Robust and Stochastic Linear MPC for Systems Subject to Multiplicative Uncertainty

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Outline

- Problem statement
- Probabilistic constraints and recursive feasibility
- Polytopic invariant sets with probabilistic constraints
- Constraint handling using Farkas’ Lemma
- Receding horizon control
Problem description

- Linear system with multiplicative uncertainty:

\[ x(k + 1) = \left( A + \Delta(k) \right)x(k) + Bu(k) \]

  \[ x \in \mathbb{R}^{nx} \]

  \[ u \in \mathbb{R}^{nu} \]

- \( A, B \) known, \( \Delta(k) \) uncertain

  \[ \Delta(k) = \sum_{j=1}^{\rho} w_j(k) \Delta^{(j)}, \quad w_j(k) \geq 0, \quad \sum_{j=1}^{\rho} w_j(k) = 1 \]

  \[ \mathbb{E}\left( \Delta(k) \right) = 0 \]

  \[ w(k) = \left( w_1(k), \ldots, w_\rho(k) \right) \text{ stochastic with known, finitely supported distribution} \]

  \[ \{ w(0), w(1), \ldots \} \text{ independent & identically distributed} \]

- \( x(k) \) measured at time \( k \)

- Constraints: \( \Pr\{ Fx(k) + Gu(k) \leq 1 \} \geq p \) for all \( k = 0, 1, \ldots \)

  \[ \Pr\{ F_j x(k) + G_j u(k) \leq 1 \} \geq p_j, \quad p_j \in (0, 1], \quad j = 1, \ldots, n_C \]
Problem description

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Expectation MPC cost

Quadratic expected value cost:

\[ J(x(k), \{u(0|k), u(1|k), \ldots\}) = \sum_{i=0}^{\infty} \mathbb{E}(\|x(i|k)\|^2_Q + \|u(i|k)\|^2_R) \]

where \( x(i|k), u(i|k) \) are the values of \( x(k+i), u(k+i) \) predicted at time \( k \)

- Quasi-closed loop predictions with horizon \( N \):
  \[ u(i|k) = Kx(i|k) + c(i|k), \quad c(i|k) = \begin{cases} \text{decision variables,} & i = 0, \ldots, N - 1 \\ 0, & i \geq N \end{cases} \]

- Optimal unconstrained control law: \( u(k) = Kx(k) \)

- \( J(x(k), \{u(0|k), u(1|k), \ldots\}) = V(x(k), c(k)) \)

- \( V(x, c) \): quadratic in \( c = (c(0), \ldots, c(N-1)) \)

- Optimal value \( V^*(x) = \min_{c \in F_c(x)} V(x, c) \)

MPC law: \( \kappa_{\text{MPC}}(x) = Kx + c^*(0) \), where \( c^* = \arg \min_{c \in F_c(x)} V(x, c) \)
Expectation MPC cost

Quadratic expected value cost:

\[ J(x(k), \{u(0|k), u(1|k), \ldots\}) = \sum_{i=0}^{\infty} \mathbb{E}\left(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\right) \]

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  - optimal unconstrained control law: \( u(k) = Kx(k) \)
  - \( J(x(k), \{u(0|k), u(1|k), \ldots\}) = V(x(k), c(k)) \)
  - \( V(x, c) \): quadratic in \( c = (c(0), \ldots, c(N - 1)) \)

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  \text{decision variables}, & i = 0, \ldots, N - 1 \\
  0, & i \geq N 
\end{cases}
  \]

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  MPC law: \(\kappa_{\text{MPC}}(x) = Kx + c^*(0)\), where \(c^* = \arg \min_{c \in \mathcal{F}_c(x)} V(x, c)\)
Prediction model dynamics

Predicted state decomposition: $x(i|k) = z(i|k) + e(i|k)$

nominal: $z(i|k) = \mathbb{E}(x(i|k))$ \hspace{1cm} $z(0|k) = x(k)$

uncertain: $e(i|k) = x(i|k) - z(i|k)$ \hspace{1cm} $e(0|k) = 0$

Quasi-closed loop predictions: $u(i|k) = Kz(i|k) + Le(i|k) + c(i|k)$

$K$, $L$: unconstrained optimal feedback gains for $z$, $e$ dynamics

Prediction model and constraints:

$z(i + 1|k) = \Phi_z z(i|k) + Bc(i|k)$ \hspace{1cm} $\Phi_z = A + BK$

$e(i + 1|k) = (\Phi_e + \Delta(k + i))e(i|k) + \Delta(k + i)z(i|k)$ \hspace{1cm} $\Phi_e = A + BL$

$\Pr\{(F + GK)z(i|k) + Gc(i|k) + (F + GL)e(i|k) \leq 1\} \geq p, \hspace{1cm} i = 0, 1, \ldots$

all expressions affine in decision variables $c(0|k), \ldots, c(N - 1|k)$
Prediction model dynamics

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▷ Quasi-closed loop predictions:  \( u(i|k) = K z(i|k) + L e(i|k) + c(i|k) \)

\( K, L \): unconstrained optimal feedback gains for \( z, e \) dynamics

▷ Prediction model and constraints:

\[
\begin{align*}
\Phi_z z(i|k) + B c(i|k) \\
\Phi_e e(i|k) + \Delta(k+i) z(i|k)
\end{align*}
\]

\[
\text{Pr}\{ (F + GK) z(i|k) + G c(i|k) + (F + GL) e(i|k) \leq 1 \} \geq p, \quad i = 0, 1, \ldots
\]

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\[ z(i+1|k) = \Phi_z z(i|k) + Bc(i|k) \quad \Phi_z = A + BK \]

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\[ \Pr\{(F + GK)z(i|k) + Gc(i|k) + (F + GL)e(i|k) \leq 1\} \geq p, \quad i = 0, 1, \ldots \]

all expressions affine in decision variables \( c(0|k), \ldots, c(N-1|k) \)
Feasibility of probabilistic constraints

Consider the general dynamics: \( x^+ = f(x, u, w) \)

and probabilistic constraint: \( \Pr\{ F(x, u, w) \leq 1 \} \geq p, \ w \sim \mathcal{D} \)

Suppose \( u(i|0), i = 0, 1 \) are such that
\[
\Pr\{ F(x(i|0), u(i|0), w(i)) \leq 1 \} \geq p, \ i = 0, 1
\]
then:

(i) it is not necessarily true that
\[
\Pr\{ F(x(1), u(1|0), w(1)) \leq 1 \} \geq p
\]
e.g. consider \( x \in \mathbb{R}, F(x, u, w) = f(x, u, w) \):
\[
\Pr\{ F(x(1|0), u(1|0), w(1)) \leq 1 \} \geq p
\]
but
\[
\Pr\{ F(x_{\text{max}}(1), u(1|0), w(1)) \leq 1 \} \not\geq p
\]
(ii) for some realizations \( w(0) \), there may not exist any feasible \( u(1) \)
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(ii) for some realizations \( w(0) \), there may not exist any feasible \( u(1) \)
Feasibility of probabilistic constraints

Define $\mathcal{T}_0 = \{x : \exists u, F(x, u, w) \leq 1 \text{ w.p. } p\}$,

then $u$ exists s.t. $F(x, u, w) \leq 1 \text{ w.p. } p$ iff $x \in \mathcal{T}_0 \text{ w.p. } 1$ ← hard constraint

$\exists \{u(0), u(1), \ldots\}$ s.t. $\begin{cases} F(x(k), u(k), w) \leq 1 \text{ w.p. } p \\ x(k) \in \mathcal{T}_0 \text{ w.p. } 1 \end{cases}$ $k = 0, 1 \ldots$ iff $x(0) \in \mathcal{R}_\infty$

$\mathcal{R}_\infty = \text{infinite-time reachability set}$ [Bertsekas 1972]

$\hat{\mathcal{T}}_0 = \{(x, u) : F(x, u, w) \leq 1 \text{ w.p. } p\}$,

$\hat{\mathcal{R}}_k = \{(x, u) : f(x, u, w) \in \mathcal{R}_{k-1} \text{ w.p. } 1\} \cap \hat{\mathcal{T}}_0$, $\mathcal{R}_k = \text{Proj}_x(\hat{\mathcal{R}}_k)$, $k = 1, 2, \ldots$

$\mathcal{R}_\infty$ exists if $\mathcal{T}_0$ and $\text{supp}(\mathcal{D})$ are compact, and $x^+ = f(x, u, w)$ is stabilizable, with minimal robust control invariant set $\subset \text{int}(\mathcal{T}_0)$.
Feasibility of probabilistic constraints

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$$\exists \{u(0), u(1), \ldots\} \text{ s.t. } \left\{ \begin{array}{l} F(x(k), u(k), w) \leq 1 \text{ w.p. } p \\ x(k) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array} \right\} k = 0, 1 \ldots \text{ iff } x(0) \in \mathcal{R}_\infty$$

$\mathcal{R}_\infty = \text{infinite-time reachability set}$

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Feasibility of probabilistic constraints

Stochastic MPC formulation:

\[
\min_{\{u(0|k), u(1|k), \ldots\}} J(x(k), \{u(0|k), u(1|k), \ldots\})
\]

subject to

\[
F(x(i|k), u(i|k), w(k + i)) \leq 1 \text{ w.p. } p
\]

\[
x(i|k) \in \mathcal{T}_0 \text{ w.p. } 1
\]

\[
i = 0, 1, \ldots
\]

Constraints at prediction step \(i = 0, \ldots, N - 1\) are invoked with:

- worst case \(\{w(k), \ldots, w(k + i - 1)\}\) via robust tubes
  
  \[\text{[e.g. Mayne 2005, Raković 2006]}\]

- stochastic \(w(k + i)\) via random sampling
  
  \[\text{[e.g. Calafiore & Campi 2005, Campi 2008]}\]

Finite horizon \(N\) implies terminal constraint \(x(N|k) \in \mathcal{T}\)

where \(x \in \mathcal{T} \implies \begin{cases} x^+ \in \mathcal{T} \text{ w.p. } 1 \\ F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p \end{cases}\)

for some terminal feedback law \(u = \kappa_T(x)\)
Feasibility of probabilistic constraints

Stochastic MPC formulation:

\[
\begin{align*}
\min_{\{u(0|k), u(1|k),\ldots\}} & \quad J(x(k), \{u(0|k), u(1|k),\ldots\}) \\
\text{subject to} & \quad F(x(i|k), u(i|k), w(k+i)) \leq 1 \text{ w.p. } p \\
& \quad \forall w(k+j) \in \text{supp}(D), j = 0,\ldots,i-1 \\
& \quad i = 0, 1, \ldots
\end{align*}
\]

- Constraints at prediction step \(i = 0,\ldots,N-1\) are invoked with:
  - worst case \(\{w(k),\ldots,w(k+i-1)\}\) via robust tubes \[\text{e.g. Mayne 2005, Raković 2006}\]
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where \(x \in \mathcal{T} \implies \begin{cases} x^+ \in \mathcal{T} \text{ w.p. } 1 \\ F(x, \kappa_{\mathcal{T}}(x), w) \leq 1 \text{ w.p. } p \end{cases} \)

for some terminal feedback law \(u = \kappa_{\mathcal{T}}(x)\)
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\[
\begin{align*}
\min_{\{u(0|k), u(1|k), \ldots\}} & \quad J(x(k), \{u(0|k), u(1|k), \ldots\}) \\
\text{subject to} & \quad F(x(i|k), u(i|k), w(k+i)) \leq 1 \text{ w.p. } p \\
& \quad \forall w(k+j) \in \text{supp}(D), \; j = 0, \ldots, i-1 \\
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\]

- Constraints at prediction step \(i = 0, \ldots, N-1\) are invoked with:
  - worst case \(\{w(k), \ldots, w(k+i-1)\}\) via robust tubes [e.g. Mayne 2005, Raković 2006]
  - stochastic \(w(k+i)\) via random sampling [e.g. Calafiore & Campi 2005, Campi 2008]

- Finite horizon \(N\) implies terminal constraint \(x(N|k) \in T\)

\[
\text{where } x \in T \implies \begin{cases} 
\{x^+ \in T \text{ w.p. } 1 \\
F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p
\end{cases}
\]

for some terminal feedback law \(u = \kappa_T(x)\)
Terminal set

Let $\mathcal{O}_\infty = \text{maximal admissible set for dynamics: } x^+ = f(x, \kappa_T(x), w)$
and constraint: $F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p$

where $\mathcal{O}_k = \bigcap_{j=0}^{k} \mathcal{P}_j$, \quad k = 0, 1, \ldots$

$\mathcal{P}_0 = \{x : F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p\}$

$\mathcal{P}_k = \{x(0) : x(k) \in \mathcal{P}_0 \text{ w.p. } 1\}$

[cf. Kolmanovsky & Gilbert 1998]
Terminal set

Let $O_\infty = \text{maximal admissible set for dynamics: } x^+ = f(x, \kappa_T(x), w)$
and constraint: $F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p$

where

\[ O_k = \bigcap_{j=0}^{k} P_j, \quad k = 0, 1, \ldots \]
\[ P_0 = \{ x : F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p \} \]
\[ P_k = \{ x(0) : x(k) \in P_0 \text{ w.p. } 1 \} \]

[cf. Kolmanovsky & Gilbert 1998]

If $O_k \subseteq P_{k+1}$, then $O_k = O_\infty$
Terminal set

Let \( \mathcal{O}_\infty = \) maximal admissible set for dynamics: 
\[
x^+ = f(x, \kappa_T(x), w)
\]
and constraint: 
\[
F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p
\]

where 
\[
\mathcal{O}_k = \bigcap_{j=0}^{k} \mathcal{P}_j, \quad k = 0, 1, \ldots
\]
\[
\mathcal{P}_0 = \{x : F(x, \kappa_T(x), w) \leq 1 \text{ w.p. } p\}
\]
\[
\mathcal{P}_k = \{x(0) : x(k) \in \mathcal{P}_0 \text{ w.p. } 1\}
\]

If \( \mathcal{O}_k \subseteq \mathcal{P}_{k+1} \), then \( \mathcal{O}_k \cap \mathcal{P}_{k+1} = \mathcal{O}_{k+1} = \mathcal{O}_k \), and

(i) \( x(0) \in \mathcal{O}_k = \mathcal{O}_{k+1} \Rightarrow \begin{cases} 
  x(0) \in \mathcal{P}_1 \\
  \vdots \\
  x(0) \in \mathcal{P}_{k+1}
\end{cases} \Rightarrow \begin{cases} 
  x(1) \in \mathcal{P}_0 \text{ w.p. } 1 \\
  \vdots \\
  x(1) \in \mathcal{P}_k \text{ w.p. } 1
\end{cases} \Rightarrow x(1) \in \mathcal{O}_k \text{ w.p. } 1
\]
i.e. \( \mathcal{O}_k \) is positively invariant with probability 1

(ii) \( x(0) \in \mathcal{O}_k \Rightarrow x(i) \in \mathcal{O}_k \ \forall i \geq 0 \Rightarrow x(i) \in \mathcal{P}_0 \text{ w.p. } 1 \ \forall i \geq 0 \Rightarrow x(0) \in \mathcal{P}_i \ \forall i \geq 0
\]

hence \( \mathcal{O}_k \subseteq \bigcap_{j=0}^{\infty} \mathcal{P}_j = \mathcal{O}_\infty \)

but \( \mathcal{O}_\infty \subseteq \mathcal{O}_k \) by definition, so \( \mathcal{O}_k = \mathcal{O}_\infty \)

[cf. Kolmanovsky & Gilbert 1998]
Terminal set

Here: \( f(\cdot), \kappa_T(\cdot) \) and \( F(\cdot) \) are linear
\[
\{(x, u) : F(x, u, w) \leq 1 \text{ w.p. } p\}
\]

is convex \[\text{[Prekopa 1995]}\]

\[\Downarrow\]

\( \mathcal{O}_\infty \) is convex but not necessarily polytopic

▷ use polytopic terminal set \( \mathcal{T} \subseteq \mathcal{O}_\infty \), \( \mathcal{T} \) invariant w.p. 1

▷ compute \( \mathcal{T} \) iteratively:

★ initialise with: \( \mathcal{T}^{(0)} \subseteq \mathcal{E} = \max \) invariant ellipsoid for \( p = 1 \)

★ grow \( \mathcal{T}^{(k)} \) at successive iterations \( k = 1, 2, \ldots \) by adding vertices \( v \) such that:
\[
\begin{align*}
    f(v, \kappa_T(v), w) &\in \mathcal{T}^{(k-1)} \text{ w.p. 1} \text{ and} \\
    F(v, \kappa_T(v), w) &\leq 1 \text{ w.p. } p
\end{align*}
\]

★ use a random sampling approach to invoke probabilistic constraints \[\text{[Calafiore & Campi 2006]}\]
Probabilistic constraints via random sampling

- One-step uncertainty sampling gives a cloud of $n_s$ points per prediction step
- Impose constraint via empirical mean of sample constraint indicator variables $\beta$

- In the limit as $n_s \to \infty$:
  \[
  \frac{1}{n_s} \sum_{j=1}^{n_s} \beta_{i|k}^{\{j\}} \geq p
  \]
  \[
  \Downarrow
  \]
  \[
  \Pr\{F(x(i|k), u(i|k), w) \leq 1\} \geq p
  \]

- Bounds on accuracy of sampled constraints for finite $n_s$

- Deterministic sampling reduces the number of particles required
Terminal set

(i). Compute $\mathcal{E}$ using SDP

(ii). inscribe $\sigma \mathcal{E}$, $\sigma \in (0, 1)$ to initialise $\mathcal{T}^{(0)}$

(iii). At iteration $j = 1, 2, \ldots$:
   
   (a). randomly choose a candidate point $v$

   (b). if $f(v, \kappa_{\mathcal{T}}(v), w) \in \mathcal{T}^{(j-1)}$
       and
       $\Pr\{F(v, \kappa_{\mathcal{T}}(v), w) \leq 1\} \geq p$
       then $\mathcal{T}^{(j)} := \text{Co}\{\mathcal{T}^{(j-1)}, v\}$
       otherwise return to (a)

Terminate if $\mathcal{T}^{(j)} - \mathcal{T}^{(j-1)} < \text{threshold}$
and $\mathcal{T}^{(j)} \supset \alpha \mathcal{E}$
Constraint handling

Probabilistic constraint at $i + 1$:

$$ e(l|k) \in \{e : Ve \leq \alpha(l|k)\} \text{ w.p. 1} \quad l = 0, \ldots, i $$

$$ \Pr\{Fx(i + 1|k) + Gu(i + 1|k) \leq 1\} \geq p $$
Constraint handling

Bound the state of the uncertain dynamics, \( e(i + 1|k) = (\Phi_e + \Delta)e(i|k) + \Delta z(i|k) \) using a robust tube with polytopic cross-section:

\[
e(i|k) \in \{ e : Ve \leq \alpha(i|k) \} \text{ w.p. 1}
\]

Using Farkas' Lemma: if \( S_i = \{ x : V_i x \leq \alpha_i \} \), then \( S_1 \subseteq S_2 \) iff:

\[
\exists H \geq 0 \text{ satisfying } HV_1 \leq V_2 \text{ and } H\alpha_1 \leq \alpha_2
\]

[e.g. Bitsoris 1988]
Constraint handling

Bound the state of the uncertain dynamics, $e(i + 1|k) = (\Phi_e + \Delta)e(i|k) + \Delta z(i|k)$ using a robust tube with polytopic cross-section:

$$e(i|k) \in \{e : Ve \leq \alpha(i|k)\} \text{ w.p. 1}$$

- **Application to tubes:**

  \[Ve(i|k) \leq \alpha(i|k) \implies V((\Phi_e + \Delta^{(j)})e(i|k) + \Delta^{(j)} z(i|k)) \leq \alpha(i + 1|k) \text{ iff:} \]

    \[\exists H^{(j)} \geq 0 \text{ satisfying } H^{(j)}V \leq V(\Phi_e + \Delta^{(j)}), \text{ and} \]

    \[H^{(j)} \alpha(i|k) \leq \alpha(i + 1|k) - V \Delta^{(j)} z(i|k)\]

- **Application to terminal constraint** $x(N|k) \in \mathcal{S} \iff V_T(e(N|k) + z(N|k)) \leq 1$:

  \[Ve(N|k) \leq \alpha(N|k) \implies V_T(e(N|k) + z(N|k)) \leq 1 \text{ iff:} \]

    \[\exists H_T \geq 0 \text{ satisfying } H_T V \leq V_T, \text{ and} \]

    \[H_T \alpha(N|k) \leq 1 - V_T z(N|k)\]
Constraint handling

Probabilistic constraint at $i + 1$:

$$e(l|k) \in \{e : Ve \leq \alpha(l|k)\} \text{ w.p. } 1 \quad l = 0, \ldots, i$$

$$\Pr\{Fx(i + 1|k) + Gu(i + 1|k) \leq 1\} \geq p$$

- 1-step-ahead probabilistic constraint is equivalent to

$$Ve(i|k) \leq \alpha(i|k) \implies \Pr\{Fz(i + 1|k) + Gc(i + 1|k) + Fe(\Phi_e + \Delta)e(i|k) + Fe\Delta z(i|k) \leq 1\} \geq p$$

where $Fz = F + GK$, $Fe = F + GL$

- Conditions based on one Farkas matrix $H^{\{j\}}$ per sample $\Delta^{\{j\}}$:

$$\exists H^{\{j\}} \geq 0 \text{ s.t. } H^{\{j\}} V = Fe(\Phi_e + \Delta^{\{j\}}), \text{ and } H^{\{j\}} \alpha(i|k) + Fz(i + 1|k) + Gc(i + 1|k) + Fe\Delta^{\{j\}} z(i|k) \leq 1$$

↑

require:

$$\frac{1}{n_s} \sum_{j=1}^{n_s} \beta^{\{j\}}_{i|k} \geq p,$$

$$\beta^{\{j\}}_{i|k} = \begin{cases} 1 & \text{if satisfied} \\ 0 & \text{if violated} \end{cases}$$
Constraint handling

- $V \in \mathbb{R}^{n_V \times n_x}$ chosen offline
  - from a regular $n_V$-partition of the $n_x$-hypersphere
    - e.g. $n_V = 10$, $n_x = 2$
    - $\{x : Vx \leq 1\}$

- If $H_T$, $H^{(j)}$, $j = 1, \ldots, \rho$, $H^{\{j\}}$, $j = 1, \ldots, n_s$ are computed offline,
  - then online constraints are linear in $\alpha(i|k)$

- Determine $H$ e.g. by solving LPs of the form:

  $$
  H = \arg\min_H \|H\|_\infty \\
  \text{subject to } H \geq 0 \text{ and } HV_1 \leq V_2
  $$

  $\uparrow$

  heuristic to relax the online constraint: $H\alpha_1 \leq \alpha_2$
MPC algorithm

▷ Offline:

1. Compute gains $K, L$, predicted cost $V(x, c)$, and samples $\{\Delta^j, j=1,\ldots,n_s\}$

2. Compute $H_T, H^{(j)}, j=1,\ldots,\rho, H^j, j=1,\ldots,n_s$

▷ Online, at $k=0,1,\ldots$:

1. $c^*(k) = \arg\min_c \quad V(x(k), c)$
   subject to $H^{(j)}\alpha(i|k) \leq \alpha(i+1|k) - V\Delta^{(j)}z(i|k)$
   $H_T\alpha(N|k) \leq 1 - V_Tz(N|k)$
   $\frac{1}{n_s} \sum_{j=1}^{n_s} \beta^j_i \geq p, \quad \beta^j_i = 0 \text{ or } 1$

2. $u(k) := Kx(k) + c^*(0|k)$

▷ Online optimisation = MIQP in $Nn_u + (N-1)n_V$ variables
   with $(N-2)n_V\rho + n_T + (N-1)pn_s n_C$ constraints
Closed loop stability

Let
\[ \mathbf{c}^*(0) = \{c^*(0), \ldots, c^*(N-1)\} = \arg \min_{\mathbf{c} \in \mathcal{F}_c(x(0))} V(x(0), \mathbf{c}) \]
\[ \tilde{\mathbf{c}}(1) = \{c^*(1), \ldots, c^*(N-1), 0\} \]

Then, by definition, for \( x(0) \in \mathcal{F}_x \):
\[ V(x(1), \tilde{\mathbf{c}}(1)) = V^*(x(0)) - (\|x(0)\|^2_Q + \|u(0)\|^2_R) \]

but \( \tilde{\mathbf{c}}(1) \in \mathcal{F}_c(x(1)) \) w.p. 1, so the optimal cost satisfies
\[ \mathbb{E}(V^*(x(1))) \leq V^*(x(0)) - (\|x(0)\|^2_Q + \|u(0)\|^2_R) \]

\[ \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}(\|x(k)\|^2_Q + \|u(k)\|^2_R) \leq \frac{1}{r} \left[ V^*(x(0)) - \mathbb{E}(V^*(x(r))) \right] \]

\[ \Rightarrow \text{quadratic stability} \]

\[ \star \text{ summing over } r \text{ time-steps:} \]

\[ \text{hence } (x(k), u(k)) \rightarrow (0, 0) \text{ as } k \rightarrow \infty \text{ w.p. 1 if } Q \succ 0 \text{ (or } R \succ 0 \text{ and observable}) \]
Closed loop stability

Let
\[ c^*(0) = \{c^*(0), \ldots, c^*(N - 1)\} = \arg \min_{c \in \mathcal{F}_c(x(0))} V(x(0), c) \]
\[ \tilde{c}(1) = \{c^*(1), \ldots, c^*(N - 1), 0\} \]

Then, by definition, for \( x(0) \in \mathcal{F}_x \):
\[ V(x(1), \tilde{c}(1)) = V^*(x(0)) - (\|x(0)\|_Q^2 + \|u(0)\|_R^2) \]
but \( \tilde{c}(1) \in \mathcal{F}_c(x(1)) \) w.p. 1, so the optimal cost satisfies
\[ \mathbb{E}(V^*(x(1))) \leq V^*(x(0)) - (\|x(0)\|_Q^2 + \|u(0)\|_R^2) \]

\( \star \) summing over \( r \) time-steps:
\[ \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E} (\|x(k)\|_Q^2 + \|u(k)\|_R^2) \leq \frac{1}{r} \left[ V^*(x(0)) - \mathbb{E}(V^*(x(r))) \right] \]

\[ \implies \text{quadratic stability} \]

\( \star \) hence \((x(k), u(k)) \to (0, 0)\) as \( k \to \infty \) w.p. 1 if \( Q \succ 0 \) (or \( R \succ 0 \) and observable)
Numerical example

- plant order: $n_x = 2$, with $\rho = 3$
- uncertainty set vertices
- horizon: $N = 5$
- number of samples of $\Delta$: $n_s = 64$
- computation time per online optimisation:
  - 122 ms (robust MPC)
  - 350 ms (stochastic MPC)
- Low degree of conservativeness constraint violation rate:
  $39.5\%$ for $p = 0.6$
Conclusions

- Avoid exponential growth in number of constraints through use of robust tubes
- Allow flexible definition of tube cross-section through individually scaled facets
- Increase size of terminal set and therefore the region of attraction offline
- Applicable to linear systems with process uncertainty with finite support and known (but arbitrary) distribution
- Low conservatism of probabilistic constraints