x \in \mathbb{X}_f$ it holds that $|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} = 0$ and $|\tilde{v}_{(0,0)}^*(x)|_{\mathcal{R}} = 0$, and, for all $x \in \mathcal{X}_N \setminus \mathbb{X}_f$ it holds that $|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} > 0$;

(ii) for all $x \in \mathcal{X}_N$ it holds that $c_1|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} \leq \operatorname{dist}(\mathcal{Q}, x, \mathbb{X}_f) \leq |\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}}$ for some scalar $c_1 \in (0,1)$ and $0 \leq \operatorname{dist}(\mathcal{R}, \kappa_N^*(x), K\mathbb{X}_f) \leq |\tilde{v}_{(0,0)}^*(x)|_{\mathcal{R}}$.

Our second observation is concerned with the desirable properties of the value function $V_N^0(\cdot)$:

Proposition 8 *Suppose Assumptions 1–4 hold. Then there exist a scalar $c_2 \in [1, \infty)$ such that for all $x \in \mathcal{X}_N$ it holds that:*

$$|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} \leq V_N^0(x) \leq c_2|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}}, \tag{6.5a}$$

$$(|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} + |\tilde{v}_{(0,0)}^*(x)|_{\mathcal{R}}) \leq V_N^0(x) \leq c_2(|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} + |\tilde{v}_{(0,0)}^*(x)|_{\mathcal{R}}), \text{ and,} \tag{6.5b}$$

$$\forall x^+ \in \mathcal{F}(x),\ V_N^0(x^+) - V_N^0(x) \leq -(|\tilde{e}_{(0,0)}^*(x)|_{\mathcal{Q}} + |\tilde{v}_{(0,0)}^*(x)|_{\mathcal{R}}), \tag{6.5c}$$

where $\mathcal{F}(\cdot)$ is given by (6.2).

Our third observation is a direct, but important, consequence of Proposition 8:

Corollary 2 *Suppose Assumptions 1–4 hold. Then there exists a scalar pair* $(\bar{a}_N, \bar{b}_N) \in [0,1) \times (0,\infty)$ *such that the inequalities*

$$\forall k \in \mathbb{N},\ V_N^0(x_k) \le \bar{a}_N^k V_N^0(x_0), \tag{6.6a}$$

$$\forall k \in \mathbb{N},\ (|\tilde{e}^*_{(0,0)}(x_k)|_{\mathcal{Q}} + |\tilde{v}^*_{(0,0)}(x_k)|_{\mathcal{R}}) \le \bar{a}_N^k \bar{b}_N (|\tilde{e}^*_{(0,0)}(x_0)|_{\mathcal{Q}} + |\tilde{v}^*_{(0,0)}(x_0)|_{\mathcal{R}}),\ \textit{and,} \tag{6.6b}$$

$$\forall k \in \mathbb{N},\ |\tilde{e}^*_{(0,0)}(x_k)|_{\mathcal{Q}} \le \bar{a}_N^k \bar{b}_N |\tilde{e}^*_{(0,0)}(x_0)|_{\mathcal{Q}}. \tag{6.6c}$$

hold true for any $x_0 \in \mathcal{X}_N$ *and any corresponding state sequence* $\{x_k\}_{k\in\mathbb{N}}$ *generated by* (6.2).

We are now able to verify the relevant and strong system theoretic properties of the developed PTMPC. This is achieved by utilizing results of Lemma 1, Propositions 7 and 8 and Corollary 2:

Theorem 1 *Suppose Assumptions 1–4 hold. Then:*

(i) For all $x \in \mathcal{X}_N$ *it holds that* $\kappa_N^*(x) \in \mathbb{U}$ *and* $\mathcal{F}(x) \subseteq \mathcal{X}_N$*, i.e. the* N*–step parameterized tubes controllability set* $\mathcal{X}_N$ *is a robust positively invariant set for the system* $x^+ = Ax + B\kappa_N^*(x) + w$ *and the constraint set* $(\mathbb{X}_{\kappa_N^*}, \mathbb{W})$ *where* $\mathbb{X}_{\kappa_N^*} := \{x \in \mathbb{X} \ :\ \kappa_N^*(x) \in \mathbb{U}\}$.

(ii) There exists a scalar pair $(a_N, b_N) \in [0,1) \times (0,\infty)$ *such that the inequalities:*

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q}, x_k, \mathcal{X}_N) = 0 \ \textit{and}\ \operatorname{dist}(\mathcal{Q}, x_k, \mathbb{X}_f) \le a_N^k b_N \operatorname{dist}(\mathcal{Q}, x_0, \mathbb{X}_f),$$
$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{R}, \kappa_N^*(x_k), \mathbb{U}) = 0 \ \textit{and}\ \operatorname{dist}(\mathcal{R}, \kappa_N^*(x_k), K\mathbb{X}_f) \le a_N^k b_N,$$

hold true for any $x_0 \in \mathcal{X}_N$ *and any corresponding state sequence* $\{x_k\}_{k\in\mathbb{N}}$ *generated by* (6.2).

(iii) The set $\mathbb{X}_f$ *is robustly exponentially stable for the system* $x^+ = Ax + B\kappa_N^*(x) + w$ *and the constraint set* $(\mathbb{X}_{\kappa_N^*}, \mathbb{W})$ *with the basin of attraction being equal to the* N*–step parameterized tubes controllability set* $\mathcal{X}_N$.

Recalling Remark 6 and the fact pointed out in (6.4) we can establish a stronger stability and convergence related properties:

Corollary 3 *Suppose Assumptions 1–4 hold. Then:*

(i) There exists a scalar pair $(\tilde{a}_N, \tilde{b}_N) \in [0,1) \times (0,\infty)$ *such that the inequalities:*

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q}, x_k, \mathcal{X}_N) = 0 \ \textit{and}\ \operatorname{dist}(\mathcal{Q}, x_k, X_\infty) \le \tilde{a}_N^k \tilde{b}_N \operatorname{dist}(\mathcal{Q}, x_0, X_\infty),$$
$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{R}, \kappa_N^*(x_k), \mathbb{U}) = 0 \ \textit{and}\ \operatorname{dist}(\mathcal{R}, \kappa_N^*(x_k), KX_\infty) \le \tilde{a}_N^k \tilde{b}_N,$$

hold true for any $x_0 \in \mathcal{X}_N$ *and any corresponding state sequence* $\{x_k\}_{k\in\mathbb{N}}$ *generated by* (6.2) *where the set* X_∞ *is the minimal robust positively invariant set for the system* $x^+ = (A+BK)x + w$ *given by* $X_\infty = \bigoplus_{i=0}^{\infty}(A+BK)^i\mathbb{W}$.

(ii) The minimal robust positively invariant set X_∞ *is also the minimal set with respect to the set inclusion which is robustly exponentially stable for the system* $x^+ = Ax + B\kappa_N^*(x) + w$ *and the constraint set* $(\mathbb{X}_{\kappa_N^*}, \mathbb{W})$ *with the basin of attraction being equal to the* N*–step parameterized tubes controllability set* $\mathcal{X}_N$.

Remark 7 *As far as the implementation of the PTMPC is concerned, the online implementation of the PTMPC can be performed via the standard convex optimization software. To this end, we provide in Section 7 a computationally relevant reformulation of the PTOC problem* $\mathbb{P}_N(x)$ *given in* (5.12) *which, for any fixed* $x \in \mathcal{X}_N$*, reduces to a standard and numerically tractable linear programming problem.*

7 Computational Issues & Illustrative Examples

In this section, we discuss computational issues relevant for efficient implementation of PTOC and PTMPC.

7.1 PTOC & PTMPC: Tractable Linear Programming Formulation & Online Implementation

As already pointed out in Remark 2, the Minkowski set addition and convex hull operations are only utilized for the analysis and their actual explicit computation is not needed for the efficient implementation of PTOC and PTMPC. Similarly as in [40], we utilize convexity and the basic properties of the support function in order to demonstrate that the set inclusions $\forall k \in \mathbb{N}_{N-1},\ U_k \subseteq \mathbb{X}$, $\forall k \in \mathbb{N}_{N-1},\ X_k \subseteq \mathbb{X}$ and $X_N \subseteq \mathbb{X}_f$, where the sets $U_k,\ k \in \mathbb{N}_{N-1}$ and $X_k,\ k \in \mathbb{N}_N$ are given via the relationships (3.1), (3.2), (5.1), (3.11) and (3.12), can be equivalently expressed as a tractable set of affine/linear inequalities without the explicit computation of the involved Minkowski set addition and convex hull operations. Namely, by utilizing convexity and the basic properties of the support function, it follows that, for any $k \in \mathbb{N}_{N-1}$, the set inclusion $\bigoplus_{j=0}^{k} U_{(j,k)} \subseteq \mathbb{U}$ holds true if and only if for all $l \in \mathbb{N}_{[1:r]}$ it holds that $\mathcal{S}(\bigoplus_{j=0}^{k} U_{(j,k)}, G_l) \leq 1$. We recall that, by the additivity of the support function in the first argument [41, 42], we have, for all $l \in \mathbb{N}_{[1:r]}$,:

$$\mathcal{S}(\bigoplus_{j=0}^{k} U_{(j,k)}, G_l) = \sum_{j=0}^{k} \mathcal{S}(U_{(j,k)}, G_l).$$

Since $U_{(0,k)} = \{K(A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)}\}$ and $U_{(j,k)} = \operatorname{convh}(\{\tilde{u}_{(i,j,k)}\ :\ i \in \mathbb{N}_{[1:q]}\})$ for $j \in \mathbb{N}_{[1:k]}$ it follows that, for all $l \in \mathbb{N}_{[1:r]}$, $\mathcal{S}(U_{(0,k)}, G_l) = G_l^T(K(A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)})$ and all $j \in \mathbb{N}_{[1:k]}$, $\mathcal{S}(U_{(j,k)}, G_l) = \max_{i \in \mathbb{N}_{[1:q]}} G_l^T \tilde{u}_{(i,j,k)}$. Hence, the set inclusion $\bigoplus_{j=0}^{k} U_{(j,k)} \subseteq \mathbb{U}$ is equivalently expressed as:

$$\forall l \in \mathbb{N}_{[1:r]},\ \exists g_{(l,k)} := \{g_{(l,j,k)} \in \mathbb{R}\ :\ j \in \mathbb{N}_{[1:k]}\} \text{ such that} \tag{7.1a}$$
$$\forall j \in \mathbb{N}_{[1:k]},\ \forall i \in \mathbb{N}_{[1:q]},\ G_l^T \tilde{u}_{(i,j,k)} \leq g_{(l,j,k)},\ \text{and,} \tag{7.1b}$$
$$G_l^T(K(A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)}) + \sum_{j=1}^{k} g_{(l,j,k)} \leq 1. \tag{7.1c}$$

By the same token, the set inclusion $X_k \subseteq \mathbb{X}$ which takes the form $\bigoplus_{j=0}^{k} X_{(j,k)} \subseteq \mathbb{X}$ is equivalently expressed as:

$$\forall l \in \mathbb{N}_{[1:p]},\ \exists f_{(l,k)} := \{f_{(l,j,k)} \in \mathbb{R}\ :\ j \in \mathbb{N}_{[1:k]}\} \text{ such that} \tag{7.2a}$$
$$\forall j \in \mathbb{N}_{[1:k]},\ \forall i \in \mathbb{N}_{[1:q]},\ F_l^T \tilde{x}_{(i,j,k)} \leq f_{(l,j,k)},\ \text{and,} \tag{7.2b}$$
$$F_l^T((A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{e}_{(0,k)}) + \sum_{j=1}^{k} f_{(l,j,k)} \leq 1. \tag{7.2c}$$

Likewise, the set inclusion $X_N \subseteq \mathbb{X}_f$ which takes the form $\bigoplus_{j=0}^{N} X_{(j,N)} \subseteq \mathbb{X}_f$ is equivalently expressed as:

$$\forall l \in \mathbb{N}_{[1:t]},\ \exists h_{(l,N)} := \{h_{(l,j,N)} \in \mathbb{R}\ :\ j \in \mathbb{N}_{[1:N]}\} \text{ such that} \tag{7.3a}$$
$$\forall j \in \mathbb{N}_{[1:N]},\ \forall i \in \mathbb{N}_{[1:q]},\ H_l^T \tilde{x}_{(i,j,N)} \leq h_{(l,j,N)},\ \text{and,} \tag{7.3b}$$
$$H_l^T((A+BK)^N(x - \tilde{e}_{(0,0)}) + \tilde{e}_{(0,N)}) + \sum_{j=1}^{N} h_{(l,j,N)} \leq 1. \tag{7.3c}$$

We can now provide a computationally relevant reformulation of PTOC which reduces to a computationally tractable linear programming problem. This is achieved by utilizing the facts stated in relationships (7.1)–(7.3) as well as the fact that minimization of the cost function $V_N(\cdot)$ specified via the relationships (5.5)–(5.10) can be achieved by minimizing the sum of an adequate number of slack variables subject of appropriate cost–reformulation constraints (these slack variables yield the values of the gauge functions involved in the definition of the cost function $V_N(\cdot)$). To this end, we introduce the decision variable:

$$\mathbf{d}_N := (\mathbf{d}_{(N,\mathbf{X})}^T, \mathbf{d}_{(N,\mathbf{U})}^T, \mathbf{d}_{(N,\mathbf{f})}^T, \mathbf{d}_{(N,\mathbf{g})}^T, \mathbf{d}_{(N,\mathbf{h})}^T, \mathbf{d}_{(N,\alpha)}^T, \mathbf{d}_{(N,\beta)}^T)^T \in \mathbb{R}^{N_{tot}},\ \text{where,} \tag{7.4a}$$
$$N_{tot} := \frac{1}{2}[q(n+m+2)+p+r]N^2 + \frac{1}{2}[(2+q)n + (2-q)m + 2t - p - r + 4]N + n + 1. \tag{7.4b}$$

As evident, the overall decision variable is composed from the variables $\mathbf{d}_{(N,\mathbf{X})}$, $\mathbf{d}_{(N,\mathbf{U})}$, $\mathbf{d}_{(N,\mathbf{f})}$, $\mathbf{d}_{(N,\mathbf{g})}$, $\mathbf{d}_{(N,\mathbf{h})}$, $\mathbf{d}_{(N,\alpha)}$ and $\mathbf{d}_{(N,\beta)}$. The vectorized form of the variable $\mathbf{d}_{(N,\mathbf{X})}$ is composed from the sequence $\mathbf{e}_0$ and the sequences $\mathbf{x}_{(i,k,N)}$ belonging to the set of extreme partial state sequences $\{\mathbf{x}_{(i,k,N)}\ :\ i \in \mathbb{N}_{[1:q]}, k \in \mathbb{N}_{[1:N]}\}$ characterizing the partial state tubes $\mathbf{X}_{(k,N)}$ and, in turn, the overall state tube $\mathbf{X}_N$. Likewise, the vectorized form of the variable $\mathbf{d}_{(N,\mathbf{U})}$ is composed from the sequence $\mathbf{v}_0$ and the sequences $\mathbf{u}_{(i,k,N-1)}$ belonging to the set of extreme partial control sequences $\{\mathbf{u}_{(i,k,N-1)}\ :\ i \in \mathbb{N}_{[1:q]}, k \in \mathbb{N}_{[1:N-1]}\}$ characterizing the partial state tubes $\mathbf{U}_{(k,N-1)}$ and in turn the overall state tube $\mathbf{U}_{N-1}$. The vectorized forms of the variables $\mathbf{d}_{(N,\mathbf{f})}$, $\mathbf{d}_{(N,\mathbf{g})}$ and $\mathbf{d}_{(N,\mathbf{h})}$ collect, respectively, all the slack variables (i.e. slack variables $f_{(l,k)},\ l \in \mathbb{N}_{[1:p]},\ k \in \mathbb{N}_{[1:N-1]},\ g_{(l,k)},\ l \in \mathbb{N}_{[1:r]},\ k \in \mathbb{N}_{[1:N-1]}$ and $h_{(l,N)},\ l \in \mathbb{N}_{[1:t]}$) needed to ensure the satisfaction of the set inclusions $\forall k \in \mathbb{N}_{N-1}, X_k \subseteq \mathbb{X}$, $\forall k \in \mathbb{N}_{N-1}, U_k \subseteq \mathbb{U}$ and $X_N \subseteq \mathbb{X}_f$ according to the relationships (7.1)–(7.3). Finally, the vectorized form of the variables $\mathbf{d}_{(N,\alpha)}$ and $\mathbf{d}_{(N,\beta)}$ collect all the slack variables utilized to obtain the equivalent cost reformulation of the cost function $V_N(\cdot)$. A comprehensive account of the corresponding algebraic details addressing the corresponding vectorization and providing precise definitions of variables $\mathbf{d}_{(N,\mathbf{X})}$, $\mathbf{d}_{(N,\mathbf{U})}$, $\mathbf{d}_{(N,\mathbf{f})}$, $\mathbf{d}_{(N,\mathbf{g})}$, $\mathbf{d}_{(N,\mathbf{h})}$, $\mathbf{d}_{(N,\alpha)}$ and $\mathbf{d}_{(N,\beta)}$ as well as the overall decision variable $\mathbf{d}_N$ is provided in Appendix $B-1$.

The computationally tractable reformulation of PTOC takes the form:

$$V_N^0(x) = \min_{\mathbf{d}_N}\{\gamma^T \mathbf{d}_N \ :\ (x, \mathbf{d}_N) \in \Gamma_N\}, \tag{7.5a}$$

$$\mathbf{d}_N^0(x) = \arg\min_{\mathbf{d}_N}\{\gamma^T \mathbf{d}_N \ :\ (x, \mathbf{d}_N) \in \Gamma_N\},\ \text{where}, \tag{7.5b}$$

$$\Gamma_N = \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ \mathcal{M}_{\text{xeq}}x + \mathcal{M}_{\text{deq}}\mathbf{d}_N = \mathcal{N}_{\text{eq}},\ \mathcal{M}_{\text{xineq}}x + \mathcal{M}_{\text{dineq}}\mathbf{d}_N \leq \mathcal{N}_{\text{ineq}}\},\ \text{and}, \tag{7.5c}$$

$$\gamma = (\mathbf{0}^T, \mathbf{1}^T)^T,\ \text{with}\ \mathbf{0} = (0, 0, \ldots, 0)^T \in \mathbb{R}^{Ntot-(qN^2+2N+1)}\ \text{and}\ \mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^{qN^2+2N+1} \tag{7.5d}$$

The detailed from of the set Γ_N is provided in Appendix $B-2$. The matrices $\mathcal{M}_{\text{xeq}}$, $\mathcal{M}_{\text{deq}}$, $\mathcal{M}_{\text{xineq}}$, $\mathcal{M}_{\text{dineq}}$ and vectors $\mathcal{N}_{\text{eq}}$ and $\mathcal{N}_{\text{eq}}$ satisfy $\mathcal{M}_{\text{xeq}} \in \mathbb{R}^{N_{eq}\times n}$, $\mathcal{M}_{\text{deq}} \in \mathbb{R}^{N_{eq}\times N_{tot}}$, $\mathcal{M}_{\text{xineq}} \in \mathbb{R}^{N_{ineq}\times n}$, $\mathcal{M}_{\text{dineq}} \in \mathbb{R}^{N_{ineq}\times N_{tot}}$ and vectors $\mathcal{N}_{\text{eq}} \in \mathbb{R}^{N_{eq}}$ and $\mathcal{N}_{\text{eq}} \in \mathbb{R}^{N_{ineq}}$ where:

$$N_{eq} = \frac{1}{2}qnN^2 + (1 + \frac{1}{2}q)nN,\ \text{and}, \tag{7.6a}$$

$$N_{ineq} = \frac{1}{2}q(p + r + s_1 + s_2)N^2 + [p + r + t + s_1 + s_2 + qs_3 - \frac{1}{2}(p + r + s_1 + s_2)]N + 2t + s_3. \tag{7.6b}$$

The equalities $\mathcal{M}_{\text{xeq}}x + \mathcal{M}_{\text{deq}}\mathbf{d}_N = \mathcal{N}_{\text{eq}}$ appearing in the definition of the set Γ_N are obtained from the relationships (5.2), (3.3b) and (3.4b). The inequalities $\mathcal{M}_{\text{xineq}}x + \mathcal{M}_{\text{dineq}}\mathbf{d}_N \leq \mathcal{N}_{\text{ineq}}$ appearing in the definition of the set Γ_N are obtained from: (i) the reformulation of the set inclusions $\forall k \in \mathbb{N}_{N-1},\ X_k \subseteq \mathbb{X}$, $\forall k \in \mathbb{N}_{N-1},\ U_k \subseteq \mathbb{U}$, $X_N \subseteq \mathbb{X}_f$ according to the relationships (7.1)–(7.3), (ii) the stabilizing constraint (5.3c) and (iii) the cost reformulation constraints. The detailed definition of the set Γ_N is provided in Appendix $B-2$.

Remark 8 *As evident, the dimension of the overall decision variable and the total number of equality and inequality constraints are all quadratic functions of the horizon length N. This implies in turn that, the computationally relevant reformulation of PTOC is numerically tractable and that it scales favorably with respect to the horizon length N.*

Remark 9 *By utilizing algebraic details provided in Appendix $B-2$, it follows that for all $x \in \mathbb{X}$ we have:*

$$\exists \theta_N \in \Theta_N(x)\ \textit{if and only if}\ \exists \mathbf{d}_N \in \mathbb{R}^{N_{tot}}\ \textit{such that}\ (x, \mathbf{d}_N) \in \Gamma_N,$$

where $\Theta(\cdot)$ and Γ_N are given, respectively, by (5.4) *and* (7.5c) *(with the understanding that Γ_N is constructed as discussed in Appendix $B-2$). Furthermore, it also holds that:*

$$\mathcal{X}_N = \text{Projection}_{\mathbb{R}^n}(\Gamma_N) = \{x \in \mathbb{R}^n \ :\ \exists \mathbf{d}_N \in \mathbb{R}^{N_{tot}}\ \textit{such that}\ (x, \mathbf{d}_N) \in \Gamma_N\},$$

which, in turn, verifies the fact (i) asserted in Proposition 6 (since $\mathcal{X}_N$ is closed polyhedral PC–set which contains the PC–polytopic set $\mathbb{X}_f$ and is contained in the PC–polytopic set $\mathbb{X}$). We also note that due to Assumption 4 the values of the slack variables collected in the variables $\mathbf{d}_{(N,\alpha)}$ and $\mathbf{d}_{(N,\beta)}$ are all lower bounded by 0 *and hence, since, Γ_N is, by construction, a closed polyhedral set in $\mathbb{R}^{n+N_{tot}}$, it follows that the linear programming problem specified in* (7.5) *is well–posed. In turn, it follows, as also asserted in Proposition 6 (ii), that the value function $V_N^0(\cdot)$ is convex, piecewise affine and continuous function [34] which is, in addition, such that $\forall x \in \mathbb{X}_f,\ V_N^0(x) = 0$ and $\forall x \in \mathcal{X}_N \setminus \mathbb{X}_f,\ V_N^0(x) > 0$ (due to Assumption 3 and the relationships* (4.4) *and* (4.5)*). Finally, the standard results [34] imply the existence of piecewise affine and continuous function $\mathbf{d}_N^*(\cdot)$ such that:*

$$\forall x \in \mathcal{X}_N,\ \mathbf{d}_N^*(x) \in \mathbf{d}_N^0(x),$$

where we note that the function $\mathbf{d}_N^0(\cdot)$ given by (7.5b) *is not necessarily single–valued. The above mentioned piecewise affine and continuous function $\mathbf{d}_N^*(\cdot)$ verifies in turn the fact (ii) asserted in Proposition 6.*

Remark 10 *The explicit form of the functions $V_N^0(\cdot)$, $\mathbf{d}_N^*(\cdot)$ and $\kappa_N^*(\cdot)$ can be, in principle, obtained by utilizing the standard computational geometry software as offered in [43–45] The explicit form of the function $\kappa_N^*(\cdot)$ when available permits for the implementation of PTMPC without online optimization; however this approach is limited to lower dimensional problems and, in general, the online optimization broadens significantly the range of the applicability of PTMPC. The online implementation of PTMPC reduces to solving the linear programming problem in* (7.5) *at the state x_k encountered in the process and evaluating the value of the PTMPC law $\kappa_N^*(x_k)$ by utilizing the value of optimizing decision variable $\mathbf{d}_N^*(x_k)$. We note that as long as the initial state x_0 is such that the corresponding* *online optimization is feasible then it is guaranteed that the online optimization will remains feasible for all the future time $k \in \mathbb{N}$ and at all possible states x_k encountered in the control process. The previous fact is true as the initial optimization is feasible for all state $x \in \mathcal{X}_N$ and the N–step parameterized tubes controllability set $\mathcal{X}_N$ is robust positively invariant set for the system $x^+ = Ax + B\kappa_N^*(x) + w$ and the constraint set $(\mathbb{X}_{\kappa_N^*}, \mathbb{W})$ as asserted in Theorem 1 (see also Proposition 7). Finally, since $\forall x \in \mathbb{X}_f,\ V_N^0(x) = 0$, and, consequently, $\forall x \in \mathbb{X}_f,\ \kappa_N^*(x) = Kx$ we note that online optimization can be terminated once the state x_k enters the set $\mathbb{X}_f$.*

Remark 11 *We also note that the variable $\mathbf{d}_{(N,\mathbf{X})}$ in* (7.5)*, can be eliminated by employing the relationships* (3.3b) *and* (3.4b) *which, in turn, reduces the number of decision variables by $\frac{1}{2}qnN(N+1)$. However, the form of constraints we have utilized seems to be both the numerically and structurally preferred option even in the case of the nominal MPC [46].*

7.2 Illustrative Examples

We provide first two examples illustrating advantages of the developed PTOC and PTMPC over the methods proposed in [13, 14] as well as in [15–18].

Illustrative Example 1 *The first illustrative example is a variant of the example used in [13]. The system is one dimensional system given by:*

$$x^+ = x + u + w,$$

with the state and control constraint sets:

$$\mathbb{X} = [-30, 30] = \{x \in \mathbb{R} \ :\ \frac{1}{30}x \leq 1,\ -\frac{1}{30}x \leq 1\},\ \textit{and, } \mathbb{U} = [-2, 2] = \{u \in \mathbb{R} \ :\ \frac{1}{2}u \leq 1,\ -\frac{1}{2}u \leq 1\}.$$

The disturbance set $\mathbb{W}$ is given by:

$$\mathbb{W} = \text{convh}(\{1, -1\}) \ \textit{so that } q = 2,\ \tilde{w}_1 = 1 \ \textit{and } \tilde{w}_2 = -1.$$

The matrices Q and R defining the sets $\mathcal{Q}$ and $\mathcal{R}$ as well as their gauge functions utilized for the cost function $V_N(\cdot)$ are given by:

$$Q = (1,\ -1)^T \ \textit{and } R = (2,\ -2)^T.$$

The local linear feedback $u = kx$, the terminal constraint set $\mathbb{X}_f$ and the matrix P defining the set $\mathcal{P}$ and its gauge function utilized for the cost function $V_N(\cdot)$ are given by:

$$k = -\frac{1}{2},\ \mathbb{X}_f = [-4, 4] = \{x \in \mathbb{R} \ :\ \frac{1}{4}x \leq 1,\ -\frac{1}{4}x \leq 1\},\ \textit{and, } P = (4,\ -4)^T.$$

In this example, any initial condition $x_0 \in \mathbb{X}$ is min–max controllable to a target set within $N = 26$ steps. In particular, for the initial conditions $x_0' = 30$ and $x_0'' = -30$ the min–max controllability to a target set $\mathbb{X}_f$ can be guaranteed if and only if $N = 26$. So, we choose the initial condition $x_0 = 30$ and consider the horizon length

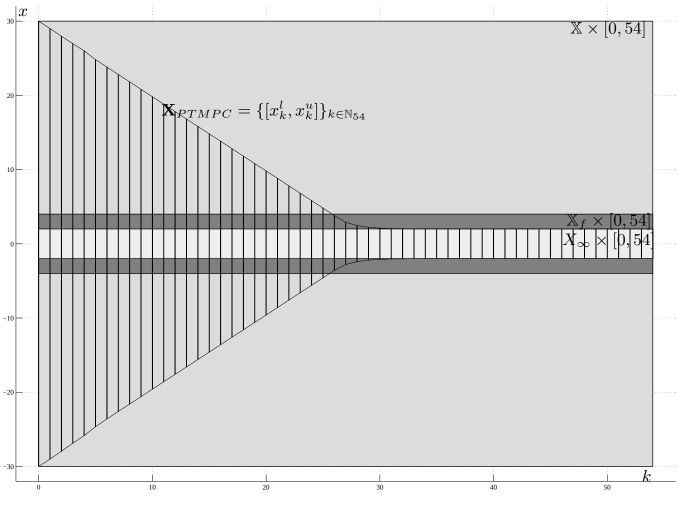


Figure 1: Parameterized Model Predictive Control State Tube.

$N = 26$. In this setting, the methods of [13, 14] require the utilization of the decision variable whose dimension is $2^{26} = 67108864$ and the corresponding optimization requires the utilization of a number of constraints that is linearly

proportional to $2^{26} = 67108864$. *Clearly, even with the most sophisticated optimization software, this would lead to an almost impossible computational task (realistically speaking, this task is, in fact, an impossible mission). On the contrary, the proposed PTMPC method requires the utilization of the decision variable* $\mathbf{d}_N$ *whose dimension is* $N_{tot} = 4162$ *while the total number of equality and inequality constraints for the underlying linear programming problem satisfy* $N_{ineq} = 5674$ *and* $N_{eq} = 728$. *This simple scalar example demonstrates clearly computational advantages of the proposed method over those suggested in [13, 14]. The performance of the PTMPC is additionally illustrated in Figure 1 where we show the tube* $\mathbf{X}_{PTMPC} := \{[x^l_k, x^u_k]\}_{k\in\mathbb{N}}$ *composed from the extreme trajectories generated from* $\{x^l_k\}_{k\in\mathbb{N}}$ *and* $\{x^u_k\}_{k\in\mathbb{N}}$ *which satisfy:*

$$x^l_{k+1} = x^l_k + \kappa^*_N(x^l_k) + w^*(x^l_k, \kappa^*_N(x^l_k)),\ \textit{and}\ x^u_{k+1} = x^u_k + \kappa^*_N(x^u_k) + w^*(x^u_k, \kappa^*_N(x^u_k)),$$

where $x^l_0 = -30$, $x^u_0 = 30$ *and the function* $w^*(\cdot,\cdot)$ *is the maximizing disturbance function taking the form* $w^*(x,u) = 1$ *if* $x+u \geq 0$, $w^*(x,u) = -1$ *if* $x+u \leq 0$ *and* $w^*(x,u)$ *can be either* 1 *or* -1 *if* $x+u = 0$. *As evident from the Figure, the PTMPC law induces the controlled, uncertain, dynamics with strong system theoretic properties. Clearly, the tube* $\mathbf{X}_{PTMPC} := \{[x^l_k, x^u_k]\}_{k\in\mathbb{N}}$ *composed from the extreme trajectories converges exponentially fast to the minimal robust positively invariant set* $X_\infty = [-2, 2]$ *(exhibiting also the exponential convergence to the maximal robust positively invariant set* $\mathbb{X}_f = [-4, 4]$*) as expected by Theorem 1 and Corollary 3.*

Illustrative Example 2 *Our second illustrative example demonstrates the advantages of the developed PTOC and PTMPC over the methods utilizing the so–called afine disturbance feedbacks discussed in [15–18]. At the conceptual level, these methods are subsumed within our method as they can be recovered from our method by invoking an additional constraint on the control tubes, namely, by requiring that:*

$$\forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ \forall j \in \mathbb{N}_{[k:N-1]},\ \tilde{u}_{(i,j,k)} = M_{(j,k)}\tilde{w}_i$$

for a set of design matrices $M_{(j,k)} \in \mathbb{R}^{m\times n}$ *and introducing these relationships as an additional set of constraints in the corresponding optimization for PTOC (where one optimizes over the set of* $M_{(j,k)}$ *which then induce the set of* $\tilde{u}_{(i,j,k)}$*). Clearly, whenever the optimization with this additional set of constraints is feasible so is the one without enforcing these additional constraints. Hence, whenever the methods of [15–18] are feasible so is our proposal. We prove that the opposite is not true by providing an example where the methods of [15–18] fail to be feasible for the RMPC synthesis while our PTMPC method allows for a feasible RMPC synthesis.*

The system is two dimensional system specified by:

$$x^+ = Ax + Bu + w,\ \textit{with},\ A = \frac{1}{2}\begin{pmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{pmatrix},\ \textit{and},\ ,\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The state and control constraint sets are given by:

$$\begin{aligned} \mathbb{X} := \{x \in \mathbb{R}^2\ :\ & (0.0176,\ 0.0655)x \leq 1,\ (-0.0428,\ 0.0428)x \leq 1,\ (-0.0747,\ -0.0200)x \leq 1, \\ & (-0.0176,\ -0.0655) \leq 1,\ (0.0428,\ -0.0428)x \leq 1,\ (0.0747,\ 0.0200)x \leq 1\},\ \textit{and}, \\ \mathbb{U} := \{u \in \mathbb{R}\ :\ & \frac{1}{3}u \leq 1,\ -\frac{1}{3}u \leq 1\}. \end{aligned}$$

The disturbance set $\mathbb{W}$ *is given by:*

$$\begin{aligned} &\mathbb{W} = \operatorname{convh}(\{\tilde{w}_i\ :\ i \in \mathbb{N}_{[1:q]}\})\ \textit{where}\ q = 18,\ \textit{and},\ \tilde{w}_1 = (-1.8660,\ -4.2321),\ \tilde{w}_2 = (-2.7321,\ -3.7321), \\ &\tilde{w}_3 = (2.7321,\ -3.7321),\ \tilde{w}_4 = (1.8660,\ -4.2321),\ \tilde{w}_5 = (0,\ -4.7321),\ \tilde{w}_6 = (4.5981,\ 0.5000), \\ &\tilde{w}_7 = (4.5981,\ -0.5000),\ \tilde{w}_8 = (4.0981,\ -2.3660),\ \tilde{w}_9 = (2.7321,\ 3.7321),\ \tilde{w}_{10} = (4.0981,\ 2.3660), \\ &\tilde{w}_{11} = (1.8660,\ 4.2321),\ \tilde{w}_{12} = (0,\ 4.7321),\ \tilde{w}_{13} = (-1.8660,\ 4.2321),\ \tilde{w}_{14} = (-4.5981,\ 0.5000), \\ &\tilde{w}_{15} = (-4.5981,\ -0.5000),\ \tilde{w}_{16} = (-4.0981,\ -2.3660),\ \tilde{w}_{17} = (-2.7321,\ 3.7321),\ \tilde{w}_{18} = (-4.0981,\ 2.3660). \end{aligned}$$

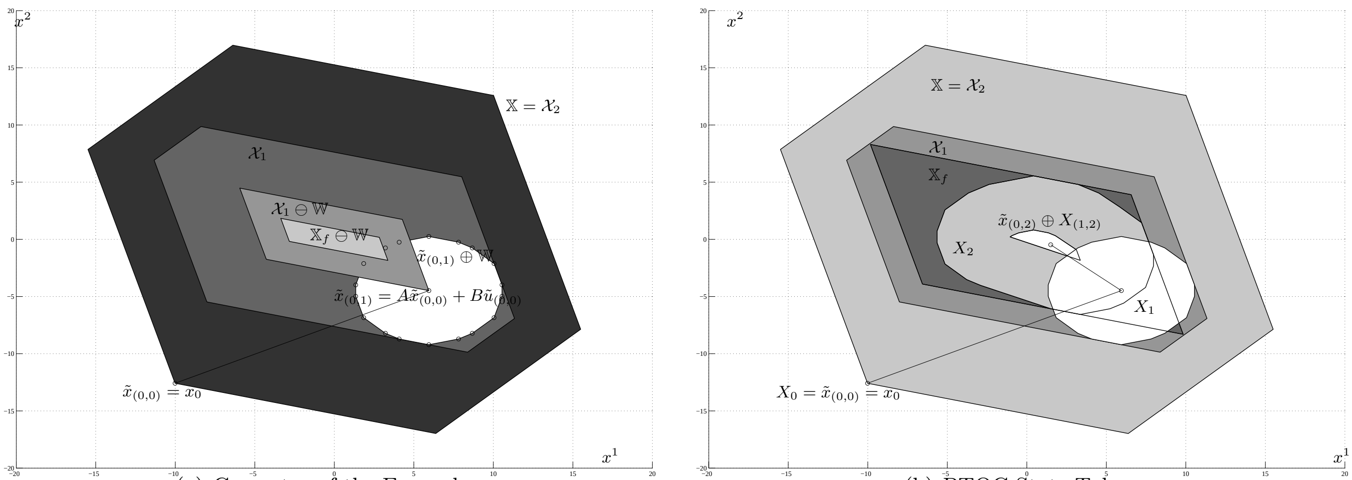
The sets $\mathcal{Q}$ *and* $\mathcal{R}$ *utilized for the cost function* $V_N(\cdot)$ *via the associated gauge functions are given by:*

$$\mathcal{Q} := \{x \in \mathbb{R}^2\ :\ (1,\ 0)x \leq 1,\ -(1,\ 0)x \leq 1,\ (0,\ 1)x \leq 1,\ -(0,\ 1)x \leq 1\},\ \textit{and},\ \mathcal{R} := \{u \in \mathbb{R}\ :\ u \leq 1,\ -u \leq 1\}.$$

The local linear feedback $u = Kx$, *the terminal constraint set* $\mathbb{X}_f$ *and the set* $\mathcal{P}$ *utilized in the cost function* $V_N(\cdot)$ *via the associated gauge function are given by:*

$$\begin{aligned} &\mathbb{X}_f = \{x \in \mathbb{R}^2\ :\ (0.0473,\ 0.1765)x \leq 1,\ -(0.0473,\ 0.1765)x \leq 1,\ (0.1314,\ 0.0352)x \leq 1,\ -(0.1314,\ 0.0352)x \leq 1\}, \\ &K = -(0.3943,\ 0.1057),\ \textit{and}, \\ &\mathcal{P} := \{x \in \mathbb{R}^2\ :\ (1.0635,\ 3.9692)x \leq 1,\ -(1.0635,\ 3.9692)x \leq 1,\ (2.9564,\ 0.7922)x \leq 1,\ -(2.9564,\ 0.7922)x \leq 1\}. \end{aligned}$$

Similarly as in the previous example, any initial condition $x_0 \in \mathbb{X}$ *is min–max controllable to a target set* $\mathbb{X}_f$ *within* $N = 2$ *steps. In particular, for the initial conditions equal to the extreme points of the set* $\mathbb{X}$ *the min–max* *controllability to a target set $\mathbb{X}_f$ can be guaranteed if and only if $N = 2$. The state constraint set $\mathbb{X}$ is in this case equal to 2–step parameterized tubes controllability set $\mathcal{X}_2$ for this example. We consider the initial condition $x_0 = -(10.0127,\ 12.5832)$ and the horizon length $N = 2$. Let $\tilde{x}_{(0,0)} = x_0$. The only feasible control $u \in \mathbb{U}$ for this initial condition permitting for the min–max controllability to a target set $\mathbb{X}_f$ is $u = 3$ so we are forced to set $\tilde{u}_{(0,0)} = 3$ and, in turn, we get $\tilde{x}_{(0,1)} = (5.9455,\ -4.4814)^T$ which is one of the extreme points of the set $\mathcal{X}_1 \ominus \mathbb{W}$. This fixes the state tube cross–section X_1 to $X_1 = \tilde{x}_{(0,1)} \oplus \mathbb{W}$. The initial condition $\tilde{x}_{(0,0)} = x_0$, the state $\tilde{x}_{(0,1)}$ and the state tube cross–section X_1 are shown in Figure 2. The 1– and 2–step parameterized tubes controllability sets $\mathcal{X}_1$, $\mathcal{X}_2$ as well as effective target sets at times 1 and 2, namely the sets $\mathcal{X}_1 \ominus \mathbb{W}$ and $\mathbb{X}_f \ominus \mathbb{W}$ are also show in Figure 2 (a) using the different levels of gray scale shading. In order to implement the proposed PTMPC we need to find the*



(a) Geometry of the Example. (b) PTOC State Tube.

Figure 2: Geometry and PTOC State Tubes for Illustrative Example 2.

partial control tube cross sections $U_{(0,1)} = \{\tilde{u}_{(0,1)}\}$ and $U_{(1,1)} = \{\tilde{u}_{(i,1,1)}\ :\ i \in \mathbb{N}_{[1:q]}\}$ satisfying the constraints:

$$\begin{aligned}
&\forall i \in \mathbb{N}_{[1:18]},\ \tilde{u}_{(0,1)} + \tilde{u}_{(i,1,1)} \in \mathbb{U},\ \textit{and,}\\
&\forall i \in \mathbb{N}_{[1:18]},\ A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} + A\tilde{w}_i + B\tilde{u}_{(i,1,1)} \in \mathbb{X}_f \ominus \mathbb{W}.
\end{aligned}$$

This task is possible to accomplish since, by inspection of Figure 2 (b), we see that the state tube cross–section X_1 satisfies $X_1 \subseteq \mathcal{X}_1$ and the set $\mathcal{X}_1$ is the 1–step parameterized tubes controllability set. We solve PTOC for this problem and in Figure 2 (b) we show the corresponding PTOC state tubes $\mathbf{X}_2 = \{X_0 = \{\tilde{x}_{(0,0)}\}, X_1 = \tilde{x}_{(0,1)} \oplus X_{(1,1)}, X_2 = \tilde{x}_{(0,2)} \oplus X_{(1,2)} \oplus X_{(2,2)}\}$ obtained from the corresponding PTOC control tubes $\mathbf{U}_1 = \{U_0 = \{\tilde{u}_{(0,0)}\}, U_1 = \tilde{u}_{(0,1)} \oplus U_{(1,1)}\}$ given by:

$$\begin{aligned}
&\tilde{u}_{(0,0)} = 3,\ \tilde{u}_{(0,1)} = -1.9251\ \textit{and}\\
&U_{(1,1)} = \operatorname{convh}(\{0.5,\ 0.8943,\ -0.6830,\ -0.5774,\ -0.1830,\ -1.0749,\ -1.0749,\ -1.0749,\ -1.0749,\\
&\qquad -1.0749,\ -1.0749,\ -0.5000,\ 0.2887,\ 1.7604,\ 1.866,\ 1.683,\ 0.683,\ 1.366\})\ \textit{so that}\\
&U_{(1,1)} = [-1.0749, 1.8660],\ \textit{and},\ \tilde{u}_{(0,1)} \oplus U_{(1,1)} = [-3, -0.059] \subseteq \mathbb{U}.
\end{aligned}$$

In Figure 2 (b), we also depict the set $\tilde{x}_{(0,2)} \oplus X_{(1,2)}$ in order to illustrate that the partial state tube cross–section $X_{(1,2)}$ has been transformed in a non linear fashion by the control rule induced from the partial control tube $U_{(1,1)}$ in order to meet the constraints $X_2 = \tilde{x}_{(0,2)} \oplus X_{(1,2)} \oplus X_{(2,2)} \subseteq \mathbb{X}_f$.

By construction it follows that for the methods of [15–18] to be applicable it is necessary to find $\tilde{u}_{(0,1)}$ and a matrix $M \in \mathbb{R}^{1\times 2}$ satisfying the constraints:

$$\begin{aligned}
&\forall i \in \mathbb{N}_{[1:18]},\ \tilde{u}_{(0,1)} + M\tilde{w}_i \in \mathbb{U},\ \textit{and,}\\
&\forall i \in \mathbb{N}_{[1:18]},\ A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} + (A + BM)\tilde{w}_i \in \mathbb{X}_f \ominus \mathbb{W}.
\end{aligned}$$

However, these constraints can not be satisfies as we have verified numerically and, hence, the methods of [15–18] are not applicable to this problem. From the inspection of the Figure, it is not surprising that we can not find an affine function of states belonging to the state tube cross–section X_1 generating the admissible controls actions for the extreme points of X_1 ensuring that the state tube cross–section $X_2 = A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} \oplus (A + BM)\mathbb{W} \oplus \mathbb{W}$ satisfies $X_2 \subseteq \mathbb{X}_f$ (or equivalently $A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} \oplus (A + BM)\mathbb{W} \subseteq \mathbb{X}_f \ominus \mathbb{W}$) and that the corresponding control tube cross–section $U_1 = \tilde{u}_{(0,1)} \oplus M\mathbb{W} \subseteq \mathbb{U}$. This should come without any surprises as there are 18 extreme points 4 of which lie on the boundary of the 1–step parameterized tubes controllability set $\mathcal{X}_1$ and we are allowed to select *only three variables namely $\tilde{u}_{(0,1)}$, m_1 and m_2 (where $M = (m_1,\ m_2)$). To sum up, our example demonstrates that the methods of [15–18] may fail to be applicable when our proposal is applicable.*

Illustrative Example 3 *Our third illustrative example is two dimensional system specified by:*

$$x^+ = Ax + Bu + w, \text{ with, } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and, , } B = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

The state and control constraint sets are given by:

$$\mathbb{X} := \{x \in \mathbb{R}^2 \ : \ (\frac{1}{50},\ 0)x \leq 1,\ (-\frac{1}{20},\ 0)x \leq 1,\ (0,\ \frac{1}{2})x \leq 1,\ (0,\ -\frac{1}{10})x \leq 1\},\ \text{and, } \mathbb{U} := \{u \in \mathbb{R} \ : \ u \leq 1,\ -u \leq 1\}.$$

The disturbance set $\mathbb{W}$ is given by:

$$\mathbb{W} = \mathrm{convh}(\{\tilde{w}_i \ : \ i \in \mathbb{N}_{[1:q]}\}) \text{ where } q = 4, \text{ and, } \tilde{w}_1 = (0.1,\ 0.1),\ \tilde{w}_2 = (0.1,\ -0.1),\ \tilde{w}_3 = -\tilde{w}_1 \text{ and } \tilde{w}_4 = -\tilde{w}_2.$$

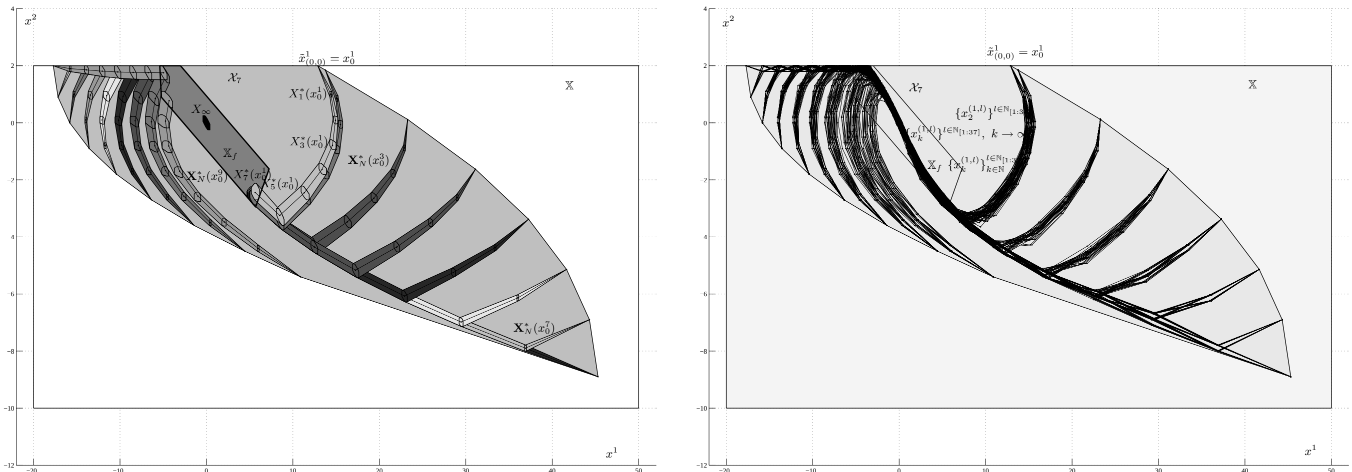
The sets $\mathcal{Q}$ and $\mathcal{R}$ utilized for the cost function $V_N(\cdot)$ via the associated gauge functions are given by:

$$\mathcal{Q} := \{x \in \mathbb{R}^2 \ : \ (1,\ 0)x \leq 1,\ -(1,\ 0)x \leq 1,\ (0,\ 1)x \leq 1,\ -(0,\ 1)x \leq 1\},\ \text{and, } \mathcal{R} := \{u \in \mathbb{R} \ : \ u \leq 1,\ -u \leq 1\}.$$

The local linear feedback $u = Kx$, the terminal constraint set $\mathbb{X}_f$ and the set $\mathcal{P}$ utilized in the cost function $V_N(\cdot)$ via the associated gauge function are given by:

$$\begin{aligned} \mathbb{X}_f = \{x \in \mathbb{R}^2 \ : \ & -(0.1974,\ 0.0329)x \leq 1,\ (0.1095,\ -0.1277)x \leq 1,\ (0,\ 0.5)x \leq 1, \\ & (0.3750,\ 1.0625)x \leq 1,\ -(0.3750,\ 1.0625)x \leq 1\}, \\ K = -(0.375,\ 1.0625),\ & \text{and,} \\ \mathcal{P} := \{x \in \mathbb{R}^2 \ : \ & (35.8103,\ 5.9684)x \leq 1,\ -(35.8103,\ 5.9684)x \leq 1,\ (0,\ 90.7194)x \leq 1, \\ & -(0,\ 90.7194)x \leq 1,\ (68.0396,\ 192.7788)x \leq 1,\ -(68.0396,\ 192.7788)x \leq 1\}. \end{aligned}$$

We set the horizon length $N = 7$. In Figure 3 (a), we show the state constraint set $\mathbb{X}$, the 7–step parameterized tubes controllability set $\mathcal{X}_7$, the terminal constraint set $\mathbb{X}_f$ and the minimal robust positively invariant set X_∞. The 7–step



(a) PTOC State Tubes. (b) Sample PTMPC Controlled Trajectories.

Figure 3: PTOC State Tubes and Sample PTMPC Controlled Trajectories from Extreme Point of the Set $\mathcal{X}_7$.

parameterized tubes controllability set $\mathcal{X}_7$ has 17 extreme points and we solve PTOC for a set of initial conditions being equal to the extreme points of $\mathcal{X}_7$. The corresponding PTOC state tubes are also shown in Figure 3 (a). As evident, they satisfy state constraints as they remain with the sets $\mathcal{X}_7$ and the final tube tube cross–sections are all included in the terminal constraint set $\mathbb{X}_f$. In Figure 3 (b), we show a number of trajectories generated by the controlled uncertain dynamics obtained from the implementation of the PTMPC law for a set of random extreme disturbance sequences. As expected in view of results established in Theorem 1 and Corollary 3, all these trajectories converge exponentially fast to the minimal robust positively invariant set X_∞.

8 Concluding Remarks

In this paper we considered the robust model predictive control synthesis problem for constrained linear discrete time systems. We introduced a parameterized tube model predictive control synthesis method which utilizes the parameterized state and control tubes and the corresponding control policy and permits for their online optimization via a single tractable linear programming problem. It was shown that the corresponding control policy possessed a higher degree of nonlinearity compared to a set of existing proposals for robust model predictive control allowing us, in turn, to demonstrate that our proposal is, in fact, also more general than a variety of recently proposed robust model predictive control synthesis methods. Furthermore, we shown that, under rather mild assumptions, the developed method is computationally efficient while it guarantees strong system theoretic properties for the controlled uncertain dynamics.

APPENDIX A: Proofs

A–1: Proof of Proposition 1

The claims of Proposition 1 summarize the discussion preceding it.

A–2: Proof of Proposition 2

As in the case of Proposition 1, the claims of Proposition 2 follow from the discussion preceding it.

A–3: Proof of Proposition 3

The fact (i) and the asserted fact in (3.14a) both follow from the definition of the Minkowski set addition since $X_k := \bigoplus_{j=0}^{k} X_{(j,k)}$. The claimed fact in (3.14b) follows from Proposition 2.

A–4: Proof of Corollary 1

Corollary 1 is a direct consequence of Proposition 3.

A–5: Proof of Proposition 4

Pick an integer $N \in \mathbb{N}_+$ and take any arbitrary $z \in \mathbb{X}_f$ and consider the local state and control tubes $\mathbf{Z}_N = \{Z_0, Z_1, \ldots, Z_N\}$ and $\mathbf{V}_{N-1} = \{V_0, V_1, \ldots, V_{N-1}\}$ specified by (4.4). We first note that by construction relationship (4.5c) is true (the relationship (4.4a) is merely obtained by the iteration of (4.5c) with $Z_0 = \{z\}$). Take a $k \in \mathbb{N}_{[0:N]}$ and assume that $Z_k \subseteq \mathbb{X}_f$. Then, by construction, $Z_{k+1} = (A + BK)Z_k \oplus \mathbb{W}$ and $(A + BK)Z_k \oplus \mathbb{W} \subseteq (A + BK)\mathbb{X}_f \oplus \mathbb{W} \subseteq \mathbb{X}_f$ and, in turn, $Z_{k+1} \subseteq \mathbb{X}_f$. Hence, since $Z_0 = \{z\} \subseteq \mathbb{X}_f$ and $\mathbb{X}_f \subseteq \mathbb{X}$, the fact that for all $k \in \mathbb{N}_{[0:N]}$, $Z_k \subseteq \mathbb{X}_f \subseteq \mathbb{X}$ is true. By (4.4b) we have that for all $k \in \mathbb{N}_{[0:N-1]}$, $V_k = KZ_k$ and, hence, since for all $k \in \mathbb{N}_N$, $Z_k \subseteq \mathbb{X}_f$ and $K\mathbb{X}_f \subseteq \mathbb{U}$, it follows that for all $k \in \mathbb{N}_{N-1}$, $V_k = KZ_k \subseteq K\mathbb{X}_f \subseteq \mathbb{U}$ as claimed.

A–6: Proof of Proposition 5

Pick an integer $N \in \mathbb{N}_+$ and take any arbitrary set $Y_N = \tilde{y}_{(0,N)} \oplus \bigoplus_{j=1}^{N} Y_{(j,N)} \subseteq \mathbb{X}_f$ as specified by (4.6). Pick any arbitrary $y_{(1,N)} \in Y_{(1,N)}$ and note that by direct inspection of (4.7) and (4.8) it follows that the fact $\hat{Y}_N(y_{(1,N)}) = (A + BK)(\tilde{y}_{(0,N)} + y_{(1,N)} \oplus \bigoplus_{j=2}^{N} Y_{(j,N)}) \oplus \mathbb{W}$ is clearly true. Now, we have that $\tilde{y}_{(0,N)} + y_{(1,N)} \oplus \bigoplus_{j=2}^{N} Y_{(j,N)} \subseteq \tilde{y}_{(0,N)} \oplus \bigoplus_{j=1}^{N} Y_{(j,N)} = Y_N$. Hence, since $Y_N \subseteq \mathbb{X}_f$, we have that $(A + BK)(\tilde{y}_{(0,N)} + y_{(1,N)} \oplus \bigoplus_{j=2}^{N} Y_{(j,N)}) \oplus \mathbb{W} \subseteq (A + BK)Y_N \oplus \mathbb{W}$. But, $(A + BK)Y_N \oplus \mathbb{W} \subseteq (A + BK)\mathbb{X}_f \oplus \mathbb{W} \subseteq \mathbb{X}_f$ and $\mathbb{X}_f \subseteq \mathbb{X}$. Hence, $(A + BK)(\tilde{y}_{(0,N)} + y_{(1,N)} \oplus \bigoplus_{j=2}^{N} Y_{(j,N)}) \oplus \mathbb{W} \subseteq \mathbb{X}_f \subseteq \mathbb{X}$ and, in turn, in view of (4.8) we have that $\hat{Y}_N(y_{(1,N)}) \subseteq \mathbb{X}_f \subseteq \mathbb{X}$ as claimed. Since $\tilde{y}_{(0,N)} + y_{(1,N)} \oplus \bigoplus_{j=2}^{N} Y_{(j,N)} \subseteq Y_N \subseteq \mathbb{X}_f$ we have that $\hat{V}_{N-1}(y_{(1,N)}) = K(\tilde{y}_{(0,N)} + y_{(1,N)} \oplus \bigoplus_{j=2}^{N} Y_{(j,N)}) \subseteq KY_N \subseteq K\mathbb{X}_f$. Hence, since $K\mathbb{X}_f \subseteq \mathbb{U}$, it follows that $\hat{V}_{N-1}(y_{(1,N)}) \subseteq K\mathbb{X}_f \subseteq \mathbb{U}$ as claimed.

A–7: Proof of Proposition 6

(i) As pointed out in Remark 9, the N–step parameterized tubes controllability set $\mathcal{X}_N$ is closed polyhedral PC–set being a projection of a closed polyhedral set Γ_N given by (7.5c) (see also Appendix $-$ B for algebraic details). By construction, the set $\mathcal{X}_N$ contains the PC–polytopic set $\mathbb{X}_f$ and is contained in the PC–polytopic set $\mathbb{X}$. Hence, it follows that $\mathcal{X}_N$ is a PC–polytopic set in $\mathbb{R}^n$ such that $\mathbb{X}_f \subseteq \mathcal{X}_N \subseteq \mathbb{X}$ as claimed. (ii) Since the equivalent PTOC problem reformulation provided in (7.5a) takes the form of a parametric linear programming problem, the value function $V_N^0(\cdot)\ :\ \mathcal{X}_N \to \mathbb{R}_+$ is convex, piecewise affine and continuous functions by standard results in parametric mathematical programming [34]. Feasibility of local state and control tubes, specified by (4.4), established in Proposition 4 combined with the facts that the cost function is non–negative and the value of the cost function for the local state and control tubes is 0 establishes that $\forall x \in \mathbb{X}_f,\ V_N^0(x) = 0$. The fact that $\forall x \in \mathcal{X}_N \setminus \mathbb{X}_f,\ V_N^0(x) > 0$ is also true as otherwise there is a contradiction. Namely, $V_N^0(x) = 0$ implies that the value of any optimal $\tilde{e}^0_{(0,0)}(x)$ must satisfy $\mathcal{G}(\mathcal{Q}, \tilde{e}^0_{(0,0)}(x)) = 0$ and, in turn, $\tilde{e}^0_{(0,0)}(x) = 0$. In view of (5.3c) it holds that $x - \tilde{e}^0_{(0,0)}(x) \in \mathbb{X}_f$ which implies that $x \in \mathbb{X}_f$ revealing a contradiction to the assumed fact that $x \in \mathcal{X}_N \setminus \mathbb{X}_f$. Hence, as claimed $\forall x \in \mathcal{X}_N \setminus \mathbb{X}_f,\ V_N^0(x) > 0$. *(iii)* As above, since the equivalent PTOC problem reformulation in (7.5b) reduces to a parametric linear programme, the piecewise affine and continuous nature of functions $\mathbf{e}_0^*(\cdot) : \mathcal{X}_N \to \mathbb{R}^{(N+1)n}$, $\mathbf{v}_0^*(\cdot) : \mathcal{X}_N \to \mathbb{R}^{Nm}$, $\mathbf{x}^*_{(i,k,N)}(\cdot) : \mathcal{X}_N \to \mathbb{R}^{(N+1-k)n},\ i \in \mathbb{N}_{[1:q]},\ k \in \mathbb{N}_{[1:N]}$ and $\mathbf{u}^*_{(i,k,N)}(\cdot) : \mathcal{X}_N \to \mathbb{R}^{(N-k)m},\ i \in \mathbb{N}_{[1:q]},\ k \in \mathbb{N}_{[1:N-1]}$ follows by the standard results in parametric mathematical programming [34]. Additional properties in *(iii)* follow directly from our construction and the feasibility of local state and control tubes, specified by (4.4), established in Proposition 4.

A–8: Proof of Proposition 7

Claimed Relationship (5.19a)

First, we note that, by construction, for all $x \in \mathcal{X}_N$, it holds that $y(x, x_{(1,1)}) := Ax + B(K(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,0)}(x)) + x_{(1,1)},\ x_{(1,1)} \in X^*_{(1,1)}(x) = \mathbb{W}$ satisfies $y(x, x_{(1,1)}) = Ax + B(K(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,0)}(x)) + x_{(1,1)} = (A + BK)(x - \tilde{e}^*_{(0,0)}(x)) + A\tilde{e}^*_{(0,0)}(x) + B\tilde{v}^*_{(0,0)}(x) + x_{(1,1)} = (A + BK)(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{e}^*_{(0,1)}(x) + x_{(1,1)} \in X^*_{(0,1)}(x) \oplus X^*_{(1,1)}(x) = X_1^*(x)$. Hence, the relation in (5.19a) holds true.

Claimed Relationship (5.19b)

Pick any arbitrary $x \in \mathcal{X}_N$ and any arbitrary $x_{(1,1)} \in X^*_{(1,1)}(x) = \mathbb{W}$ and consider the corresponding $y(x, x_{(1,1)}) = Ax + B(K(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,0)}(x)) + x_{(1,1)}$ and $\hat{\theta}_N(y(x, x_{(1,1)}))$ specified via (5.16)–(5.18). We need to show that $\hat{\theta}_N(y(x, x_{(1,1)})) \in \Theta_N(y(x, x_{(1,1)}))$. By construction relationships (3.1), (3.2), (3.3b), (3.4b), (3.11), (3.12) and (5.1) are satisfied so we need to verify the relationships (5.2) and (5.3).

Relationship (5.2)

We note that, for all $l \in \mathbb{N}_{N-2}$ it holds that:

$$
\begin{aligned}
&A\hat{e}_{(0,l)}(y(x, x_{(1,1)})) + B\hat{v}_{(0,l)}(y(x, x_{(1,1)})) = \\
&A\left(\tilde{e}^*_{(0,l+1)}(x) + x^*_{(1,l+1)}(x_{(1,1)}) - (A + BK)^l x_{(1,1)}\right) + B\left(\tilde{v}^*_{(0,l+1)}(x) + u^*_{(1,l+1)}(x_{(1,1)}) - K(A + BK)^l x_{(1,1)}\right) = \\
&\tilde{e}^*_{(0,l+2)}(x) + x^*_{(1,l+2)}(x_{(1,1)}) - (A + BK)^{l+1} x_{(1,1)} = \hat{e}_{(0,l+1)}(y(x, x_{(1,1)})).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&A\hat{e}_{(0,N-1)}(y(x, x_{(1,1)})) + B\hat{v}_{(0,N-1)}(y(x, x_{(1,1)})) = \\
&A\left(\tilde{e}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^{N-1} x_{(1,1)}\right) + BK\left(\tilde{e}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^{N-1} x_{(1,1)}\right) = \\
&(A + BK)(\tilde{e}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^{N-1} x_{(1,1)}) = \hat{e}_{(0,N)}(y(x, x_{(1,1)})).
\end{aligned}
$$

Hence, the relationships specified in (5.2) hold true.

Relationship (5.3a)

To verify the relationships in (5.3a) we note that:

$$
\begin{aligned}
&\forall l \in \mathbb{N}_{[N-1]},\ \hat{X}_{(0,l)}(y(x, x_{(1,1)})) = x^*_{(1,l+1)}(x_{(1,1)}) \oplus X^*_{(0,l+1)}(x) = \{\tilde{x}^*_{(0,l+1)}(x) + x^*_{(1,l+1)}(x_{(1,1)})\}, \text{ and,} \\
&\hat{X}_{(0,N)}(y(x, x_{(1,1)})) = (A + BK)x^*_{(1,N)}(x_{(1,1)}) \oplus (A + BK)X^*_{(0,N)}(x) = \{(A + BK)(\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}))\},
\end{aligned}
$$

where

$$\forall l \in \mathbb{N}_{[1:N]},\ \tilde{x}^*_{(0,l)}(x) = (A + BK)^l (x - \tilde{e}^*_{(0,0)}(x)) + \tilde{e}^*_{(0,l)}(x),$$

and, for all $l \in \mathbb{N}_{[1:N]}$, $x^*_{(1,l+1)}(x_{(1,1)})$ is specified in (5.16b). Furthermore,

$$\forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ \hat{X}_{(k,l)}(y(x, x_{(1,1)})) = X^*_{(k+1,l+1)}(x),\ \text{and,}$$
$$\forall k \in \mathbb{N}_{[1:N-1]},\ \hat{X}_{(k,N)}(y(x, x_{(1,1)})) = (A + BK)X^*_{(k+1,N)}(x),\ \text{and, } \hat{X}_{(N,N)}(y(x, x_{(1,1)})) = \mathbb{W}.$$

In turn, since $\mathbb{X}_f \subseteq \mathbb{X}$, it follows that for all $k \in \mathbb{N}_{N-1}$ it holds that:

$$\hat{X}_k(y(x, x_{(1,1)})) = \bigoplus_{j=0}^{k} \hat{X}_{(j,k)}(y(x, x_{(1,1)})) = \tilde{x}^*_{(0,k+1)}(x) + x^*_{(1,k+1)}(x_{(1,1)}) \oplus \bigoplus_{j=1}^{k} X^*_{(j+1,k+1)}(x) \subseteq$$
$$\tilde{x}^*_{(0,k+1)}(x) \oplus X^*_{(1,k+1)}(x) \oplus \bigoplus_{j=1}^{k} X^*_{(j+1,k+1)}(x) = X^*_{k+1}(x) \subseteq \mathbb{X}.$$

In addition, we also have:

$$\hat{X}_N(y(x, x_{(1,1)})) = \bigoplus_{j=0}^{N} \hat{X}_{(j,N)}(y(x, x_{(1,1)})) = (A + BK)(\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)})) \oplus (A + BK) \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) \oplus \mathbb{W}$$
$$(A + BK) \left(\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) \oplus \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) \right) \oplus \mathbb{W}$$

which, since $\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) \oplus \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) \subseteq \tilde{x}^*_{(0,N)}(x) \oplus X^*_{(1,N)}(x) \oplus \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) = X^*_N(x) \subseteq \mathbb{X}_f$ and $x^*_{(1,N)}(x_{(1,1)}) \in X^*_{(1,N)}(x)$, by Proposition 5 yields that:

$$\hat{X}_N(y(x, x_{(1,1)})) \subseteq \mathbb{X}_f.$$

Therefore, the relationships in (5.3a) hold true.

Relationship (5.3b)

A similar argument to the one deployed to verify the relationships in (5.3a) demonstrates that the relationships in (5.3b) also hold true. We note that:

$$\forall l \in \mathbb{N}_{[N-2]},\ \hat{U}_{(0,l)}(y(x, x_{(1,1)})) = u^*_{(1,l+1)}(x_{(1,1)}) \oplus U^*_{(0,l+1)}(x) = \{\tilde{u}^*_{(0,l+1)}(x) + u^*_{(1,l+1)}(x_{(1,1)})\},\ \text{and,}$$
$$\hat{U}_{(0,N-1)}(y(x, x_{(1,1)})) = Kx^*_{(1,N)}(x_{(1,1)}) \oplus KX^*_{(0,N)}(x) = \{K(\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}))\},$$

where

$$\forall l \in \mathbb{N}_{[1:N-1]},\ \tilde{u}^*_{(0,l)}(x) = K(A + BK)^l(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,l)}(x),$$

and, for all $l \in \mathbb{N}_{[1:N-1]}$, $u^*_{(1,l+1)}(x_{(1,1)})$ is specified in (5.16a). Furthermore,

$$\forall k \in \mathbb{N}_{[1:N-2]},\ \forall l \in \mathbb{N}_{[k:N-2]},\ \hat{U}_{(k,l)}(y(x, x_{(1,1)})) = U^*_{(k+1,l+1)}(x),\ \text{and,}$$
$$\forall k \in \mathbb{N}_{[1:N-2]},\ \hat{U}_{(k,N-1)}(y(x, x_{(1,1)})) = KX^*_{(k+1,N)}(x),\ \text{and, } \hat{U}_{(N-1,N-1)}(y(x, x_{(1,1)})) = K\mathbb{W}.$$

In turn, it follows that for all $k \in \mathbb{N}_{N-2}$ it holds that:

$$\hat{U}_k(y(x, x_{(1,1)})) = \bigoplus_{j=0}^{k} \hat{U}_{(j,k)}(y(x, x_{(1,1)})) = \tilde{u}^*_{(0,k+1)}(x) + u^*_{(1,k+1)}(x_{(1,1)}) \oplus \bigoplus_{j=1}^{k} U^*_{(j+1,k+1)}(x) \subseteq$$
$$\tilde{u}^*_{(0,k+1)}(x) \oplus U^*_{(1,k+1)}(x) \oplus \bigoplus_{j=1}^{k} U^*_{(j+1,k+1)}(x) = U^*_{k+1}(x) \subseteq \mathbb{U}.$$

In addition, since $X^*_{(N,N)}(x) = \mathbb{W}$, we also have:

$$\hat{U}_{N-1}(y(x, x_{(1,1)})) = \bigoplus_{j=0}^{N-1} \hat{U}_{(j,N)}(y(x, x_{(1,1)})) = K(\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)})) \oplus K \bigoplus_{j=2}^{N-1} X^*_{(j,N)}(x)$$
$$= K \left(\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) \oplus \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) \right)$$

which, since $\tilde{x}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) \oplus \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) \subseteq \tilde{x}^*_{(0,N)}(x) \oplus X^*_{(1,N)}(x) \oplus \bigoplus_{j=2}^{N} X^*_{(j,N)}(x) = X^*_N(x) \subseteq \mathbb{X}_f$ and $x^*_{(1,N)}(x_{(1,1)}) \in X^*_{(1,N)}(x)$, by Proposition 5 yields that:

$$\hat{U}_{N-1}(y(x, x_{(1,1)})) \subseteq K\hat{X}_N(y(x, x_{(1,1)})) \subseteq K\mathbb{X}_f \subseteq \mathbb{U}.$$

Therefore, the relationships in (5.3b) hold true.

Relationship (5.3c)

We note that:

$$y(x, x_{(1,1)}) - \hat{e}_{(0,0)}(y(x, x_{(1,1)})) = (A + BK)(x - \tilde{e}^*_{(0,0)}) + x_{(1,1)}$$

so that due to Assumption 3 (in particular, the robust positive invariance property of the set $\mathbb{X}_f$) and since $(x - \tilde{e}^*_{(0,0)}) \in \mathbb{X}_f$ and $x_{(1,1)} \in \mathbb{W}$ we have that $(A + BK)(x - \tilde{e}^*_{(0,0)}) + x_{(1,1)} \in \mathbb{X}_f$, i.e.:

$$y(x, x_{(1,1)}) - \hat{e}_{(0,0)}(y(x, x_{(1,1)})) \in \mathbb{X}_f.$$

Finally, since $x \in \mathcal{X}_N$ and $x_{(1,1)} \in X^*_{(1,1)}(x) = \mathbb{W}$ were arbitrary, it follows that for all $x \in \mathcal{X}_N$ and all $y(x, x_{(1,1)}) := Ax + B(K(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,0)}(x)) + x_{(1,1)},\ x_{(1,1)} \in X^*_{(1,1)}(x) = \mathbb{W}$ it holds that:

$$\hat{\theta}_N(y(x, x_{(1,1)})) \in \Theta_N(y(x, x_{(1,1)})),\ \text{and, consequently,}\ X^*_1(x) \subseteq \mathcal{X}_N,$$

as claimed in (5.19b).

Claimed Relationship (5.19c)

To prove the claimed inequality in (5.19c) we first recall that the gauge (Minkowski) functions are subadditive and convex functions [41, 42]. Before proceeding with the proof, let for typographical convenience:

$$\hat{\mathbf{e}}_0 := \{\hat{e}_{(0,k)}(y(x, x_{(1,1)}))\}_{k\in\mathbb{N}_N},\ \hat{\mathbf{v}}_0 := \{\hat{v}_{(0,k)}(y(x, x_{(1,1)}))\}_{k\in\mathbb{N}_{N-1}},$$

and, for all $k \in \mathbb{N}_{[1:N]}$:

$$\hat{\mathbf{X}}_{(k,N)} := \{\hat{X}_{(k,l)}(y(x, x_{(1,1)}))\}_{l\in\mathbb{N}_{[k:N]}},$$

and, for all $k \in \mathbb{N}_{[1:N-1]}$:

$$\hat{\mathbf{U}}_{(k,N-1)} := \{\hat{U}_{(k,l)}(y(x, x_{(1,1)}))\}_{l\in\mathbb{N}_{[k:N-1]}}.$$

We now proceed with the proof. By construction we have that:

$$V_{(0,N)}(\hat{\mathbf{e}}_0, \hat{\mathbf{v}}_0) = \sum_{l=0}^{N-1} \big(\mathcal{G}(\mathcal{Q}, \hat{e}_{(0,l)}(y(x, x_{(1,1)}))) + \mathcal{G}(\mathcal{R}, \hat{v}_{(0,l)}(y(x, x_{(1,1)})))\big) + \mathcal{G}(\mathcal{P}, \hat{e}_{(0,N)}(y(x, x_{(1,1)}))).$$

But, for all $l \in \mathbb{N}_{N-1}$, we have:

$$\begin{aligned}\mathcal{G}(\mathcal{Q}, \hat{e}_{(0,l)}(y(x, x_{(1,1)}))) &= \mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,l+1)}(x) + x^*_{(1,l+1)}(x_{(1,1)}) - (A + BK)^l x_{(1,1)}) \\ &\leq \mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,l+1)}(x)) + \mathcal{G}(\mathcal{Q}, x^*_{(1,l+1)}(x_{(1,1)}) - (A + BK)^l x_{(1,1)}).\end{aligned}$$

by subadditivity of the gauge (Minkowski) function $\mathcal{G}(\mathcal{Q}, \cdot)$. By the same token, for all $l \in \mathbb{N}_{N-2}$, we also have:

$$\begin{aligned}\mathcal{G}(\mathcal{R}, \hat{v}_{(0,l)}(y(x, x_{(1,1)}))) &= \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,l+1)}(x) + u^*_{(1,l+1)}(x_{(1,1)}) - K(A + BK)^l x_{(1,1)}) \\ &\leq \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,l+1)}(x)) + \mathcal{G}(\mathcal{R}, u^*_{(1,l+1)}(x_{(1,1)}) - K(A + BK)^l x_{(1,1)}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\mathcal{R}, \hat{v}_{(0,N-1)}(y(x, x_{(1,1)}))) &= \mathcal{G}(\mathcal{R}, K(\tilde{e}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^{N-1} x_{(1,1)})) \\ &\leq \mathcal{G}(\mathcal{R}, K\tilde{e}^*_{(0,N)}(x)) + \mathcal{G}(\mathcal{R}, K(x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^l x_{(1,1)})).\end{aligned}$$

Similarly, by subadditivity of the gauge (Minkowski) function $\mathcal{G}(\mathcal{P}, \cdot)$ we have that:

$$\begin{aligned}\mathcal{G}(\mathcal{P}, \hat{e}_{(0,N)}(y(x, x_{(1,1)}))) &= \mathcal{G}(\mathcal{P}, (A + BK)(\tilde{e}^*_{(0,N)}(x) + x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^{N-1} x_{(1,1)})) \\ &\leq \mathcal{G}(\mathcal{P}, (A + BK)\tilde{e}^*_{(0,N)}(x)) + \mathcal{G}(\mathcal{P}, (A + BK)(x^*_{(1,N)}(x_{(1,1)}) - (A + BK)^{N-1} x_{(1,1)})).\end{aligned}$$

Hence, it follows that:

$$
\begin{aligned}
V_{(0,N)}(\hat{\mathbf{e}}_0, \hat{\mathbf{v}}_0) \leq & -(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,0)}(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,0)}(x))) + \sum_{l=0}^{N-1} \Big(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,l)}(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,l)}(x))\Big) + \mathcal{G}(\mathcal{P}, \tilde{e}^*_{(0,N)}(x)) \\
& - \mathcal{G}(\mathcal{P}, \tilde{e}^*_{(0,N)}(x)) + \Big(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,N)}(x)) + \mathcal{G}(\mathcal{R}, K\tilde{e}^*_{(0,N)}(x))\Big) + \mathcal{G}(\mathcal{P}, (A+BK)\tilde{e}^*_{(0,N)}(x)) \\
& + \sum_{l=1}^{N-1} \Big(\mathcal{G}(\mathcal{Q}, x^*_{(1,l)}(x_{(1,1)}) - (A+BK)^{l-1}x_{(1,1)}) + \mathcal{G}(\mathcal{R}, u^*_{(1,l)}(x_{(1,1)}) - K(A+BK)^{l-1}x_{(1,1)})\Big) \\
& + \mathcal{G}(\mathcal{P}, (x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)})) - \mathcal{G}(\mathcal{P}, (x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)})) \\
& + \Big(\mathcal{G}(\mathcal{Q}, x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)}) + \mathcal{G}(\mathcal{R}, K(x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)}))\Big) \\
& + \mathcal{G}(\mathcal{P}, (A+BK)(x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)}))
\end{aligned}
$$

By Assumption 3 we have that:

$$
-\mathcal{G}(\mathcal{P}, \tilde{e}^*_{(0,N)}(x)) + \Big(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,N)}(x)) + \mathcal{G}(\mathcal{R}, K\tilde{e}^*_{(0,N)}(x))\Big) + \mathcal{G}(\mathcal{P}, (A+BK)\tilde{e}^*_{(0,N)}(x)) \leq 0
$$

and

$$
\begin{aligned}
& \mathcal{G}(\mathcal{P}, (A+BK)(x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)})) - \mathcal{G}(\mathcal{P}, (x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)})) \\
& + \Big(\mathcal{G}(\mathcal{Q}, x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)}) + \mathcal{G}(\mathcal{R}, K(x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)}))\Big) \\
& \leq 0.
\end{aligned}
$$

Hence, it follows that:

$$
V_{(0,N)}(\hat{\mathbf{e}}_0, \hat{\mathbf{v}}_0) \leq -(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,0)}(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,0)}(x))) + V_{(0,N)}(\tilde{\mathbf{e}}^*_0(x), \tilde{\mathbf{v}}^*_0(x)) + V'_{(1,N)}(\mathbf{x}^*_{(1,N)}, \mathbf{u}^*_{(1,N-1)}),
$$

where $\mathbf{x}^*_{(1,N)} := \{x^*_{(1,l)}(x_{(1,1)})\}_{(l \in \mathbb{N}_{[1:N]})}$, $\mathbf{u}^*_{(1,N-1)} := \{u^*_{(1,l)}(x_{(1,1)})\}_{(l \in \mathbb{N}_{[1:N-1]})}$, and

$$
\begin{aligned}
V'_{(1,N)}(\mathbf{x}^*_{(1,N)}, \mathbf{u}^*_{(1,N-1)}) = & \sum_{l=1}^{N-1} \Big(\mathcal{G}(\mathcal{Q}, x^*_{(1,l)}(x_{(1,1)}) - (A+BK)^{l-1}x_{(1,1)}) + \mathcal{G}(\mathcal{R}, u^*_{(1,l)}(x_{(1,1)}) - K(A+BK)^{l-1}x_{(1,1)})\Big) \\
& + \mathcal{G}(\mathcal{P}, (x^*_{(1,N)}(x_{(1,1)}) - (A+BK)^{N-1}x_{(1,1)})).
\end{aligned}
$$

By the convexity of the gauge functions $\mathcal{G}(\mathcal{Q}, \cdot)$, $\mathcal{G}(\mathcal{R}, \cdot)$ and $\mathcal{G}(\mathcal{P}, \cdot)$ it follows that the function $V_{(1,N)}(\cdot)$ specified in (5.7) is convex and hence, in view of (5.7), (5.8) and (5.16), it follows that:

$$
V'_{(1,N)}(\mathbf{x}^*_{(1,N)}, \mathbf{u}^*_{(1,N-1)}) \leq V_{(1,N)}(\mathbf{X}^*_{(1,N)}(x), \mathbf{U}^*_{(1,N-1)}(x)).
$$

Therefore we have that:

$$
V_{(0,N)}(\hat{\mathbf{e}}_0, \hat{\mathbf{v}}_0) \leq -(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,0)}(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,0)}(x))) + V_{(0,N)}(\tilde{\mathbf{e}}^*_0(x), \tilde{\mathbf{v}}^*_0(x)) + V_{(1,N)}(\mathbf{X}^*_{(1,N)}(x), \mathbf{U}^*_{(1,N-1)}(x)).
$$

Now, for all $k \in \mathbb{N}_{[1:N-1]}$, we have by construction and a direct application of the standard argument [31] that:

$$
V_{(k,N)}(\hat{\mathbf{X}}_{(k,N)}, \hat{\mathbf{U}}_{(k,N-1)}) \leq V_{(k+1,N)}(\mathbf{X}^*_{(k+1,N)}(x), \mathbf{U}^*_{(k+1,N-1)}(x)),
$$

and, by construction,

$$
V_{(N,N)}(\hat{X}_{(N,N)}) = 0.
$$

Hence, it follows that:

$$
\begin{aligned}
V_{(0,N)}(\hat{\mathbf{e}}_0, \hat{\mathbf{v}}_0) + \sum_{k=1}^{N-1} V_{(k,N)}(\hat{\mathbf{X}}_{(k,N)}, \hat{\mathbf{U}}_{(k,N-1)}) + V_{(N,N)}(\hat{\mathbf{X}}_{(N,N)}) \leq -(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,0)}(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,0)}(x))) \\
+ V_{(0,N)}(\tilde{\mathbf{e}}^*_0(x), \tilde{\mathbf{v}}^*_0(x)) + \sum_{k=1}^{N-1} V_{(k,N)}(\mathbf{X}^*_{(k,N)}(x), \mathbf{U}^*_{(k,N-1)}(x)) + V_{(N,N)}(\mathbf{X}^*_{(N,N)}(x)).
\end{aligned}
$$

Hence, we can now conclude that for all $x \in \mathcal{X}_N$ and all $y(x, x_{(1,1)}) := Ax + B(K(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,0)}(x)) + x_{(1,1)}$, $x_{(1,1)} \in X^*_{(1,1)}(x) = \mathbb{W}$ it holds that:

$$
V_N(\hat{\theta}_N(y(x, x_{(1,1)}))) - V^0_N(x) \leq -\Big(\mathcal{G}(\mathcal{Q}, \tilde{e}^*_{(0,0)}(x)) + \mathcal{G}(\mathcal{R}, \tilde{v}^*_{(0,0)}(x))\Big)
$$

But, by optimality of $V_N^0(\cdot)$, $V_N^0(y(x,x_{(1,1)})) \leq V_N(\hat{\theta}_N(y(x,x_{(1,1)})))$ and hence it follows that for all $x \in \mathcal{X}_N$ and all $y(x,x_{(1,1)}) := Ax + B(K(x - \tilde{e}^*_{(0,0)}(x)) + \tilde{v}^*_{(0,0)}(x)) + x_{(1,1)},\ x_{(1,1)} \in X^*_{(1,1)}(x) = \mathbb{W}$ it holds that:

$$V_N^0(y(x,x_{(1,1)})) - V_N^0(x) \leq V_N(\hat{\theta}_N(y(x,x_{(1,1)}))) - V_N^0(x) \leq -\left(\mathcal{G}(\mathcal{Q},\tilde{e}^*_{(0,0)}(x)) + \mathcal{G}(\mathcal{R},\tilde{v}^*_{(0,0)}(x))\right),$$

as claimed in (5.19c). The proof is completed.

A–9: Proof of Lemma 1

(i) The facts that for all $x \in \mathbb{X}_f$ it holds that $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} = 0$ and $|\tilde{v}^*_{(0,0)}(x)|_{\mathcal{R}} = 0$ follows from the fact (iii) established in Proposition 6. Since, by (5.3c), $x - \tilde{e}^*_{(0,0)}(x) \in \mathbb{X}_f$ it follows that for all $x \in \mathcal{X}_N \backslash \mathbb{X}_f$ it holds that $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} > 0$. (ii) We note that $\operatorname{dist}(\mathcal{Q},x,\mathbb{X}_f) = \operatorname{dist}(\mathcal{Q},x-e+e,\mathbb{X}_f)$ and hence, since $x - \tilde{e}^*_{(0,0)}(x) \in \mathbb{X}_f$, we have that, for all $x \in \mathcal{X}_N$, $\operatorname{dist}(\mathcal{Q},x,\mathbb{X}_f) = \operatorname{dist}(\mathcal{Q},\tilde{e}^*_{(0,0)}(x) + (x - \tilde{e}^*_{(0,0)}(x)),\mathbb{X}_f) \leq |\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}}$. Since for all $x \in \mathbb{X}_f$ it holds that $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} = 0$ and $\operatorname{dist}(\mathcal{Q},x,\mathbb{X}_f) = 0$ as well as for all $x \in \mathcal{X}_N \backslash \mathbb{X}_f$ it holds that $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} > 0$ and $\operatorname{dist}(\mathcal{Q},x,\mathbb{X}_f) > 0$, the continuity of the functions $|\cdot|_{\mathcal{Q}}$, $\tilde{e}^*_{(0,0)}(\cdot)$ as well as $\operatorname{dist}(\mathcal{Q},\cdot,\mathbb{X}_f)$ together with the compactness of the sets $\mathbb{X}_f$ and $\mathcal{X}_N$ guarantee the existence of a scalar $c_1 \in (0,1)$ such that $c_1|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} \leq \operatorname{dist}(\mathcal{Q},x,\mathbb{X}_f)$. Similarly, we have that, for all $x \in \mathcal{X}_N$, $0 \leq \operatorname{dist}(\mathcal{R},\kappa^*_N(x),K\mathbb{X}_f) = \operatorname{dist}(\mathcal{R},\tilde{v}^*_{(0,0)}(x) + K(x - \tilde{e}^*_{(0,0)}(x)),K\mathbb{X}_f)$. But $x - \tilde{e}^*_{(0,0)} \in \mathbb{X}_f$ and so $K(x - \tilde{e}^*_{(0,0)}) \in K\mathbb{X}_f$. Hence, as claimed, for all $x \in \mathcal{X}_N$ it holds that $c_1|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} \leq \operatorname{dist}(\mathcal{Q},x,\mathbb{X}_f) \leq |\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}}$ for some $c_1 \in (0,1)$ and $0 \leq \operatorname{dist}(\mathcal{R},\kappa^*_N(x),K\mathbb{X}_f) \leq |\tilde{v}^*_{(0,0)}(x)|_{\mathcal{R}}$.

A–10: Proof of Proposition 8

By construction we have that, for all $x \in \mathcal{X}_N$ it holds that $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} \leq V_N^0(x)$ and $(|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} + |\tilde{v}^*_{(0,0)}(x)|_{\mathcal{R}}) \leq V_N^0(x)$. By Proposition 6 and Lemma 1, the functions $V_N^0(\cdot)$ and $\tilde{e}^*_{(0,0)}(\cdot)$ are continuous functions satisfying that for all $x \in \mathbb{X}_f$ it holds that $V_N^0(x) = 0$, and $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} = 0$ as well as that for all $x \in \mathcal{X}_N \setminus \mathbb{X}_f$ it holds that $V_N^0(x) > 0$ and $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} > 0$. Hence, since $|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} \leq V_N^0(x)$, the continuity of the functions $V_N^0(\cdot)$, $|\cdot|_{\mathcal{Q}}$ and $\tilde{e}^*_{(0,0)}(\cdot)$ and the compactness of the sets $\mathbb{X}_f$ and $\mathcal{X}_N$ guarantee the existence of a scalar $c_2 \in [1,\infty)$ such that, for all $x \in \mathcal{X}_N$ it holds that $V_N^0(x) \leq c_2|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}}$ and, in turn, $V_N^0(x) \leq c_2(|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} + |\tilde{v}^*_{(0,0)}(x)|_{\mathcal{R}})$. By Proposition 7 we have that for all $x \in \mathcal{X}_N$ and all $x^+ \in \mathcal{F}(x)$ it holds that $V_N^0(x^+) - V_N^0(x) \leq -(|\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} + |\tilde{v}^*_{(0,0)}(x)|_{\mathcal{R}})$. Consequently, the claimed facts are affirmative.

A–11: Proof of Corollary 2

The claimed facts are direct consequence of the inequalities established in Proposition 8. In particular, the scalar pair $(\bar{a}_N,\bar{b}_N) \in [0,1) \times (0,\infty)$ can be taken as $\bar{a}_N = \frac{c_2-1}{c_2}$ and $\bar{b}_N = c_2$ where $c_2 \in [1,\infty)$ is the scalar appearing in the statement of Proposition 8.

A–12: Proof of Theorem 1

(i) By construction and (6.1), it follows that for all $x \in \mathcal{X}_N$ it holds that $\kappa^*_N(x) \in \mathbb{U}$. Since for all $x \in \mathcal{X}_N$ it holds that $\mathcal{F}(x) = X^*_1(x)$ and since by Proposition 7 it holds that $X^*_1(x) \subseteq \mathcal{X}_N$ it follows that for all $x \in \mathcal{X}_N$ we have that $\mathcal{F}(x) \subseteq \mathcal{X}_N$. Hence, since $\mathcal{X}_N \subseteq \mathbb{X}$ and due to the definition of the map $\mathcal{F}(\cdot)$ in (6.2), it follows that the N–step parameterized tubes controllability set $\mathcal{X}_N$ is a robust positively invariant set for the system $x^+ = Ax + B\kappa^*_N(x) + w$ and the constraint set $(\mathbb{X}_{\kappa^*_N},\mathbb{W})$ where $\mathbb{X}_{\kappa^*_N} := \{x \in \mathbb{X}\ :\ \kappa^*_N(x) \in \mathbb{U}\}$. (ii) By (i) it follows that the relationships:

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q},x_k,\mathcal{X}_N) = 0 \text{ and } \operatorname{dist}(\mathcal{R},\kappa^*_N(x_k),\mathbb{U}) = 0$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2). By Corollary 2 we have that the inequalities:

$$|\tilde{e}^*_{(0,0)}(x_k)|_{\mathcal{Q}} \leq \bar{a}_N^k\bar{b}_N|\tilde{e}^*_{(0,0)}(x_0)|_{\mathcal{Q}}$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2) where $(\bar{a}_N,\bar{b}_N) \in [0,1) \times (0,\infty)$ is the scalar pair appearing in the statement of Corollary 2. In view of Lemma 1 it follows that the inequalities:

$$\operatorname{dist}(\mathcal{Q},x_k,\mathbb{X}_f) \leq \bar{a}_N^k\bar{b}_N\frac{1}{c_1}\operatorname{dist}(\mathcal{Q},x_0,\mathbb{X}_f)$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2) where $c_1 \in (0,1)$ is the scalar appearing in the statement of Lemma 1. Similarly, by Lemma 1 we have that the inequalities

$$\operatorname{dist}(\mathcal{R},\kappa^*_N(x_k),K\mathbb{X}_f) \leq |\tilde{v}^*_{(0,0)}(x_k)|_{\mathcal{R}} \leq |\tilde{v}^*_{(0,0)}(x_k)|_{\mathcal{R}} + |\tilde{e}^*_{(0,0)}(x_k)|_{\mathcal{Q}} \leq \bar{a}_N^k\bar{b}_N(|\tilde{v}^*_{(0,0)}(x_0)|_{\mathcal{R}} + |\tilde{e}^*_{(0,0)}(x_0)|_{\mathcal{Q}})$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2). Let:

$$c_3 := \max_x \{|\tilde{v}^*_{(0,0)}(x)|_{\mathcal{R}} + |\tilde{e}^*_{(0,0)}(x)|_{\mathcal{Q}} \;:\; x \in \mathcal{X}_N\}$$

and note that $c_3 \in (0,\infty)$ since $\tilde{v}^*_{(0,0)}(\cdot)$ and $\tilde{e}^*_{(0,0)}(\cdot)$ are continuous and the set $\mathcal{X}_N$ is compact. Hence, the inequalities:

$$\operatorname{dist}(\mathcal{R}, \kappa^*_N(x_k), K\mathbb{X}_f) \le \bar{a}^k_N \bar{b}_N c_3$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2). Define:

$$a_N = \bar{a}_N \text{ and } b_N := \max\{\bar{b}_N \frac{1}{c_1},\ \bar{b}_N c_3\}$$

and note that $(a_N, b_N) \in [0,1) \times (0,\infty)$. Finally, we have that the inequalities:

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q}, x_k, \mathbb{X}_f) \le a^k_N b_N \operatorname{dist}(\mathcal{Q}, x_0, \mathbb{X}_f) \text{ and } \operatorname{dist}(\mathcal{R}, \kappa^*_N(x_k), K\mathbb{X}_f) \le a^k_N b_N$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2). Our second claim is verified. (iii) In view of Definition 2, our third claim follows from the facts established in (ii).

A–13: Proof of Corollary 3

(i) We first note that KX_∞, $K\mathbb{X}_f$, X_∞ and $\mathbb{X}_f$ are compact sets such that $X_\infty \subseteq \mathbb{X}_f$ and $KX_\infty \subseteq K\mathbb{X}_f$. Hence, since the functions $\kappa^*_N(\cdot)$, $\operatorname{dist}(\mathcal{Q}, \cdot, X_\infty)$, $\operatorname{dist}(\mathcal{R}, \cdot, KX_\infty)$, $\operatorname{dist}(\mathcal{Q}, \cdot, \mathbb{X}_f)$ and $\operatorname{dist}(\mathcal{R}, \cdot, K\mathbb{X}_f)$ are continuous and the sets $\mathbb{X}_f$ and $\mathcal{X}_N$ are compact there exists a scalar $c_4 \in [1,\infty)$ such that for all $x \in \mathcal{X}_N \setminus \mathbb{X}_f$ it holds that:

$$\begin{aligned}
&\operatorname{dist}(\mathcal{Q}, x, \mathbb{X}_f) \le \operatorname{dist}(\mathcal{Q}, x, X_\infty) \le c_4 \operatorname{dist}(\mathcal{Q}, x, \mathbb{X}_f), \text{ and,}\\
&\operatorname{dist}(\mathcal{R}, \kappa^*_N(x_k), K\mathbb{X}_f) \le \operatorname{dist}(\mathcal{R}, \kappa^*_N(x), KX_\infty) \le c_4 \operatorname{dist}(\mathcal{R}, \kappa^*_N(x), K\mathbb{X}_f).
\end{aligned}$$

Hence, in view of the inequalities established in Theorem 1 (ii), it follows that the inequalities:

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q}, x_k, X_\infty) \le a^k_N b_N c_4 \operatorname{dist}(\mathcal{Q}, x_0, X_\infty) \text{ and } \operatorname{dist}(\mathcal{R}, \kappa^*_N(x_k), KX_\infty) \le a^k_N b_N c_4$$

hold true for any $x_0 \in \mathcal{X}_N \setminus \mathbb{X}_f$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2) where the scalar pair $(a_N, b_N) \in [0,1) \times (0,\infty)$ is the scalar pair appearing in the statement of Theorem 1. We recall that for all $x \in \mathbb{X}_f$ the PTMPC control law $\kappa^*_N(\cdot)$ and the corresponding controlled uncertain dynamics take the form given by (6.4), i.e. for all $x \in \mathbb{X}_f$ it holds that $\kappa^*_N(x) = Kx$ and $\mathcal{F}(x) = (A+BK)x \oplus \mathbb{W}$. With this in mind, the well–known results [35, 38, 39] imply that the inequalities:

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q}, x_k, X_\infty) \le \hat{a}^k_N \hat{b}_N \operatorname{dist}(\mathcal{Q}, x_0, X_\infty) \text{ and } \operatorname{dist}(\mathcal{R}, Kx_k, KX_\infty) \le \hat{a}^k_N \hat{b}_N$$

hold true for any $x_0 \in \mathbb{X}_f$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2) for some scalar pair $(\hat{a}_N, \hat{b}_N) \in [0,1) \times (0,\infty)$. Let:

$$\tilde{a}_N = \max\{a_N, \hat{a}_N\} \text{ and } \tilde{b}_N = \max\{b_N c_4, \hat{b}_N\}$$

and note that $(\tilde{a}_N, \tilde{b}_N) \in [0,1) \times (0,\infty)$. Finally, it follows that:

$$\forall k \in \mathbb{N},\ \operatorname{dist}(\mathcal{Q}, x_k, X_\infty) \le \tilde{a}^k_N \tilde{b}_N \operatorname{dist}(\mathcal{Q}, x_0, X_\infty) \text{ and } \operatorname{dist}(\mathcal{R}, \kappa^*_N(x_k), KX_\infty) \le \tilde{a}^k_N \tilde{b}_N$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (6.2). (ii) In view of Definition 2, our second claim follows from the facts established in (i).

APPENDIX B: Appropriate Vectorization & Detailed Form of the Set Γ_N

B–1: Appropriate Vectorization

For any $N \in \mathbb{N}_+$, let $\mathbf{e}_0 := (\tilde{e}^T_{(0,0)}, \tilde{e}^T_{(0,1)}, \ldots, \tilde{e}^T_{(0,N)})^T \in \mathbb{R}^{(N+1)n}$, and, for all $k \in \mathbb{N}_{[1:N]}$ and all $i \in \mathbb{N}_{[1:q]}$, let $\mathbf{x}_{(i,k)} := (\tilde{x}^T_{(i,k,k)}, \tilde{x}^T_{(i,k,k+1)}, \ldots, \tilde{x}^T_{(i,k,N)})^T \in \mathbb{R}^{(N+1-k)n}$ and $\mathbf{x}_k := (\mathbf{x}^T_{(1,k)}, \mathbf{x}^T_{(2,k)}, \ldots, \mathbf{x}^T_{(q,k)})^T \in \mathbb{R}^{q(N+1-k)n}$. With this in mind, the parameterized state tube $\mathbf{X}_N$ is vectorized via the variable $\mathbf{d}_{(N,\mathbf{X})}$ specified by:

$$\mathbf{d}_{(N,\mathbf{X})} := (\mathbf{e}^T_0, \mathbf{x}^T_1, \mathbf{x}^T_2, \ldots, \mathbf{x}^T_N)^T \in \mathbb{R}^{N_1},\ N_1 := (1 + \frac{qN}{2})(N+1)n.$$

For any $N \in \mathbb{N}_+$, let $\mathbf{v}_0 := (\tilde{v}_{(0,0)}^T, \tilde{v}_{(0,1)}^T, \ldots, \tilde{v}_{(0,N-1)}^T)^T \in \mathbb{R}^{Nm}$, and, for all $k \in \mathbb{N}_{[1:N-1]}$ and all $i \in \mathbb{N}_{[1:q]}$, let $\mathbf{u}_{(i,k)} := (\tilde{u}_{(i,k,k)}^T, \tilde{u}_{(i,k,k+1)}^T, \ldots, \tilde{u}_{(i,k,N-1)}^T)^T \in \mathbb{R}^{(N-k)m}$ and $\mathbf{u}_k := (\mathbf{u}_{(1,k)}^T, \mathbf{u}_{(2,k)}^T, \ldots, \mathbf{u}_{(q,k)}^T)^T \in \mathbb{R}^{q(N-k)m}$. As above, the parameterized control tube $\mathbf{U}_{N-1}$ is vectorized via the variable $\mathbf{d}_{(N,\mathbf{U})}$ specified by:

$$\mathbf{d}_{(N,\mathbf{U})} := (\mathbf{v}_0^T, \mathbf{u}_1^T, \mathbf{u}_2^T, \ldots, \mathbf{u}_{N-1}^T)^T \in \mathbb{R}^{N_2},\ N_2 := (1 + \frac{q(N-1)}{2})Nm.$$

For any $N \in \mathbb{N}_+$ and for all $k \in \mathbb{N}_{[1:N-1]}$ and all $l \in \mathbb{N}_{[1:p]}$ let $f_{(l,k)} := (f_{(l,1,k)}, f_{(l,2,k)}, \ldots, f_{(l,k,k)})^T \in \mathbb{R}^k$ and $f_k := (f_{(1,k)}^T, f_{(2,k)}^T, \ldots, f_{(p,k)}^T)^T \in \mathbb{R}^{pk}$. The variable $\mathbf{d}_{(N,\mathbf{f})}$ given by:

$$\mathbf{d}_{(N,\mathbf{f})} := (f_1^T, f_2^T, \ldots, f_{N-1}^T)^T \in \mathbb{R}^{N_3},\ N_3 := \frac{N(N-1)}{2}p,$$

represents the slack variables associated with the reformulation of the set inclusion constraints $\forall k \in \mathbb{N}_{N-1},\ X_k \subseteq \mathbb{X}$ as outlined in (7.2). For any $N \in \mathbb{N}_+$ and for all $k \in \mathbb{N}_{[1:N-1]}$ and all $l \in \mathbb{N}_{[1:r]}$ let $g_{(l,k)} := (g_{(l,1,k)}, g_{(l,2,k)}, \ldots, g_{(l,k,k)})^T \in \mathbb{R}^k$ and $g_k := (g_{(1,k)}^T, g_{(2,k)}^T, \ldots, g_{(r,k)}^T)^T \in \mathbb{R}^{rk}$. As above, the variable $\mathbf{d}_{(N,\mathbf{g})}$ given by:

$$\mathbf{d}_{(N,\mathbf{g})} := (g_1^T, g_2^T, \ldots, g_{N-1}^T)^T \in \mathbb{R}^{N_4},\ N_4 := \frac{N(N-1)}{2}r,$$

represents the slack variables associated with the reformulation of the set inclusion constraints $\forall k \in \mathbb{N}_{N-1},\ U_k \subseteq \mathbb{U}$ as outlined in (7.1). For any $N \in \mathbb{N}_+$ and all $l \in \mathbb{N}_{[1:t]}$ let $h_{(l,N)} := (h_{(l,1,N)}, h_{(l,2,N)}, \ldots, h_{(l,N,N)})^T \in \mathbb{R}^N$ and $h_N := (h_{(1,N)}^T, h_{(2,N)}^T, \ldots, h_{(t,N)}^T)^T \in \mathbb{R}^{tN}$. As above, the variable $\mathbf{d}_{(N,\mathbf{h})}$ given by:

$$\mathbf{d}_{(N,\mathbf{h})} := h_N \in \mathbb{R}^{N_5},\ N_5 := Nt,$$

represents the slack variables associated with the reformulation of the set inclusion constraints $X_N \subseteq \mathbb{X}_f$ as outlined in (7.3). For any $N \in \mathbb{N}_+$, let $\alpha_0 := (\tilde{\alpha}_{(0,0)}, \tilde{\alpha}_{(0,1)}, \ldots, \tilde{\alpha}_{(0,N)})^T \in \mathbb{R}^{(N+1)}$, and, for all $k \in \mathbb{N}_{[1:N]}$ and all $i \in \mathbb{N}_{[1:q]}$, let $\alpha_{(i,k)} := (\tilde{\alpha}_{(i,k,k)}, \tilde{\alpha}_{(i,k,k+1)}, \ldots, \tilde{\alpha}_{(i,k,N)})^T \in \mathbb{R}^{(N+1-k)}$ and $\alpha_k := (\alpha_{(1,k)}^T, \alpha_{(2,k)}^T, \ldots, \alpha_{(q,k)}^T)^T \in \mathbb{R}^{q(N+1-k)}$. With this in mind, the variable $\mathbf{d}_{(N,\alpha)}$ specified by:

$$\mathbf{d}_{(N,\alpha)} := (\alpha_0^T, \alpha_1^T, \alpha_2^T, \ldots, \alpha_N^T)^T \in \mathbb{R}^{N_6},\ N_6 := (1 + \frac{qN}{2})(N+1),$$

represents the state tube cost function related slack variables. For any $N \in \mathbb{N}_+$, let $\beta_0 := (\tilde{\beta}_{(0,0)}, \tilde{\beta}_{(0,1)}, \ldots, \tilde{\beta}_{(0,N-1)})^T \in \mathbb{R}^N$, and, for all $k \in \mathbb{N}_{[1:N-1]}$ and all $i \in \mathbb{N}_{[1:q]}$, let $\beta_{(i,k)} := (\tilde{\beta}_{(i,k,k)}, \tilde{\beta}_{(i,k,k+1)}, \ldots, \tilde{\beta}_{(i,k,N-1)})^T \in \mathbb{R}^{(N-k)}$ and $\beta_k := (\beta_{(1,k)}^T, \beta_{(2,k)}^T, \ldots, \beta_{(q,k)}^T)^T \in \mathbb{R}^{q(N-k)}$. As above, the variable $\mathbf{d}_{(N,\beta)}$ specified by:

$$\mathbf{d}_{(N,\beta)} := (\beta_0^T, \beta_1^T, \beta_2^T, \ldots, \beta_{N-1}^T)^T \in \mathbb{R}^{N_7},\ N_7 := (1 + \frac{q(N-1)}{2})N,$$

represents the control tube cost function related slack variables. Finally, for any $N \in \mathbb{N}_+$, we consider the decision variable $\mathbf{d}_N$ specified by:

$$\begin{aligned}
\mathbf{d}_N &:= (\mathbf{d}_{(N,\mathbf{X})}^T, \mathbf{d}_{(N,\mathbf{U})}^T, \mathbf{d}_{(N,\mathbf{f})}^T, \mathbf{d}_{(N,\mathbf{g})}^T, \mathbf{d}_{(N,\mathbf{h})}^T, \mathbf{d}_{(N,\alpha)}^T, \mathbf{d}_{(N,\beta)}^T)^T \in \mathbb{R}^{N_{tot}},\ \text{where,}\\
N_{tot} &:= \frac{1}{2}[q(n+m+2)+p+r]N^2 + \frac{1}{2}[(2+q)n + (2-q)m + 2t - p - r + 4]N + n + 1.
\end{aligned}$$

The introduced decision variable $\mathbf{d}_N$ is utilized to provide an equivalent and computationally tractable reformulation of the graph of the set of admissible pairs of the parameterized state and control tubes as outlined next.

B–2: Detailed Form of the Set Γ_N

The dynamics of the parameterized state and control tubes leads to a set of equality constraints according to the relationships (5.2), (3.3b) and (3.4b) as specified by the set $\Gamma_{(N,EC)}$:

$$\begin{aligned}
\Gamma_{(N,EC)} := \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ & \forall k \in \mathbb{N}_{N-1},\ \tilde{e}_{(0,k+1)} = A\tilde{e}_{(0,k)} + B\tilde{v}_{(0,k)},\\
& \forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N]},\ \tilde{x}_{(i,k,k)} = \tilde{w}_i,\\
& \forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ \forall l \in \mathbb{N}_{[k:N-1]},\ \tilde{x}_{(i,k,l+1)} = A\tilde{x}_{(i,k,l)} + B\tilde{u}_{(i,k,l)}\}.
\end{aligned}$$

Evidently, the total number of equality constraints N_{eq} is:

$$N_{eq} := \frac{1}{2}qnN^2 + (1 + \frac{1}{2}q)nN.$$

The set inclusions $\forall k \in \mathbb{N}_{N-1},\ X_K \subseteq \mathbb{X}$ specified in (5.3a) lead to the constraint set $\Gamma_{(N,SC)}$ given by:

$$\begin{aligned}\Gamma_{(N,SC)} := \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ & \forall l \in \mathbb{N}_{[1:p]},\ F_l^T x \leq 1,\\ & \forall l \in \mathbb{N}_{[1:p]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ \forall j \in \mathbb{N}_{[1:k]},\ F_l^T \tilde{x}_{(i,j,k)} \leq f_{(l,j,k)},\\ & \forall l \in \mathbb{N}_{[1:p]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ F_l^T((A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{e}_{(0,k)}) + \sum_{j=1}^{k} f_{(l,j,k)} \leq 1\}.\end{aligned}$$

The total number of state constraints related inequalities N_{sineq} is:

$$N_{sineq} := \frac{1}{2}qpN^2 + (1 - \frac{1}{2}q)pN.$$

The set inclusion $X_N \subseteq \mathbb{X}_f$ specified in (5.3a) leads to the constraint set $\Gamma_{(N,TC)}$ given by:

$$\begin{aligned}\Gamma_{(N,TC)} := \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ & \forall l \in \mathbb{N}_{[1:t]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall j \in \mathbb{N}_{[1:N]},\ H_l^T \tilde{x}_{(i,j,N)} \leq h_{(l,j,N)},\\ & \forall l \in \mathbb{N}_{[1:t]},\ H_l^T((A+BK)^N(x - \tilde{e}_{(0,0)}) + \tilde{e}_{(0,N)}) + \sum_{j=1}^{N} h_{(l,j,N)} \leq 1\}.\end{aligned}$$

The total number of terminal state constraints related inequalities N_{tineq} is:

$$N_{tineq} := tqN + t.$$

The set inclusions $\forall k \in \mathbb{N}_{N-1},\ U_K \subseteq \mathbb{U}$ specified in (5.3b) lead to the constraint set $\Gamma_{(N,CC)}$ given by:

$$\begin{aligned}\Gamma_{(N,CC)} := \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ & \forall l \in \mathbb{N}_{[1:r]},\ G_l^T(K(x - \tilde{e}_{(0,0)}) + \tilde{v}_{(0,0)}) \leq 1,\\ & \forall l \in \mathbb{N}_{[1:r]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ \forall j \in \mathbb{N}_{[1:k]},\ G_l^T \tilde{u}_{(i,j,k)} \leq g_{(l,j,k)},\\ & \forall l \in \mathbb{N}_{[1:r]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ G_l^T(K(A+BK)^k(x - \tilde{e}_{(0,0)}) + \tilde{v}_{(0,k)}) + \sum_{j=1}^{k} g_{(l,j,k)} \leq 1\}.\end{aligned}$$

The total number of control constraints related inequalities N_{cineq} is:

$$N_{cineq} := \frac{1}{2}qrN^2 + (1 - \frac{1}{2}q)rN.$$

The stabilizing constraint specified in (5.3c) leads to the constraint set $\Gamma_{(N,IC)}$ given by:

$$\Gamma_{(N,IC)} := \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ \forall l \in \mathbb{N}_{[1:t]},\ H_l^T(x - \tilde{e}_{(0,0)}) \leq 1\}.$$

The total number of stabilizing constraints related inequalities N_{iineq} is:

$$N_{iineq} := t.$$

Finally, the minimization of the cost function $V_N(\cdot)$ specified via the relationships (5.5)–(5.10) is, as is customary, achieved by introducing appropriate constraints on slack variables $\mathbf{d}_{(N,\alpha)}$ and $\mathbf{d}_{(N,\beta)}$ and minimizing their sum. The corresponding constraints lead to the constraint set $\Gamma_{(N,VC)}$ given by:

$$\begin{aligned}\Gamma_{(N,VC)} := \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ :\ & \forall k \in \mathbb{N}_{N-1},\ \forall l \in \mathbb{N}_{[1:s_1]},\ Q_l^T \tilde{e}_{(0,k)} \leq \tilde{\alpha}_{(0,k)},\\ & \forall k \in \mathbb{N}_{N-1},\ \forall l \in \mathbb{N}_{[1:s_2]},\ R_l^T \tilde{v}_{(0,k)} \leq \tilde{\beta}_{(0,k)},\\ & \forall l \in \mathbb{N}_{[1:s_3]},\ P_l^T \tilde{e}_{(0,N)} \leq \tilde{\alpha}_{(0,N)},\\ & \forall k \in \mathbb{N}_{[1:N-1]},\ \forall j \in \mathbb{N}_{[k:N-1]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall l \in \mathbb{N}_{[1:s_1]},\ Q_l^T(\tilde{x}_{(i,k,j)} - (A+BK)^{j-k}\tilde{w}_i) \leq \tilde{\alpha}_{(i,k,j)},\\ & \forall k \in \mathbb{N}_{[1:N-1]},\ \forall j \in \mathbb{N}_{[k:N-1]},\ \forall i \in \mathbb{N}_{[1:q]},\ \forall l \in \mathbb{N}_{[1:s_2]},\ R_l^T(\tilde{u}_{(i,k,l)} - K(A+BK)^{j-k}\tilde{w}_i) \leq \tilde{\beta}_{(i,k,j)},\ \text{and},\\ & \forall k \in \mathbb{N}_{[1:N]},\ \forall l \in \mathbb{N}_{[1:s_3]},\ \forall i \in \mathbb{N}_{[1:q]},\ P_l^T(\tilde{x}_{(i,k,N)} - (A+BK)^{N-k}\tilde{w}_i) \leq \tilde{\alpha}_{(i,k,N)}\}.\end{aligned}$$

The total number of cost function reformulation related constraints results in N_{vineq} inequalities, where N_{vineq} is given by:

$$N_{vineq} := \frac{1}{2}q(s_1 + s_2)N^2 + [s_1 + s_2 + qs_3 - \frac{1}{2}q(s_1 + s_2)]N + s_3.$$

Thus the total number of inequalities describing the sets $\Gamma_{(N,SC)}$, $\Gamma_{(N,TC)}$, $\Gamma_{(N,CC)}$, $\Gamma_{(N,IC)}$ and $\Gamma_{(N,VC)}$ is upper bounded by $N_{ineq} = N_{sineq} + N_{tineq} + N_{cineq} + N_{iineq} + N_{vineq}$ (as some of inequalities in the description of these sets might be redundant) and it satisfies:

$$N_{ineq} = \frac{1}{2}q(p + r + s_1 + s_2)N^2 + [p + r + t + s_1 + s_2 + qs_3 - \frac{1}{2}(p + r + s_1 + s_2)]N + 2t + s_3.$$

Finally, the set Γ_N is given by:

$$\Gamma_N := \Gamma_{(N,EC)} \bigcap \Gamma_{(N,SC)} \bigcap \Gamma_{(N,TC)} \bigcap \Gamma_{(N,CC)} \bigcap \Gamma_{(N,IC)} \bigcap \Gamma_{(N,VC)},$$

and it is, evidently from its definition, a closed polyhedral set taking the form:

$$\Gamma_N = \{(x, \mathbf{d}_N) \in \mathbb{R}^{n+N_{tot}} \ : \ \mathcal{M}_{\text{xeq}} x + \mathcal{M}_{\text{deq}} \mathbf{d}_N = \mathcal{N}_{\text{eq}}, \ \mathcal{M}_{\text{xineq}} x + \mathcal{M}_{\text{dineq}} \mathbf{d}_N \leq \mathcal{N}_{\text{ineq}}\},$$

for suitable matrices $\mathcal{M}_{\text{xeq}} \in \mathbb{R}^{N_{eq} \times n}$, $\mathcal{M}_{\text{deq}} \in \mathbb{R}^{N_{eq} \times N_{tot}}$, $\mathcal{M}_{\text{xineq}} \in \mathbb{R}^{N_{ineq} \times n}$, $\mathcal{M}_{\text{dineq}} \in \mathbb{R}^{N_{ineq} \times N_{tot}}$ and vectors $\mathcal{N}_{\text{eq}} \in \mathbb{R}^{N_{eq}}$ and $\mathcal{N}_{\text{eq}} \in \mathbb{R}^{N_{ineq}}$ (all these matrices are easily constructed by utilizing the definition of the sets $\Gamma_{(N,EC)}$, $\Gamma_{(N,SC)}$, $\Gamma_{(N,TC)}$, $\Gamma_{(N,CC)}$, $\Gamma_{(N,IC)}$ and $\Gamma_{(N,VC)}$).

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