Robust and Stochastic Linear MPC for Systems Subject to Multiplicative Uncertainty

Martin Evans Mark Cannon Basil Kouvaritakis

SuS1SP.1, 26 August 2012

Outline

- Problem statement
- Probabilistic constraints and recursive feasibility
- Polytopic invariant sets with probabilistic constraints
- Constraint handling using Farkas' Lemma
- Receding horizon control

Problem description

- Linear system with multiplicative uncertainty:

$$x(k+1) = \Big(A + \Delta(k)\Big)x(k) + Bu(k)$$

state: $x \in \mathbb{R}^{n_x}$

control input: $u \in \mathbb{R}^{n_u}$

- A, B known, $\Delta(k)$ uncertain
  - $\star$ $\Delta(k) = \sum_{j=1}^{\rho} w_j(k)\Delta^{(j)},\ w_j(k) \geq 0,\ \sum_{j=1}^{\rho} w_j(k) = 1$
  - $\star$ $\mathbb{E}\Big(\Delta(k)\Big) = 0$
  - $\star$ $\mathbf{w}(k) = \Big(w_1(k), \ldots, w_\rho(k)\Big)$ stochastic with known, finitely supported distribution
  - $\star$ $\{\mathbf{w}(0), \mathbf{w}(1), \ldots\}$ independent & identically distributed
- $x(k)$ measured at time k
- Constraints: $\Pr\{Fx(k) + Gu(k) \leq 1\} \geq p$ for all $k = 0, 1, \ldots$

$$\Updownarrow$$

$$\Pr\{F_j x(k) + G_j u(k) \leq 1\} \geq p_j,\ \ p_j \in (0, 1],\ \ j = 1, \ldots, n_C$$

Problem description

- Linear system with multiplicative uncertainty:

$$x(k+1) = \Big(A + \Delta(k)\Big)x(k) + Bu(k)$$

state: $x \in \mathbb{R}^{n_x}$

control input: $u \in \mathbb{R}^{n_u}$

- A, B known, $\Delta(k)$ uncertain
  - ⋆ $\Delta(k) = \sum_{j=1}^{\rho} w_j(k)\Delta^{(j)},\ w_j(k) \geq 0,\ \sum_{j=1}^{\rho} w_j(k) = 1$
  - ⋆ $\mathbb{E}\Big(\Delta(k)\Big) = 0$
  - ⋆ $\mathbf{w}(k) = \Big(w_1(k), \ldots, w_\rho(k)\Big)$ stochastic with known, finitely supported distribution
  - ⋆ $\{\mathbf{w}(0), \mathbf{w}(1), \ldots\}$ independent & identically distributed

- $x(k)$ measured at time k

- Constraints: $\Pr\{Fx(k) + Gu(k) \leq 1\} \geq p$ for all $k = 0, 1, \ldots$

$$\Updownarrow$$

$$\Pr\{F_j x(k) + G_j u(k) \leq 1\} \geq p_j,\ \ p_j \in (0, 1],\ \ j = 1, \ldots, n_C$$

Problem description

- Linear system with multiplicative uncertainty:

$$x(k+1) = \Big(A + \Delta(k)\Big)x(k) + Bu(k)$$

state: $x \in \mathbb{R}^{n_x}$

control input: $u \in \mathbb{R}^{n_u}$

- A, B known, $\Delta(k)$ uncertain
  - ⋆ $\Delta(k) = \sum_{j=1}^{\rho} w_j(k)\Delta^{(j)},\ w_j(k) \geq 0,\ \sum_{j=1}^{\rho} w_j(k) = 1$
  - ⋆ $\mathbb{E}\Big(\Delta(k)\Big) = 0$
  - ⋆ $\mathbf{w}(k) = \Big(w_1(k), \ldots, w_\rho(k)\Big)$ stochastic with known, finitely supported distribution
  - ⋆ $\{\mathbf{w}(0), \mathbf{w}(1), \ldots\}$ independent & identically distributed

- $x(k)$ measured at time k

- Constraints: $\Pr\big\{Fx(k) + Gu(k) \leq 1\big\} \geq p$ for all $k = 0, 1, \ldots$

$$\Updownarrow$$

$$\Pr\big\{F_j x(k) + G_j u(k) \leq 1\big\} \geq p_j,\ \ p_j \in (0, 1],\ \ j = 1, \ldots, n_C$$

Expectation MPC cost

Quadratic expected value cost:

$$J(x(k), \{u(0|k), u(1|k), \ldots\}) = \sum_{i=0}^{\infty} \mathbb{E}\big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\big)$$

where $x(i|k), u(i|k)$ are the values of $x(k+i), u(k+i)$ predicted at time k

- Quasi-closed loop predictions with horizon N:

$$u(i|k) = Kx(i|k) + c(i|k), \quad c(i|k) = \begin{cases} \text{decision variables}, & i = 0, \ldots, N-1 \\ 0, & i \geq N \end{cases}$$

  - optimal unconstrained control law: $u(k) = Kx(k)$
  - $J(x(k), \{u(0|k), u(1|k), \ldots\}) = V\big(x(k), \mathbf{c}(k)\big)$
  - $V(x, \mathbf{c})$: quadratic in $\mathbf{c} = (c(0), \ldots, c(N-1))$

- Optimal value $V^*(x) = \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

  MPC law: $\kappa_{\mathrm{MPC}}(x) = Kx + c^*(0)$, where $\mathbf{c}^* = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

Expectation MPC cost

Quadratic expected value cost:

$$J(x(k), \{u(0|k), u(1|k), \ldots\}) = \sum_{i=0}^{\infty} \mathbb{E}\big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\big)$$

where $x(i|k), u(i|k)$ are the values of $x(k+i), u(k+i)$ predicted at time k

- Quasi-closed loop predictions with horizon N:

$$u(i|k) = Kx(i|k) + c(i|k), \quad c(i|k) = \begin{cases} \text{decision variables}, & i = 0, \ldots, N-1 \\ 0, & i \geq N \end{cases}$$

  - optimal unconstrained control law: $u(k) = Kx(k)$
  - $J(x(k), \{u(0|k), u(1|k), \ldots\}) = V\big(x(k), \mathbf{c}(k)\big)$
  - $V(x, \mathbf{c})$: quadratic in $\mathbf{c} = (c(0), \ldots, c(N-1))$

- Optimal value $V^*(x) = \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

MPC law: $\kappa_{\mathrm{MPC}}(x) = Kx + c^*(0)$, where $\mathbf{c}^* = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

Expectation MPC cost

Quadratic expected value cost:

$$J(x(k), \{u(0|k), u(1|k), \ldots\}) = \sum_{i=0}^{\infty} \mathbb{E}\big(\|x(i|k)\|_Q^2 + \|u(i|k)\|_R^2\big)$$

where $x(i|k), u(i|k)$ are the values of $x(k+i), u(k+i)$ predicted at time k

- ▷ Quasi-closed loop predictions with horizon N:

$$u(i|k) = Kx(i|k) + c(i|k), \quad c(i|k) = \begin{cases} \text{decision variables}, & i = 0, \ldots, N-1 \\ 0, & i \geq N \end{cases}$$

  - ⋆ optimal unconstrained control law: $u(k) = Kx(k)$
  - ⋆ $J(x(k), \{u(0|k), u(1|k), \ldots\}) = V\big(x(k), \mathbf{c}(k)\big)$
  - ⋆ $V(x, \mathbf{c})$: quadratic in $\mathbf{c} = (c(0), \ldots, c(N-1))$

- ▷ Optimal value $V^*(x) = \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

  MPC law: $\kappa_{\mathrm{MPC}}(x) = Kx + c^*(0)$, where $\mathbf{c}^* = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x)} V(x, \mathbf{c})$

Prediction model dynamics

Predicted state decomposition: $x(i|k) = z(i|k) + e(i|k)$

nominal: $z(i|k) = \mathbb{E}\big(x(i|k)\big)$ $\quad z(0|k) = x(k)$

uncertain: $e(i|k) = x(i|k) - z(i|k)$ $\quad e(0|k) = 0$

- Quasi-closed loop predictions: $u(i|k) = Kz(i|k) + Le(i|k) + c(i|k)$

  K, L: unconstrained optimal feedback gains for z, e dynamics

- Prediction model and constraints:

$$z(i+1|k) = \Phi_z z(i|k) + Bc(i|k) \qquad \Phi_z = A + BK$$

$$e(i+1|k) = \big(\Phi_e + \Delta(k+i)\big)e(i|k) + \Delta(k+i)z(i|k) \qquad \Phi_e = A + BL$$

$$\Pr\big\{(F+GK)z(i|k) + Gc(i|k) + (F+GL)e(i|k) \leq 1\big\} \geq p, \quad i = 0, 1, \ldots$$

all expressions affine in decision variables $c(0|k), \ldots, c(N-1|k)$

Prediction model dynamics

Predicted state decomposition: $x(i|k) = z(i|k) + e(i|k)$

nominal: $z(i|k) = \mathbb{E}\big(x(i|k)\big)$ $\quad z(0|k) = x(k)$

uncertain: $e(i|k) = x(i|k) - z(i|k)$ $\quad e(0|k) = 0$

- Quasi-closed loop predictions: $u(i|k) = Kz(i|k) + Le(i|k) + c(i|k)$

  K, L: unconstrained optimal feedback gains for z, e dynamics

- Prediction model and constraints:

$$z(i+1|k) = \Phi_z z(i|k) + Bc(i|k) \qquad \Phi_z = A + BK$$

$$e(i+1|k) = \big(\Phi_e + \Delta(k+i)\big)e(i|k) + \Delta(k+i)z(i|k) \qquad \Phi_e = A + BL$$

$$\Pr\big\{(F+GK)z(i|k) + Gc(i|k) + (F+GL)e(i|k) \leq 1\big\} \geq p, \quad i = 0, 1, \ldots$$

all expressions affine in decision variables $c(0|k), \ldots, c(N-1|k)$

Prediction model dynamics

Predicted state decomposition: $x(i|k) = z(i|k) + e(i|k)$

nominal: $z(i|k) = \mathbb{E}\big(x(i|k)\big)$ $\quad z(0|k) = x(k)$

uncertain: $e(i|k) = x(i|k) - z(i|k)$ $\quad e(0|k) = 0$

- Quasi-closed loop predictions: $u(i|k) = Kz(i|k) + Le(i|k) + c(i|k)$

  K, L: unconstrained optimal feedback gains for z, e dynamics

- Prediction model and constraints:

$$z(i+1|k) = \Phi_z z(i|k) + Bc(i|k) \qquad \Phi_z = A + BK$$
$$e(i+1|k) = \big(\Phi_e + \Delta(k+i)\big)e(i|k) + \Delta(k+i)z(i|k) \qquad \Phi_e = A + BL$$
$$\Pr\big\{(F+GK)z(i|k) + Gc(i|k) + (F+GL)e(i|k) \leq 1\big\} \geq p, \quad i = 0, 1, \ldots$$

all expressions affine in decision variables $c(0|k), \ldots, c(N-1|k)$

Feasibility of probabilistic constraints

Consider the general dynamics: $x^+ = f(x, u, w)$

and probabilistic constraint: $\Pr\{F(x, u, w) \leq 1\} \geq p, \;\; w \sim \mathfrak{D}$

Suppose $u(i|0)$, $i = 0, 1$ are such that

$$\Pr\big\{F(x(i|0), u(i|0), w(i)) \leq 1\big\} \geq p, \;\; i = 0, 1$$

then:

(i) it is not necessarily true that

$$\Pr\big\{F(x(1), u(1|0), w(1)) \leq 1\big\} \geq p$$

e.g. consider $x \in \mathbb{R}$, $F(x, u, w) = f(x, u, w)$:

$$\Pr\big\{F(x(1|0), u(1|0), w(1)) \leq 1\big\} \geq p$$

but

$$\Pr\big\{F(x_{\max}(1), u(1|0), w(1)) \leq 1\big\} \ngeq p$$

(ii) for some realizations $w(0)$, there may not exist $u(1|1)$

Feasibility of probabilistic constraints

Consider the general dynamics: $x^+ = f(x, u, w)$

and probabilistic constraint: $\Pr\{F(x, u, w) \leq 1\} \geq p, \;\; w \sim \mathfrak{D}$

Suppose $u(i|0)$, $i = 0, 1$ are such that

$$\Pr\big\{F(x(i|0), u(i|0), w(i)) \leq 1\big\} \geq p, \;\; i = 0, 1$$

then:

(i) it is not necessarily true that

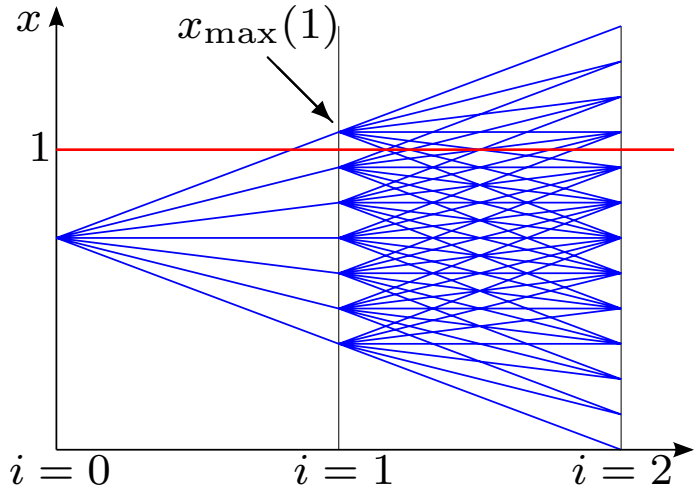
$$\Pr\big\{F(x(1), u(1|0), w(1)) \leq 1\big\} \geq p$$

e.g. consider $x \in \mathbb{R}$, $F(x, u, w) = f(x, u, w)$:

$$\Pr\big\{F(x(1|0), u(1|0), w(1)) \leq 1\big\} \geq p$$

but

$$\Pr\big\{F(x_{\max}(1), u(1|0), w(1)) \leq 1\big\} \not\geq p$$



(ii) for some realizations $w(0)$, there may not exist any feasible $u(1)$

Feasibility of probabilistic constraints

Consider the general dynamics: $x^+ = f(x, u, w)$

and probabilistic constraint: $\Pr\{F(x, u, w) \leq 1\} \geq p, \;\; w \sim \mathfrak{D}$

Suppose $u(i|0)$, $i = 0, 1$ are such that

$$\Pr\big\{F(x(i|0), u(i|0), w(i)) \leq 1\big\} \geq p, \;\; i = 0, 1$$

then:

(i) it is not necessarily true that

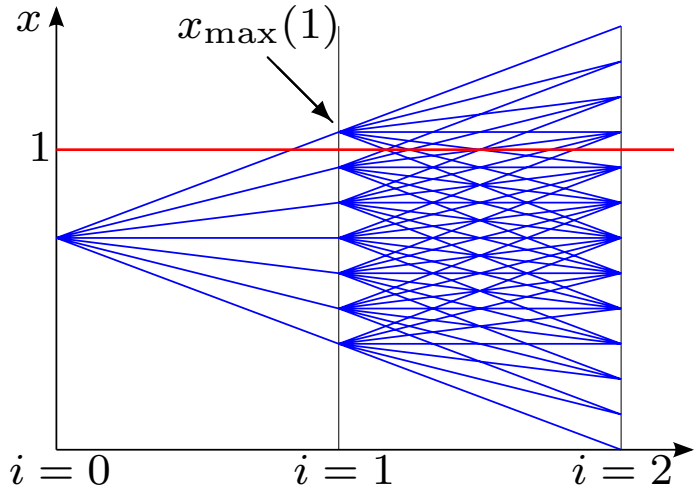
$$\Pr\big\{F(x(1), u(1|0), w(1)) \leq 1\big\} \geq p$$

e.g. consider $x \in \mathbb{R}$, $F(x, u, w) = f(x, u, w)$:

$$\Pr\big\{F(x(1|0), u(1|0), w(1)) \leq 1\big\} \geq p$$

but

$$\Pr\big\{F(x_{\max}(1), u(1|0), w(1)) \leq 1\big\} \ngeq p$$



(ii) for some realizations $w(0)$, there may not exist any feasible $u(1)$

Feasibility of probabilistic constraints

Define $\mathcal{T}_0 = \{x : \exists u,\ F(x,u,w) \leq 1 \text{ w.p. } p\}$,

then u exists s.t. $F(x,u,w) \leq 1$ w.p. p iff $x \in \mathcal{T}_0$ w.p. $1 \longleftarrow$ hard constraint

$\exists \{u(0), u(1), \ldots\}$ s.t. $\left\{ \begin{array}{l} F(x(k), u(k), w) \leq 1 \text{ w.p. } p \\ x(k) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array} \right\}$ $k = 0, 1 \ldots$ iff $x(0) \in \mathcal{R}_\infty$

$\mathcal{R}_\infty =$ infinite-time reachability set [Bertsekas 1972]

$\hat{\mathcal{T}}_0 = \{(x,u) : F(x,u,w) \leq 1 \text{ w.p. } p\}$, $\mathcal{R}_0 = \mathcal{T}_0 = \mathrm{Proj}_x(\hat{\mathcal{T}}_0)$

$\hat{\mathcal{R}}_k = \{(x,u) : f(x,u,w) \in \mathcal{R}_{k-1} \text{ w.p. } 1\} \bigcap \hat{\mathcal{T}}_0$, $\mathcal{R}_k = \mathrm{Proj}_x(\hat{\mathcal{R}}_k)$, $k = 1, 2, \ldots$

$\mathcal{R}_\infty$ exists if $\mathcal{T}_0$ and $\mathrm{supp}(\mathfrak{D})$ are compact, and
$x^+ = f(x,u,w)$ is stabilizable, with
minimal robust control invariant set $\subset \mathrm{int}(\mathcal{T}_0)$.

Feasibility of probabilistic constraints

Define $\mathcal{T}_0 = \{x : \exists u,\ F(x,u,w) \leq 1 \text{ w.p. } p\}$,

then u exists s.t. $F(x,u,w) \leq 1$ w.p. p iff $x \in \mathcal{T}_0$ w.p. 1 $\longleftarrow$ hard constraint

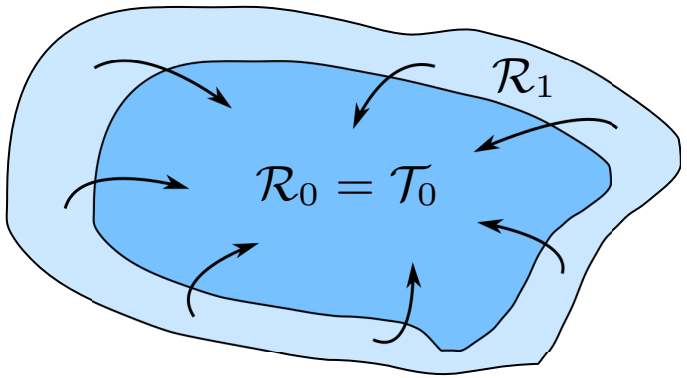
$$\exists \{u(0), u(1), \ldots\} \text{ s.t. } \left\{ \begin{array}{l} F(x(k), u(k), w) \leq 1 \text{ w.p. } p \\ x(k) \in \mathcal{T}_0 \text{ w.p. } 1 \end{array} \right\} k = 0, 1 \ldots \text{ iff } x(0) \in \mathcal{R}_\infty$$

$\mathcal{R}_\infty =$ infinite-time reachability set [Bertsekas 1972]

$\hat{\mathcal{T}}_0 = \{(x,u) : F(x,u,w) \leq 1 \text{ w.p. } p\}$, $\quad \mathcal{R}_0 = \mathcal{T}_0 = \mathrm{Proj}_x(\hat{\mathcal{T}}_0)$

$\hat{\mathcal{R}}_k = \{(x,u) : f(x,u,w) \in \mathcal{R}_{k-1} \text{ w.p. } 1\} \bigcap \hat{\mathcal{T}}_0$, $\quad \mathcal{R}_k = \mathrm{Proj}_x(\hat{\mathcal{R}}_k)$, $k = 1, 2, \ldots$

$\mathcal{R}_\infty$ exists if $\mathcal{T}_0$ and $\mathrm{supp}(\mathfrak{D})$ are compact, and $x^+ = f(x,u,w)$ is stabilizable, with minimal robust control invariant set $\subset \mathrm{int}(\mathcal{T}_0)$.



Feasibility of probabilistic constraints

Stochastic MPC formulation:

$$\begin{aligned} \min_{\{u(0|k),u(1|k),\ldots\}} \quad & J\big(x(k), \{u(0|k), u(1|k), \ldots\}\big) \\ \text{subject to} \quad & F(x(i|k), u(i|k), w(k+i)) \leq 1 \text{ w.p. } p \\ & x(i|k) \in \mathcal{T}_0 \text{ w.p. } 1 \\ & \qquad\qquad\qquad\qquad\qquad\qquad i = 0, 1, \ldots \end{aligned}$$

- Constraints at prediction step $i = 0, \ldots, N-1$ are invoked with:
  - ⋆ worst case $\{w(k), \ldots, w(k+i-1)\}$ via robust tubes [e.g. Mayne 2005, Raković 2006]
  - ⋆ stochastic $w(k+i)$ via random sampling [e.g. Calafiori & Campi 2005, Campi 2008]
- Finite horizon N implies terminal constraint $x(N|k) \in \mathbb{T}$

$$\text{where } x \in \mathbb{T} \implies \begin{cases} x^+ \in \mathbb{T} \text{ w.p. } 1 \\ F(x, \kappa_{\mathbb{T}}(x), w) \leq 1 \text{ w.p. } p \end{cases}$$

for some terminal feedback law $u = \kappa_{\mathbb{T}}(x)$

Feasibility of probabilistic constraints

Stochastic MPC formulation:

$$\min_{\{u(0|k),u(1|k),\ldots\}} \; J\big(x(k), \{u(0|k), u(1|k), \ldots\}\big)$$
$$\text{subject to} \quad F(x(i|k), u(i|k), w(k+i)) \leq 1 \text{ w.p. } p$$
$$\forall w(k+j) \in \mathrm{supp}(\mathfrak{D}),\; j = 0, \ldots, i-1$$
$$i = 0, 1, \ldots$$

- Constraints at prediction step $i = 0, \ldots, N-1$ are invoked with:
  - ⋆ worst case $\{w(k), \ldots, w(k+i-1)\}$ via robust tubes [e.g. Mayne 2005, Raković 2006]
  - ⋆ stochastic $w(k+i)$ via random sampling [e.g. Calafiori & Campi 2005, Campi 2008]
- Finite horizon N implies terminal constraint $x(N|k) \in \mathbb{T}$

$$\text{where } x \in \mathbb{T} \implies \begin{cases} x^+ \in \mathbb{T} \text{ w.p. } 1 \\ F(x, \kappa_{\mathbb{T}}(x), w) \leq 1 \text{ w.p. } p \end{cases}$$

for some terminal feedback law $u = \kappa_{\mathbb{T}}(x)$

Feasibility of probabilistic constraints

Stochastic MPC formulation:

$$\min_{\{u(0|k),u(1|k),\ldots\}} J\big(x(k),\{u(0|k),u(1|k),\ldots\}\big)$$
$$\text{subject to} \quad F(x(i|k),u(i|k),w(k+i)) \leq 1 \text{ w.p. } p$$
$$\forall w(k+j) \in \mathrm{supp}(\mathfrak{D}),\ j=0,\ldots,i-1$$
$$i=0,1,\ldots$$

- Constraints at prediction step $i=0,\ldots,N-1$ are invoked with:
  - ⋆ worst case $\{w(k),\ldots,w(k+i-1)\}$ via robust tubes [e.g. Mayne 2005, Raković 2006]
  - ⋆ stochastic $w(k+i)$ via random sampling [e.g. Calafiori & Campi 2005, Campi 2008]
- Finite horizon N implies terminal constraint $x(N|k) \in \mathbb{T}$

$$\text{where } x \in \mathbb{T} \implies \begin{cases} x^+ \in \mathbb{T} \text{ w.p. } 1 \\ F(x,\kappa_{\mathbb{T}}(x),w) \leq 1 \text{ w.p. } p \end{cases}$$

for some terminal feedback law $u=\kappa_{\mathbb{T}}(x)$

Terminal set

Let $\mathcal{O}_\infty$ = maximal admissible set for dynamics: $x^+ = f(x, \kappa_{\mathbb{T}}(x), w)$
and constraint: $F(x, \kappa_{\mathbb{T}}(x), w) \leq 1$ w.p. p

where $\mathcal{O}_k = \bigcap_{j=0}^{k} \mathcal{P}_j, \quad k = 0, 1, \ldots$ [cf. Kolmanovsky & Gilbert 1998]

$$\mathcal{P}_0 = \{x : F(x, \kappa_{\mathbb{T}}(x), w) \leq 1 \text{ w.p. } p\}$$

$$\mathcal{P}_k = \{x(0) : x(k) \in \mathcal{P}_0 \text{ w.p. } 1\}$$

Terminal set

Let $\mathcal{O}_\infty$ = maximal admissible set for dynamics: $x^+ = f(x, \kappa_{\mathbb{T}}(x), w)$
and constraint: $F(x, \kappa_{\mathbb{T}}(x), w) \leq 1$ w.p. p

where $\mathcal{O}_k = \bigcap_{j=0}^{k} \mathcal{P}_j, \quad k = 0, 1, \ldots$ [cf. Kolmanovsky & Gilbert 1998]

$$\mathcal{P}_0 = \{x : F(x, \kappa_{\mathbb{T}}(x), w) \leq 1 \text{ w.p. } p\}$$

$$\mathcal{P}_k = \{x(0) : x(k) \in \mathcal{P}_0 \text{ w.p. } 1\}$$

If $\mathcal{O}_k \subseteq \mathcal{P}_{k+1}$, then $\mathcal{O}_k = \mathcal{O}_\infty$

Terminal set

Let $\mathcal{O}_\infty$ = maximal admissible set for dynamics: $x^+ = f(x, \kappa_{\mathbb{T}}(x), w)$
and constraint: $F(x, \kappa_{\mathbb{T}}(x), w) \leq 1$ w.p. p

where $\mathcal{O}_k = \bigcap_{j=0}^{k} \mathcal{P}_j, \quad k = 0, 1, \ldots$ [cf. Kolmanovsky & Gilbert 1998]

$$\mathcal{P}_0 = \{x : F(x, \kappa_{\mathbb{T}}(x), w) \leq 1 \text{ w.p. } p\}$$

$$\mathcal{P}_k = \{x(0) : x(k) \in \mathcal{P}_0 \text{ w.p. } 1\}$$

If $\mathcal{O}_k \subseteq \mathcal{P}_{k+1}$, then $\mathcal{O}_k \bigcap \mathcal{P}_{k+1} = \mathcal{O}_{k+1} = \mathcal{O}_k$, and

(i) $$x(0) \in \mathcal{O}_k = \mathcal{O}_{k+1} \Rightarrow \left\{ \begin{array}{c} x(0) \in \mathcal{P}_1 \\ \vdots \\ x(0) \in \mathcal{P}_{k+1} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} x(1) \in \mathcal{P}_0 \text{ w.p. } 1 \\ \vdots \\ x(1) \in \mathcal{P}_k \text{ w.p. } 1 \end{array} \right\} \Rightarrow x(1) \in \mathcal{O}_k \text{ w.p. } 1$$

i.e. $\mathcal{O}_k$ is positively invariant with probability 1

(ii) $$x(0) \in \mathcal{O}_k \Rightarrow x(i) \in \mathcal{O}_k \ \forall i \geq 0 \Rightarrow x(i) \in \mathcal{P}_0 \text{ w.p. } 1 \ \forall i \geq 0$$
$$\Rightarrow x(0) \in \mathcal{P}_i \ \forall i \geq 0$$

hence $\mathcal{O}_k \subseteq \bigcap_{j=0}^{\infty} \mathcal{P}_j = \mathcal{O}_\infty$

but $\mathcal{O}_\infty \subseteq \mathcal{O}_k$ by definition, so $\mathcal{O}_k = \mathcal{O}_\infty$

Terminal set

Here: $f(\cdot)$, $\kappa_{\mathbb{T}}(\cdot)$ and $F(\cdot)$ are linear

$\{(x,u) : F(x,u,w) \leq 1 \text{ w.p. } p\}$ is convex [Prekopa 1995]

$$\Downarrow$$

$\mathcal{O}_\infty$ is convex but not necessarily polytopic

- ▷ use polytopic terminal set $\mathbb{T} \subseteq \mathcal{O}_\infty$, $\mathbb{T}$ invariant w.p. 1
- ▷ compute $\mathbb{T}$ iteratively:
  - ⋆ initialise with: $\mathbb{T}^{(0)} \subset \mathcal{E} =$ max invariant ellipsoid for $p = 1$
  - ⋆ grow $\mathbb{T}^{(k)}$ at successive iterations $k = 1, 2, \ldots$ by adding vertices v such that:

$$f(v, \kappa_{\mathbb{T}}(v), w) \in \mathbb{T}^{(k-1)} \text{ w.p. } 1 \text{ and}$$
$$F(v, \kappa_{\mathbb{T}}(v), w) \leq 1 \text{ w.p. } p$$

  - ⋆ use a random sampling approach to invoke probabilistic constraints

[Calafiori & Campi 2006]

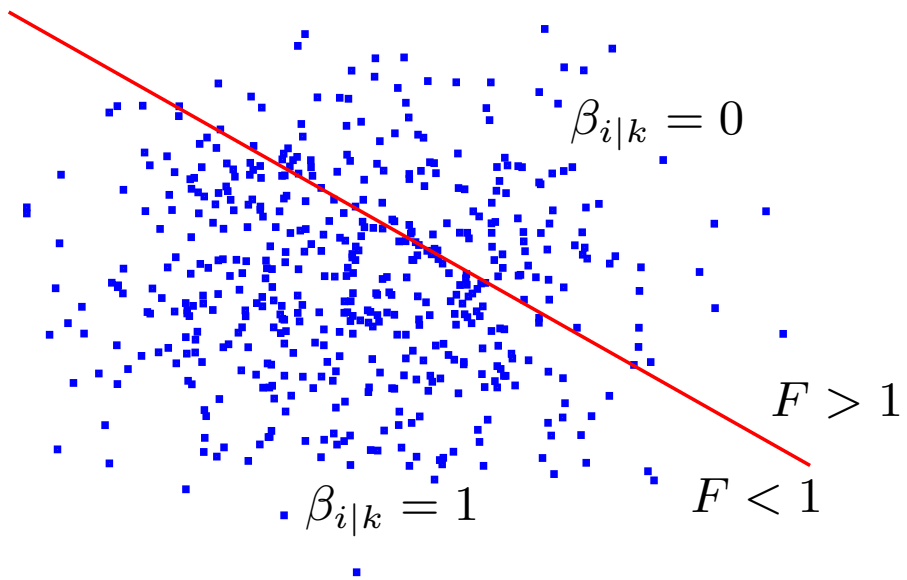
Probabilistic constraints via random sampling

- One-step uncertainty sampling gives a cloud of n_s points per prediction step
- Inpose constraint via empirical mean of sample constraint indicator variables β
  - In the limit as $n_s \to \infty$:

$$\frac{1}{n_s}\sum_{j=1}^{n_s}\beta_{i|k}^{\{j\}} \geq p$$

$$\Downarrow$$

$$\Pr\{F(x(i|k), u(i|k), w) \leq 1\} \geq p$$



  - Bounds on accuracy of sampled constraints for finite n_s
- Deterministic sampling reduces the number of particles required

Terminal set

(i). Compute $\mathcal{E}$ using SDP

(ii). inscribe $\sigma\mathcal{E}$, $\sigma \in (0,1)$ to initialise $\mathbb{T}^{(0)}$

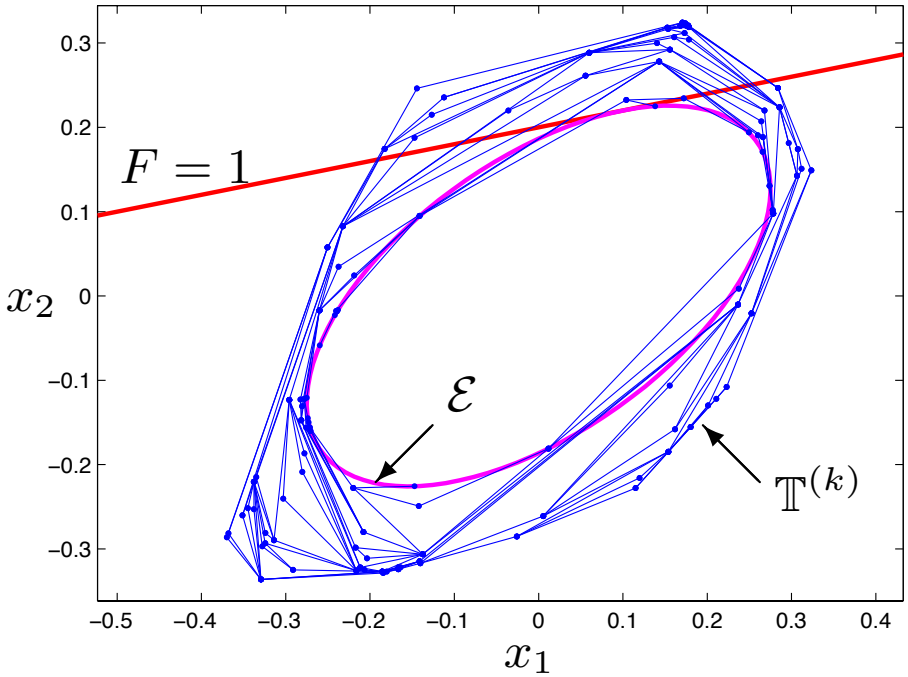
(iii). At iteration $j = 1, 2, \ldots$:

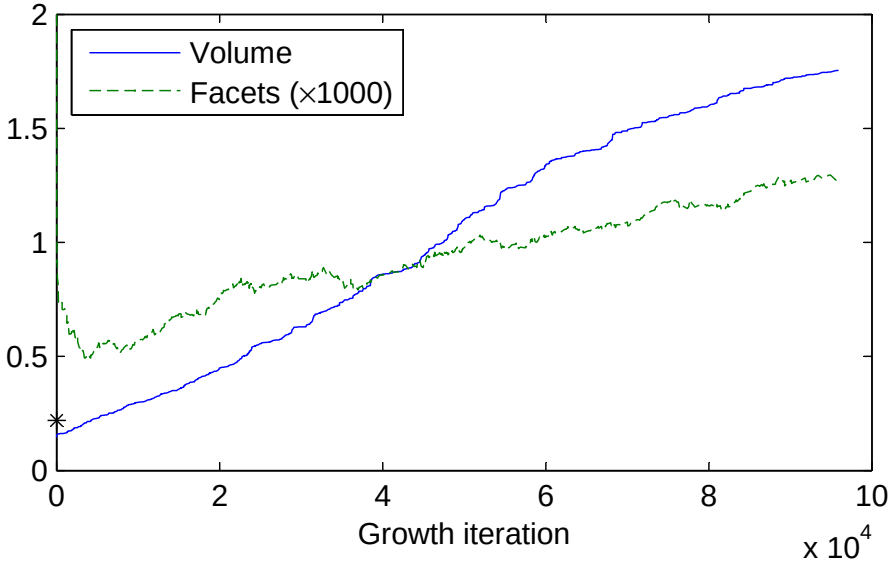
(a). randomly choose a candidate point v

(b). if $f(v, \kappa_{\mathbb{T}}(v), w) \in \mathbb{T}^{(j-1)}$ and $\Pr\{F(v, \kappa_{\mathbb{T}}(v), w) \leq 1\} \geq p$ then $\mathbb{T}^{(j)} := \mathrm{Co}\{\mathbb{T}^{(j-1)}, v\}$

otherwise return to (a)

Terminate if $\mathbb{T}^{(j)} - \mathbb{T}^{(j-1)} <$ threshold and $\mathbb{T}^{(j)} \supset \alpha\mathcal{E}$





Constraint handling

Probabilistic constraint at $i+1$:

$$e(l|k) \in \{e : Ve \leq \alpha(l|k)\} \text{ w.p. } 1 \quad l = 0, \ldots, i$$

$$\Pr\big\{Fx(i+1|k) + Gu(i+1|k) \leq 1\big\} \geq p$$

Constraint handling

Bound the state of the uncertain dynamics, $e(i+1|k) = (\Phi_e + \Delta)e(i|k) + \Delta z(i|k)$ using a robust tube with polytopic cross-section:

$$e(i|k) \in \{e : Ve \leq \alpha(i|k)\} \text{ w.p. } 1$$

- Using Farkas' Lemma: if $\mathcal{S}_i = \{x : V_i x \leq \alpha_i\}$, then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ iff:

$$\exists H \geq 0 \text{ satisfying } HV_1 \leq V_2 \text{ and } H\alpha_1 \leq \alpha_2$$ [e.g. Bitsoris 1988]

Constraint handling

Bound the state of the uncertain dynamics, $e(i+1|k) = (\Phi_e + \Delta)e(i|k) + \Delta z(i|k)$ using a robust tube with polytopic cross-section:

$$e(i|k) \in \{e : Ve \leq \alpha(i|k)\} \text{ w.p. } 1$$

- Application to tubes:

$$Ve(i|k) \leq \alpha(i|k) \implies V\big((\Phi_e + \Delta^{(j)})e(i|k) + \Delta^{(j)} z(i|k)\big) \leq \alpha(i+1|k) \text{ iff:}$$

$$\exists H^{(j)} \geq 0 \text{ satisfying } H^{(j)}V \leq V(\Phi_e + \Delta^{(j)}), \text{ and}$$
$$H^{(j)}\alpha(i|k) \leq \alpha(i+1|k) - V\Delta^{(j)} z(i|k)$$

- Application to terminal constraint $x(N|k) \in \mathcal{S} \iff V_{\mathbb{T}}\big(e(N|k) + z(N|k)\big) \leq 1$:

$$Ve(N|k) \leq \alpha(N|k) \implies V_{\mathbb{T}}\big(e(N|k) + z(N|k)\big) \leq 1 \text{ iff:}$$

$$\exists H_{\mathbb{T}} \geq 0 \text{ satisfying } H_{\mathbb{T}}V \leq V_{\mathbb{T}}, \text{ and}$$
$$H_{\mathbb{T}}\alpha(N|k) \leq 1 - V_{\mathbb{T}} z(N|k)$$

Constraint handling

Probabilistic constraint at $i+1$:

$$e(l|k) \in \{e : Ve \leq \alpha(l|k)\} \text{ w.p. } 1 \quad l = 0, \ldots, i$$

$$\Pr\big\{Fx(i+1|k) + Gu(i+1|k) \leq 1\big\} \geq p$$

- 1-step-ahead probabilistic constraint is equivalent to

  $Ve(i|k) \leq \alpha(i|k) \Longrightarrow$

  $$\Pr\big\{F_z z(i+1|k) + Gc(i+1|k) + F_e(\Phi_e + \Delta)e(i|k) + F_e \Delta z(i|k) \leq 1\big\} \geq p$$

  where $F_z = F + GK$, $F_e = F + GL$

- Conditions based on one Farkas matrix $H^{\{j\}}$ per sample $\Delta^{\{j\}}$:

  $\exists H^{\{j\}} \geq 0$ s.t. $H^{\{j\}} V = F_e(\Phi_e + \Delta^{\{j\}})$, and

  $$H^{\{j\}}\alpha(i|k) + F_z z(i+1|k) + Gc(i+1|k) + F_e \Delta^{\{j\}} z(i|k) \leq 1$$

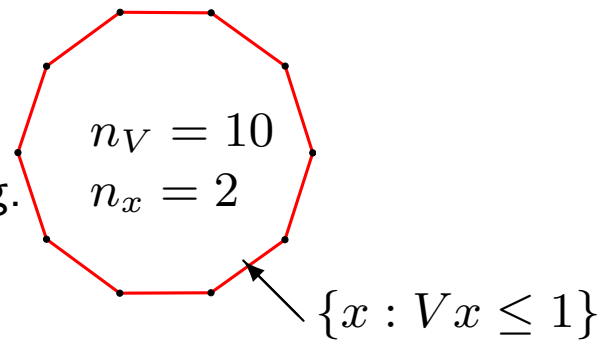
$$\uparrow$$

require: $$\frac{1}{n_s}\sum_{j=1}^{n_s} \beta_{i|k}^{\{j\}} \geq p, \qquad \beta_{i|k}^{\{j\}} = \begin{cases} 1 & \text{if satisfied} \\ 0 & \text{if violated} \end{cases}$$

Constraint handling

- $V \in \mathbb{R}^{n_V \times n_x}$ chosen offline

  from a regular n_V-partition of the n_x-hypersphere e.g.



- If $H_{\mathbb{T}}$, $H^{(j)}$, $j = 1, \ldots, \rho$, $H^{\{j\}}$, $j = 1, \ldots, n_s$ are computed offline,

  then online constraints are linear in $\alpha(i|k)$

- Determine H e.g. by solving LPs of the form:

$$H = \begin{array}{ll} \arg\min\limits_{H} & \|H\|_\infty \\ \text{subject to} & H \geq 0 \text{ and } HV_1 \leq V_2 \end{array}$$

$$\uparrow$$

heuristic to relax the online constraint: $H\alpha_1 \leq \alpha_2$

MPC algorithm

- Offline:
  1. Compute gains K, L, predicted cost $V(x, \mathbf{c})$, and samples $\{\Delta^{\{j\}},\ j = 1, \ldots, n_s\}$
  2. Compute $H_{\mathbb{T}}$, $H^{(j)}$, $j = 1, \ldots, \rho$, $H^{\{j\}}$, $j = 1, \ldots, n_s$

- Online, at $k = 0, 1, \ldots$:
  1. $$\begin{aligned} \mathbf{c}^*(k) = \ & \arg\min_{\mathbf{c}} && V(x(k), \mathbf{c}) \\ & \text{subject to} && H^{(j)}\alpha(i|k) \leq \alpha(i+1|k) - V\Delta^{(j)} z(i|k) \\ & && H_{\mathbb{T}}\alpha(N|k) \leq 1 - V_{\mathbb{T}} z(N|k) \\ & && \frac{1}{n_s}\sum_{j=1}^{n_s} \beta_{i|k}^{\{j\}} \geq p, \quad \beta_{i|k}^{\{j\}} = 0 \text{ or } 1 \end{aligned}$$
  2. $u(k) := Kx(k) + c^*(0|k)$

- Online optimisation = MIQP in $Nn_u + (N-1)n_V$ variables
  with $(N-2)n_V\rho + n_{\mathbb{T}} + (N-1)pn_s n_C$ constraints

Closed loop stability

Let

$$\mathbf{c}^*(0) = \{c^*(0), \dots, c^*(N-1)\} = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x(0))} V(x(0), \mathbf{c})$$

$$\tilde{\mathbf{c}}(1) = \{c^*(1), \dots, c^*(N-1), 0\}$$

Then, by definition, for $x(0) \in \mathcal{F}_x$:

$$V(x(1), \tilde{c}(1)) = V^*(x(0)) - \left(\|x(0)\|_Q^2 + \|u(0)\|_R^2\right)$$

but $\tilde{\mathbf{c}}(1) \in \mathcal{F}_{\mathbf{c}}(x(1))$ w.p. 1, so the optimal cost satisfies

$$\boxed{\mathbb{E}\big(V^*(x(1))\big) \leq V^*(x(0)) - \left(\|x(0)\|_Q^2 + \|u(0)\|_R^2\right)}$$

- ⋆ summing over r time-steps:

$$\frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\left(\|x(k)\|_Q^2 + \|u(k)\|_R^2\right) \leq \frac{1}{r}\Big[V^*(x(0)) - \mathbb{E}\big(V^*(x(r))\big)\Big]$$

$\Longrightarrow$ quadratic stability

- ⋆ hence $(x(k), u(k)) \to (0,0)$ as $k \to \infty$ w.p. 1 if $Q \succ 0$ (or $R \succ 0$ and observable)

Closed loop stability

Let

$$\mathbf{c}^*(0) = \{c^*(0), \ldots, c^*(N-1)\} = \arg \min_{\mathbf{c} \in \mathcal{F}_{\mathbf{c}}(x(0))} V(x(0), \mathbf{c})$$

$$\tilde{\mathbf{c}}(1) = \{c^*(1), \ldots, c^*(N-1), 0\}$$

Then, by definition, for $x(0) \in \mathcal{F}_x$:

$$V(x(1), \tilde{c}(1)) = V^*(x(0)) - \left(\|x(0)\|_Q^2 + \|u(0)\|_R^2\right)$$

but $\tilde{\mathbf{c}}(1) \in \mathcal{F}_{\mathbf{c}}(x(1))$ w.p. 1, so the optimal cost satisfies

$$\boxed{\mathbb{E}\big(V^*(x(1))\big) \leq V^*(x(0)) - \left(\|x(0)\|_Q^2 + \|u(0)\|_R^2\right)}$$

- ⋆ summing over r time-steps:

$$\frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}\left(\|x(k)\|_Q^2 + \|u(k)\|_R^2\right) \leq \frac{1}{r}\Big[V^*(x(0)) - \mathbb{E}\big(V^*(x(r))\big)\Big]$$

$\Longrightarrow$ quadratic stability

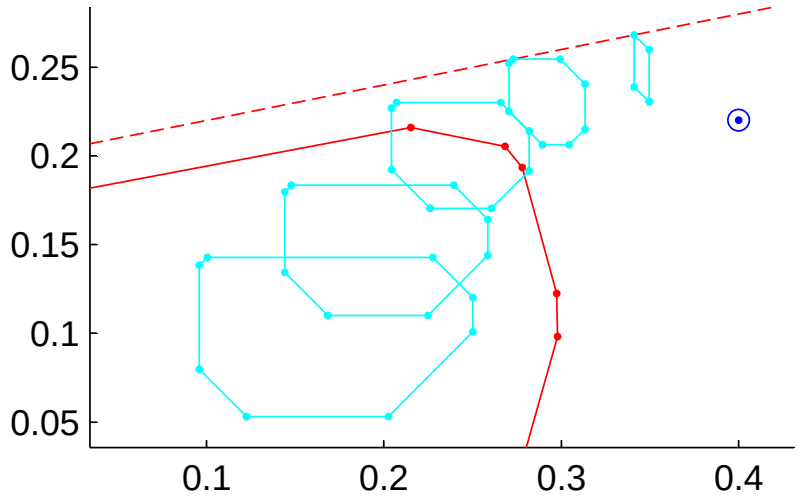
- ⋆ hence $(x(k), u(k)) \to (0,0)$ as $k \to \infty$ w.p. 1 if $Q \succ 0$ (or $R \succ 0$ and observable)

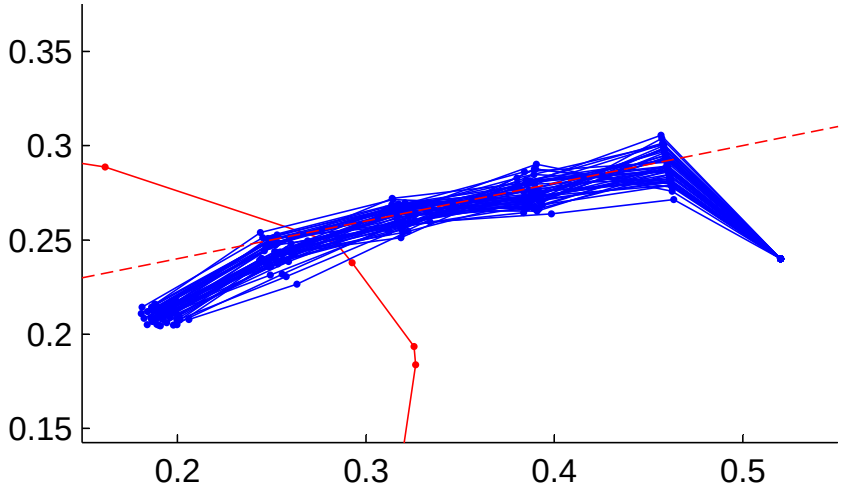
Numerical example

- plant order: $n_x = 2$, with $\rho = 3$ uncertainty set vertices
- horizon: $N = 5$
- number of samples of Δ: $n_s = 64$
- computation time per online optimisation:
  - 122 ms (robust MPC)
  - 350 ms (stochastic MPC)
- Low degree of conservativeness

  constraint violation rate:

  39.5% for $p = 0.6$





Conclusions

- Avoid exponential growth in number of constraints through use of robust tubes
- Allow flexible definition of tube cross-section through individually scaled facets
- Increase size of terminal set and therefore the region of attraction offline
- Applicable to linear systems with process uncertainty with finite support and known (but arbitrary) distribution
- Low conservatism of probabilistic constraints