Affine Feedback Policies for Robust Control with Constraints

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Abstract

This thesis is concerned with the optimal control of linear discrete-time systems with convex state and input constraints and subject to bounded disturbances. It is shown that the nonconvex problem of finding a constraint admissible affine state feedback policy over a finite horizon can be converted to an equivalent convex problem, where the input at each time is modelled as an affine function of prior disturbances. This implies that a broad class of constrained finite horizon robust and optimal control problems can be solved efficiently using convex optimization methods. These policies can be then used in the design of robust receding horizon control (RHC) laws such that the system constraints are satisfied for all time and for all allowable disturbance sequences.

By choosing a control policy from this class that minimizes the expected value of a quadratic function of the states and inputs at each time, it is possible to provide sufficient conditions under which the policy optimization problem is convex at each time step, and for which such an RHC control law renders the closed-loop system input-to-state stable. Alternatively, using a quadratic cost function where the disturbance is negatively weighted as in $\mathcal{H}_\infty$ control, one can provide conditions under which the finite-horizon min-max control problem to be solved at each time step can be rendered convex-concave, and provide conditions guaranteeing that the ℓ_2 gain of the resulting closed-loop system is bounded.

When all of the system constraints are linear, the complexity of solving these problems grows polynomially with the problem size for a wide variety of disturbance classes, making their solution tractable using standard techniques in convex optimization. In the particular case that the cost function is a quadratic function of the states and inputs and the disturbance set is ∞-norm bounded, a sparse problem structure can be recovered via introduction of state-like variables and decomposition of the problem into a set of coupled finite horizon control problems. This decomposed problem can then be formulated as a highly structured quadratic program, solvable by a primal-dual interior-point method for which each iteration requires a number of operations that increases cubicly with horizon length.

Finally, it is shown how the ideas presented can be extended to the output feedback case. A similar convex reparameterization is applied to the problem of finding a constraint admissible affine output feedback policy over a finite horizon, to be used in conjunction with a fixed linear state observer. A time-invariant control law is developed using these policies that can be computed by solving a finite-dimensional, tractable optimization problem at each time step, and that guarantees that the closed-loop system satisfies the system constraints for all time.

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Declaration

As required by the University Statute, I hereby declare that this dissertation is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other university. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specified explicitly in the text.

I also declare that the length of this dissertation is less than 65,000 words and that the number of figures is less than 150.

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Contents

Notation

Scalar Sets

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}_+$	the nonnegative real numbers
$\bar{\mathbb{R}}$	the extended real numbers: $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{Z}_{[i,j]}$	the set of integers $\{i, \dots, j\}$

Definitions and Inequalities

$A := B$	A is defined by B
$A =: B$	B defines A
$A \leq B$	element-wise inequality between A and B
$A < B$	strict element-wise inequality between A and B
$A \preceq B$	matrix inequality between symmetric matrices: $B - A$ is positive semidefinite
$A \prec B$	strict matrix inequality between symmetric matrices: $B - A$ is positive definite
$A \preceq_K B$	conic inequality: $B - A \in K$
$A \prec_K B$	strict conic inequality: $B - A \in \operatorname{int}(K)$

Norms

$\lVert\cdot\rVert$	vector norm
$\lVert x\rVert_2$	2–norm of the vector x: $\lVert x\rVert_2 := \sqrt{x^\top x}$
$\lVert x\rVert_p$	ℓ_p norm of the vector x
$\lVert x\rVert_Q$	weighted 2–norm of the vector x: $\lVert x\rVert_Q := \sqrt{x^\top Q x}$

Topology and Convex Sets

$\operatorname{conv} C$	convex hull of the set C
$\operatorname{int} C$	interior of the set C
$\operatorname{lin} C$	linear hull of the set C
$\operatorname{rint} C$	relative interior of the set C
∂C	boundary of the set C
C°	polar of the set C
K^*	dual cone of the conic set K
σ_C	support function of the set C
γ_C	gauge function of the set C
$\operatorname{dom} f$	effective domain of the function f

Vectors and Matrices

$\mathbf{1}$	vector of ones of appropriate dimension: $\mathbf{1} := [1 \ldots 1]^\top$
$\langle x, y \rangle$	inner product of vectors x and y
$\operatorname{vec}(x, y)$	vertical concatenation of vectors x and y: $\operatorname{vec}(x, y) := \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$
$x \perp y$	vectors x and y are orthogonal: $x^\top y = 0$
$\operatorname{vec}(A)$	vertical concatenation of columns of the matrix A: if $A = [a_1, \ldots, a_n]$, then $\operatorname{vec}(A) = \operatorname{vec}(a_1, \ldots, a_n)$
$A^\top$	transpose of the matrix A
$A^\dagger$	pseudo-inverse of matrix A
$\operatorname{tr}(A)$	trace of the matrix A
$(A)_i$	i^{th} row of the matrix A
$(A)_{(i)}$	i^{th} column of the matrix A
$A \otimes B$	Kronecker product of matrices A and B
I_n	identity matrix in $\mathbb{R}^{n \times n}$
$\mathcal{N}(A)$	nullspace of A
$\mathcal{R}(A)$	range of A

Set Operations

$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
$A \oplus B$	Minkowski sum of sets A and B: $A \oplus B := \{a + b \mid a \in A,\ b \in B\}$
$A \sim B$	Pontryagin difference of sets A and B: $A \sim B := \{a \mid a + b \in \mathcal{A},\ \forall b \in B\}$
$A \backslash B$	Relative complement of sets A and B: $A \setminus B := \{a \mid a \in A,\ a \notin B\}$

Other Notation

$\mathcal{B}_p^n$	p-norm unit ball in $\mathbb{R}^n$: $\mathcal{B}_p^n := \{x \in \mathbb{R}^n \mid \|x\|_p^n \leq 1\}$
$\mathbb{E}[x]$	Expected value of random vector x
$\mathcal{P}[X]$	Probability of event X

Acronyms

ISS	Input-to-State Stable
LMI	Linear Matrix Inequality
LTI	Linear Time Invariant
LP	Linear Program(ming)
LQR	Linear Quadratic Regulator
mRPI	Minimal Robust Positively Invariant
QMI	Quadratic Matrix Inequality
QP	Quadratic Program(ming)
RHC	Receding Horizon Control
RPI	Robust Positively Invariant
SDP	Semidefinite Program(ming)
SOCP	Second-Order Cone Program(ming)

Chapter 1. Introduction

This dissertation is concerned with the control of constrained linear systems subject to bounded disturbances. In particular, we consider the problem of designing a stabilizing control law for a discrete-time linear dynamical system of the form

$$x^+ = Ax + Bu + Gw, \tag{1.1}$$

while guaranteeing that the states x and control inputs u remain inside some constraint set Z, i.e.

$$(x, u) \in Z \tag{1.2}$$

for all sequences of disturbances w arising from some known set W.

The above problem is motivated by the fact that for many real control applications, optimal operation nearly always occurs on or close to some constraints [Mac02]. These constraints typically arise, for example, due to actuator limitations, safe regions of operation or performance specifications. For safety-critical applications, it is crucial that some or all of these constraints are met despite the presence of disturbances or modelling inaccuracies.

For such applications, it is appropriate to treat the control design problem in a *worst-case* fashion, i.e. to assume that the uncertainty will be realized in such a way as to force the system to violate its constraints if it is possible to do so. It is therefore necessary to consider both constraints and uncertainty explicitly in the control design, in order to create a robust control law with stability and constraint satisfaction guarantees that is minimally conservative.

1.1 Background

Taken separately, the issues of robustness and constraint satisfaction for linear systems are generally well understood. The field of linear robust control, which is mainly motivated by frequency-domain performance criteria [Zam81] and which does *not* explicitly consider time-domain constraints as in the above problem formulation, is considered to be mature and a number of excellent references are available on the subject [GL95, ZDG96, DP00].

On the other hand, the problem of controlling a constrained linear system without disturbances has been the subject of intensive research since the early 1980s. A technique that has proven particularly suitable for the design of nonlinear controllers for such systems is *predictive control* [Mac02, CB04]. Predictive control is not a control method *per se*, but rather a family of optimal control techniques where, at each time instant, a finite horizon constrained optimal control problem is solved using tools from mathematical programming. The solution to this optimization problem is usually implemented in a receding horizon fashion — at each time instant, a measurement (or estimate) of the system states is obtained, the associated optimization problem is solved and only the first control input in the optimal policy is implemented. The rather myopic strategy of successively planning sequences of control moves over a finite horizon can result in instability or constraint violations if proper care is not taken, but these issues have largely been resolved in the undisturbed case [MRRS00].

Taken together, the requirements that the control law must satisfy a set of time-domain state and inputs constraints, and that it must do so *robustly* with respect to some unknown external disturbance or modelling error, can cause considerable difficulty. In the case of linear controller design, there are only a handful of design methods for constrained problems, even if all the constraint sets are considered to be polytopes or ellipsoids; see, for example, the literature on set invariance theory [Bla99] or ℓ_1 optimal control [DD95, Sha96, FG97, SB98]. In any case, such design methods are typically computationally intractable or suffer from excessive conservativeness for all but a very limited set of problems.

If one wishes to apply the general methodology of predictive control to the design of robust nonlinear control laws for constrained systems, then an initial requirement is to specify a method for solving finite horizon robust control problems for the system (1.1)–(1.2). Problems of this type are of long standing interest in the control literature; see, for example, [Wit68, BR73] for some seminal work on the subject.

It is generally accepted that if disturbances are to be accounted for in such problems, then the optimization has to be done over *feedback policies*, rather than over open-loop *input sequences* as in conventional predictive control, otherwise problems of infeasibility will quickly arise [MRRS00]. In the most general case, one would like to find, over a finite horizon of length N, a feedback policy

$$\pi := \{\mu_0(\cdot), \ldots, \mu_{N-1}(\cdot)\}$$

for the discrete-time linear dynamical system

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \qquad && \forall i \in \mathbb{Z}_{[0,N-1]} \\ u_i &= \mu_i(x_0, \ldots, x_i) && \forall i \in \mathbb{Z}_{[0,N-1]} \end{aligned}$$

that guarantees satisfaction of the system constraints for every possible sequence of disturbances, where each of the functions $\mu_i(\cdot)$ is a potentially nonlinear function mapping the sequence of observed states $\{x_0, \ldots, x_i\}$ to a control input u_i, and the initial state x_0 is known. However, optimization over arbitrary nonlinear feedback policies is extremely difficult in most cases, since there is no known method for even parameterizing the family of nonlinear functions over which one must search for a solution.

Proposals that take this approach, such as those based on robust dynamic programming [BR71, BB91, BBM03, DB04, MRVK06], or those based on enumeration of extreme disturbance sequences generated from the set W, as in [SM98], are typically limited to situations where the constraint and disturbance sets are polyhedral, and are generally intractable for all but the smallest problems. A number of analytical results are also available in the polyhedral case, if the cost function is suitably chosen, that show that the solution turns out to be a time-varying piecewise affine state feedback control policy [MS97, RC03, Bor03, BBM03, DB04, KM04a].

Unfortunately, the practicality of these results is also limited to small problem sizes, since the solution complexity grows exponentially with the size of the problem data, in general. The problem is even more acute in the case of non-polyhedral constraint or disturbance sets, where the solution to problems of infinite dimension is generally required.

1.2 Affine Feedback Policies

An obvious sub-optimal strategy is to restrict the class of functions from which the control policy π might be composed. The most straightforward choice is to restrict the functions constituting π to those which are *affine* functions of the sequence of states, i.e. to parameterize each control input u_i as

$$u_i = g_i + \sum_{j=0}^{i} K_{i,j} x_j, \tag{1.3}$$

where the matrices $K_{i,j}$ and vectors g_i are decision variables. There are two advantages of such a parameterization. First, the control policy is characterized by a tractable number of decision variables. Second, the close relationship to linear control laws means that finite horizon policies of this type, if calculable, fit naturally within the framework generally used in predictive control to guarantee stability and invariance of the closed-loop system. However, for a given starting state x the set of constraint admissible parameters $\{\{K_{i,j}\},\{g_i\}\}$ is easily shown to be *nonconvex* in general, making control policies in this form entirely unsuitable for on-line calculation as part of a receding horizon control strategy.

As a result, most proposals that take this approach [Bem98, CRZ01, KM03, LK99, MSR05] fix a stabilizing feedback gain K, then parameterize the control sequence as

$$u_i = g_i + K x_i$$

and optimize the design parameters g_i. Though tractable, this approach is essentially *ad hoc* and is in any case problematic since it is unclear how one should select the gain K to minimize conservativeness.

An alternative to (1.3) is to define a class of affine *disturbance feedback* control policies in the form

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j. \tag{1.4}$$

The parameterization (1.4) was recently proposed as a means for finding solutions to a general class of robust optimization problems, called affinely adjustable robust counterpart (AARC) problems [Gus02, BTGGN04]. The same parameterization has also appeared specifically in application to robust model predictive control problems in [LÖ3a, LÖ3b, vHB02, vH04], and appears to have originally been suggested within the context of stochastic programs with recourse [GW74].

The advantage of the policy model (1.4) is that, in contrast to (1.3), the set of constraint admissible parameters $\{\{M_{i,j}\},\{v_i\}\}$ for control policies in this form is *convex* when all of the relevant constraint sets are convex; as a result, one can reasonably expect that policies in the form (1.4) can be found reliably and efficiently, generally using off-the-shelf software packages. However, the policy formulation (1.4) does not fit naturally within the predictive control framework, since it is not obvious how one can employ existing methods for ensuring stability and invariance properties in combination with policies of this type.

1.3 Organization and Highlights of this Dissertation

The idea underpinning all of the results in this dissertation is that the policy parameterizations (1.3) and (1.4) are *equivalent*, in a sense to be precisely defined in Chapter 3. As a result, one can exploit the analytical properties of the state feedback parameterization (1.3), while simultaneously enjoying all of the computational advantages inherent to the disturbance feedback parameterization (1.4). From this central idea, a wealth of results relating to the stability and efficient computation of receding horizon control laws constructed from these parameterizations can be derived.

We here outline the contributions of each of the remaining chapters. Where indicated, some of the material represents an extension of work previously published by the author in collaboration with one or more co-authors. In all such cases, the present author is the principle author of these publications.

Background and Policy Parameterizations

Chapter 2: Convexity plays a central role throughout the dissertation. This chapter brings together some fundamental definitions and results from the theory of convex sets and functions, convex optimization and variational analysis that are critical to the development of later chapters.

Chapter 3: In this chapter, definitions and basic results relating to the affine control policies (1.3) and (1.4) are introduced, forming the foundation for much of the work in later chapters. The key equivalence result relating the parameterizations (1.3) and (1.4) is presented, and it is shown that the set of constraint admissible policies in the form (1.4) is convex when all of the relevant constraint sets are convex, while the set of admissible policies in the form (1.3) is not — these equivalence and convexity results can be viewed as a special case of the well-known Youla parameterization in linear system theory [YJB76]. Additional results relating to the invariance of receding horizon controls are also developed, and these ideas are central to the subsequent development of stabilizing control laws. Portions of this chapter have appeared in [GK05c, GKM05, GKM06].

Stability and Receding Horizon Control

Chapter 4: In this chapter, we consider the problem of finding a finite horizon control policy in the form (1.3) that minimizes the expected value of a quadratic cost function. It is shown that, using the equivalent parameterization (1.4), this problem can be posed as a convex optimization problem. It is then shown how a receding horizon control law synthesized from policies that are optimal in this sense can guarantee that the resulting closed-loop system is input-to-state stable, and that the behavior of such a control law matches that of a classical linear-quadratic or $\mathcal{H}_2$ control law when the system is operating far from its constraints. General results relating to input-to-state stability of constrained systems with convex Lyapunov functions are also developed in support of the main results. This chapter is based largely on results appearing in [GK05b, GK06a].

Chapter 5: In this chapter, we employ an alternative quadratic cost function where the disturbance is negatively weighted as in $\mathcal{H}_\infty$ control [BB91, GL95], and consider the problem of finding a control policy that minimizes the maximum value of this function. By imposing additional convex constraints on the set of policies introduced in Chapter 3, we show that this min-max optimization problem can be rendered convex-concave, making its solution amenable in principle to standard techniques in convex optimization. We further show how one can guarantee that if these policies are used in the synthesis of receding horizon control laws, then the ℓ_2 gain of the resulting closed-loop system is bounded, and the achievable bound decreases with the length of the planning horizon used by the controller. This chapter expands on results appearing in [GKA06, GK06b].

Computational Methods

Chapter 6: In the theoretical results of Chapters 4 and 5, receding horizon control laws are proposed that require the repeated solution of finite horizon control problems that are solvable in principle using convex optimization techniques. In this chapter we consider the problem of calculating such optimal policies in practice. It is argued that the problem of finding such optimal policies is generally only feasible when the state and input constraints are characterized by linear inequalities, though the disturbance set can be characterized in a large variety of ways. Of particular interest in engineering applications are polytopic or norm-bounded disturbance sets, and each of these is considered in turn for the problems posed in Chapters 3–5. In all of the cases considered, the central result is that a feasible or optimal affine state feedback policy can be found by solving a single convex optimization problem, in one of a variety of standard forms, whose size is polynomially bounded in the size of the problem data.

Chapter 7: In this chapter, the solution to one of the convex optimization problems presented in Chapter 6 — the problem of finding a policy that minimizes a quadratic function of the nominal state and input sequences of a system subject to ∞–norm bounded disturbances — is considered in significantly greater detail. In its original form, this optimization problem is a dense convex quadratic program with $O(N^2)$ variables, assuming that the horizon length N dominates the number of states and control inputs at each stage. Hence each iteration of an interior-point optimization method would require the solution of a dense linear system in $O(N^6)$ operations.

We show how structure can be exploited to devise a sparse formulation of this problem, thereby realizing a substantial reduction in computational effort to $O(N^3)$ work per interior-point iteration. This sparse formulation is the result of a decomposition technique that can be used to separate the problem into a coupled set of finite horizon control problems. The reduction of effort is the analogue, for robust control, to the situation in classical unconstrained optimal control in which Linear Quadratic Regulator (LQR) problems can be solved in $O(N)$ time, using a Riccati [AM90, Sec. 2.4] or Differential Dynamic Programming [JM70] technique in which the state feedback equation $x^+ = Ax + Bu$ is explicit in every stage, compared to $O(N^3)$ time for the more compact formulation in which states are eliminated from the system. More direct motivation for these results comes from [Wri93, Ste95, RWR98, Bie00, DBS05], which describe efficient implementations of optimization methods for solving optimal control problems with state and control constraints, though without disturbances. Much of the work in this chapter is based on [GK05a, GKR07].

Output Feedback Extensions and Conclusion

Chapter 8: All of the results of Chapters 3–7 relate to the application of the policy parameterization (1.3) and its associated reparameterization (1.4) to problems where a complete measurement of the state is available. In this chapter, an analogous reparameterization for *output* feedback control is employed in conjunction with a fixed linear state observer and a corresponding bound on the state estimation error.

The main aim of the chapter is to provide conditions under which receding horizon control laws synthesized from this parameterization can guarantee constraint satisfaction for all time. When the state estimation error bound matches the minimal robust positively invariant (mRPI) set for the system error dynamics, we show that the control law is actually time-invariant, but its calculation requires the solution of an infinite-dimensional optimization problem when the mRPI set is not finitely determined. By employing an invariant outer approximation to the mRPI error set [RKKM05], we develop a time-invariant control law that can be computed by solving a finite-dimensional tractable optimization problem at each time step. The computational complexity of the proposed control law does not differ greatly from the state feedback results of previous chapters, so the main technical difficulties encountered with the output feedback problem considered here relate to the specification of appropriate conditions on the initial error set and terminal state such that the closed-loop system is robust positively invariant under a finitely determined time-invariant controller. This work has also appeared in [GK07, GK06c].

Chapter 9: This chapter summarizes the main contributions of the dissertation and suggests some directions for future research.

In this chapter we collect various definitions and useful results relating to convexity and convex optimization. The selection of results presented is dictated entirely by their use in subsequent chapters; for a thorough review of convex analysis and optimization, the reader is referred to the excellent texts [Roc70, RW98, BNO03, BV04], from which many of the results and definitions are drawn.

2.1 Convex Sets

Definition 2.1 (Convex Set). *A set $C \subseteq \mathbb{R}^n$ is a* convex set *if, for every pair of points $x \in C$ and $y \in C$, every point on the line connecting them is also contained in C, i.e.*

$$(1-\tau)x + \tau y \in C, \quad \textit{for all } \tau \in (0,1). \tag{2.1}$$

Convex sets play a central role in almost all of the results to be presented in this dissertation. Although many of the theoretical results to be presented will be based on abstract convex sets without any special structure, several specific classes of convex sets will also be used, particularly when dealing with computational problems.

Example 2.2 (Polyhedra and Polytopes). *A set is $C \subseteq \mathbb{R}^n$ is a* polyhedron *if it can be defined by a set of affine inequalities*

$$C := \{x \in \mathbb{R}^n \mid Ax \leq b\} \tag{2.2}$$

for some matrix $A \in \mathbb{R}^{t \times n}$ and vector $b \in \mathbb{R}^t$, where t is the number of inequalities defining the set. The set C is a polytope *if it is a bounded polyhedron. Both polyhedra and polytopes are convex sets.*

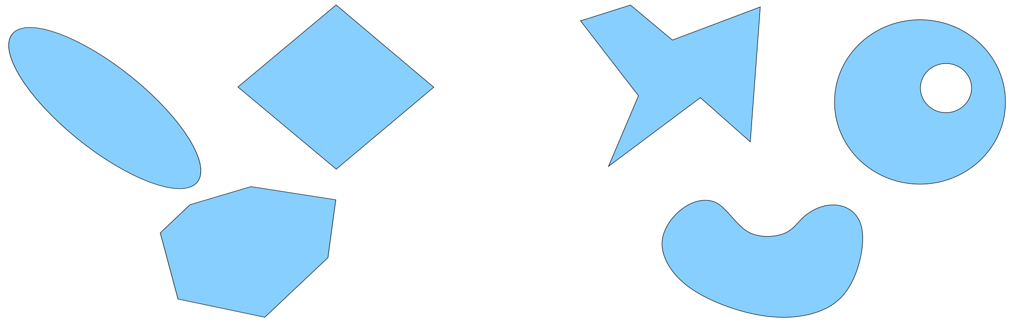

Figure 2.1: Examples of convex and nonconvex sets

Example 2.3 (Norm Balls). *Given any norm $\|\cdot\|$ in $\mathbb{R}^n$, the set*

$$\mathcal{B} := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

is a convex set. The set is called a norm ball *for the norm $\|\cdot\|$.*

We will often use norm balls defined by the p–norms $\|\cdot\|_p$, and so use the notation

$$\mathcal{B}_p^n := \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$$

for these sets. Some examples of convex and nonconvex sets are shown in Figure 2.1.

A particular class of convex set that we will encounter when dealing with certain optimization problems is the *convex cone*:

Definition 2.4 (Convex Cones). *A set $K \subseteq \mathbb{R}^n$ is a* convex cone *if it is convex and if, for every pair of points $x \in K$ and $y \in K$ and scalars τ_1 and τ_2,*

$$\tau_1 x + \tau_2 y \in C, \quad \textit{for all } (\tau_1, \tau_2) \geq 0. \tag{2.3}$$

A set K is called a *proper cone* if it is a closed convex cone with nonempty interior that does not contain any line, i.e. the only $x \in K$ also satisfying $-x \in K$ is the origin.

Example 2.5 (Semidefinite Cone). *The set of symmetric positive semidefinite matrices in $\mathbb{R}^{n\times n}$*

$$\left\{Q \in \mathbb{R}^{n\times n} \mid Q \succeq 0\right\}$$

is called the semidefinite cone*.*

Example 2.6 (Norm Cone). *Given any norm* $\|\cdot\|$ *in* $\mathbb{R}^n$*, the set*

$$\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \;\middle|\; \|x\| \leq t \right\} \subseteq \mathbb{R}^{n+1}$$

is called the norm cone *associated with the norm* $\|\cdot\|$.

Both the semidefinite cone and the norm cones for every norm are proper cones.

2.1.1 Convex Hulls

If a set C is not convex, then it can be 'convexified' by taking its *convex hull*, i.e. the smallest convex set containing C, denoted conv (C). An alternative (and equivalent) definition of the convex hull of a set C is the set of all *convex combinations* of points in C, where a convex combination of vectors $\{x_1, \ldots, x_n\}$ is any linear combination $\sum_{i=1}^n \lambda_i x_i$ with nonnegative weights λ_i satisfying $\sum_{i=1}^n \lambda_i = 1$.

A useful result for relating points in $\operatorname{conv}(C)$ to points in C is *Carathéodory's Theorem*:

Theorem 2.7 (Carathéodory's Theorem). *If the set* $C \in \mathbb{R}^n$ *is nonempty, then every point* $x \in \operatorname{conv}(C)$ *can be written as a convex combination of* $n+1$ *points (not necessarily different) in* C.

2.1.2 Operations on Convex Sets

Of special interest will be those operations that, when applied to a closed and convex set C, preserve convexity and closedness. We here outline a number of such operations that are most relevant to subsequent results:

Proposition 2.8 (Set Intersections).

i. *The intersection of an arbitrary collection of convex sets is convex.*

ii. *The intersection of an arbitrary collection of closed sets is closed.*

iii. *The intersection of a finite collection of polyhedral sets is polyhedral.*

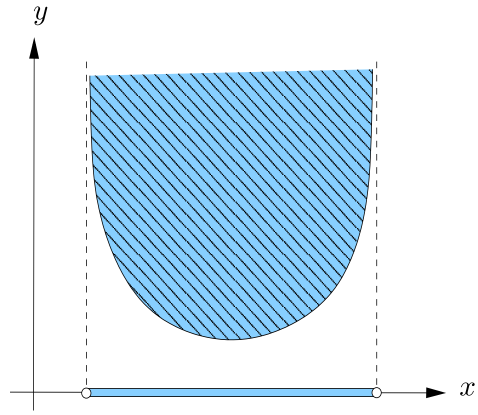

Figure 2.2: Loss of closedness under the linear mapping $f(x, y) = x$.

Proposition 2.9 (Set Addition).

i. *$C \oplus D$ is a convex set if C and D are convex.*

ii. *$C \oplus D$ is a closed set if C and D are closed and at least one of the sets is nonempty and bounded.*

iii. *$C \oplus D$ is a closed set if C and D are orthogonal, i.e. if $c \perp d$ for all $c \in C$ and all $d \in D$.*

Proposition 2.10 (Linear Mappings). *If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping and $C \subseteq \mathbb{R}^n$ is a convex set, then*

i. *$L(C)$ is a convex set.*

ii. *If C is also closed, then a sufficient condition for $L(C)$ to be closed is that there does not exist any nonzero y such that $L(y) = 0$ and such that $x + \lambda y \in C$ for all $x \in C$ and all $\lambda \geq 0$.*

iii. *If C is polyhedral, then $L(C)$ is polyhedral.*

It is important to note that it is possible for a set to lose closedness under a linear mapping if the conditions of Prop. 2.10(ii) do not hold, though these conditions are not necessary ones. An example of such a situation is shown in Figure 2.2.

2.1.3 Polar Sets and Dual Cones

In application to the robust control problems considered in subsequent chapters, a useful type of convex set is the *polar set* of a (not necessarily convex) set C:

Definition 2.11 (Polar Sets). *Given a set $C \subseteq \mathbb{R}^n$ with $0 \in C$, the* polar *of C is defined as:*

$$C^\circ := \{v \mid \langle v, x\rangle \leq 1, \ \forall x \in C\}.$$

Proposition 2.12 (Properties of Polar Sets). *Given a set $C \subseteq \mathbb{R}^n$ containing the origin, the following properties hold:*

i. *C° is closed and convex with $0 \in C^\circ$.*

ii. *$C^\circ = (\operatorname{conv} C)^\circ$.*

iii. *If C is closed and convex, then $(C^\circ)^\circ = C$.*

iv. *For $\lambda > 0$, $(\lambda C)^\circ = \lambda^{-1} C^\circ$.*

v. *If in addition $D \subseteq \mathbb{R}^n$, then $(C \bigcup D)^\circ = C^\circ \bigcap D^\circ$.*

vi. *For the p–norm unit ball $\mathcal{B}_p$,*

$$\begin{aligned}
(\mathcal{B}_p)^\circ &= \mathcal{B}_q, \quad 1 < p < \infty, \ p^{-1} + q^{-1} = 1 \\
(\mathcal{B}_1)^\circ &= \mathcal{B}_\infty, \quad (\mathcal{B}_\infty)^\circ = \mathcal{B}_1.
\end{aligned}$$

Several sets with their associated polars are shown in Figure 2.3.

If the set K is a convex cone, then K° is called the *polar cone* of K. Of greater use for our purposes, however, will be the *dual cone* of K:

Definition 2.13 (Dual Cone). *Given a convex cone $K \subseteq \mathbb{R}^n$, the* dual cone *of K is defined as:*

$$K^* := \{v \mid \langle v, x\rangle \geq 0, \ \forall x \in K\}.$$

Note that it is easily shown that if K is a convex cone, then $K^\circ = -K^*$.

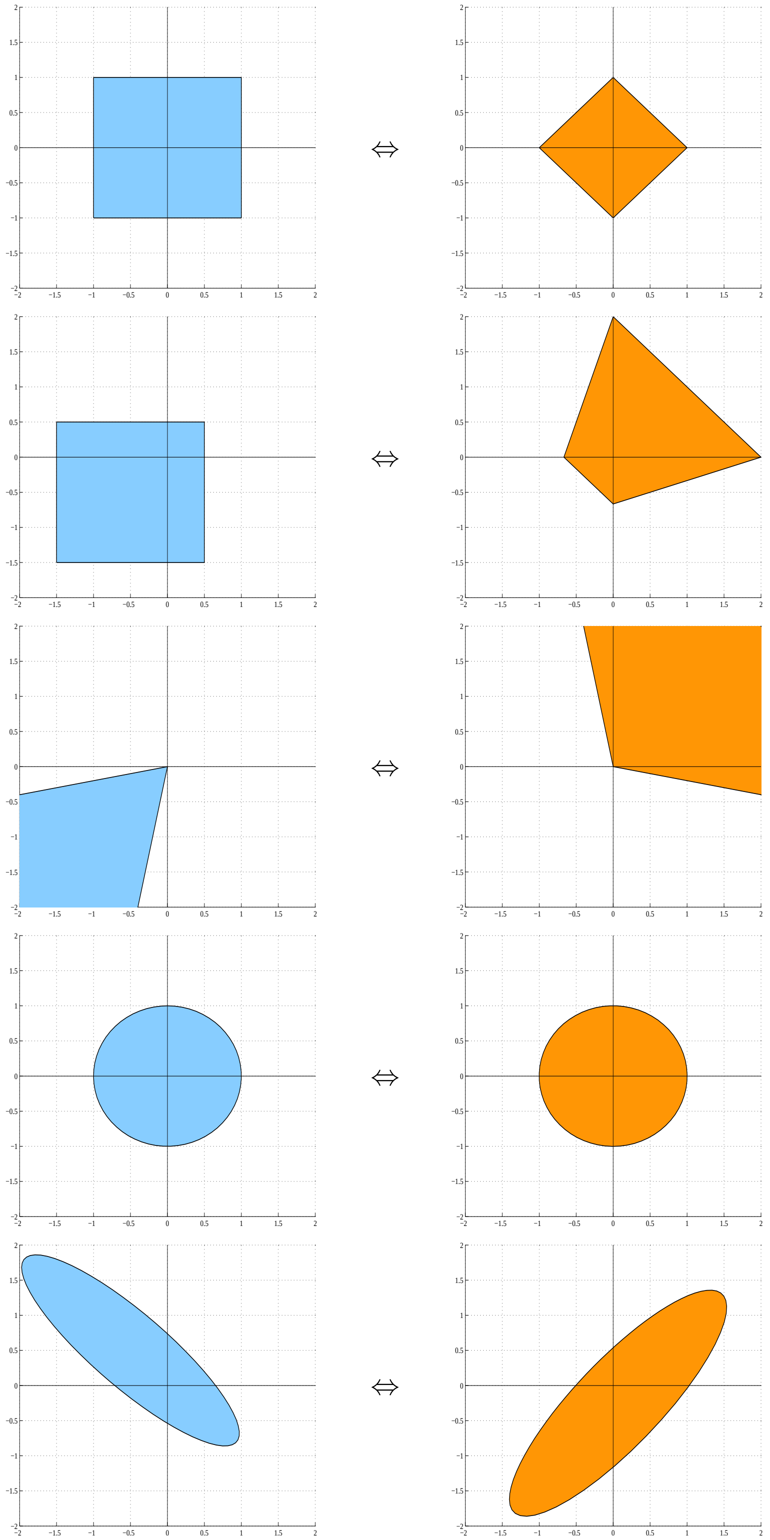

Figure 2.3: Convex Sets and Polar Sets

2.2 Convex Functions

Definition 2.14 (Convex Function). *A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a* convex function *relative to the set $C \subseteq \mathbb{R}^n$ if, for every pair of points $x \in C$ and $y \in C$, the following inequality holds:*

$$f\left((1-\tau)x + \tau y\right) \leq (1-\tau)f(x) + \tau f(y), \quad \textit{for all } \tau \in (0,1). \tag{2.4}$$

The function f is strictly convex *if the inequality* (2.4) *is strict.*

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called *concave* if $-f$ is convex.

Note that the function f in Def. 2.14 assigns a value on the extended real line $\bar{\mathbb{R}}$ (i.e. the set $\mathbb{R} \cup \{-\infty, \infty\}$) to every value in $\mathbb{R}^n$. The *effective domain* of f, denoted $\operatorname{dom} f$, is defined as

$$\operatorname{dom} f := \{x \mid f(x) < \infty\}.$$

The function f is *proper* if $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$ (i.e. $\operatorname{dom} f$ is nonempty) and the function never takes the value $-\infty$. Note that if a convex function $g : C \to \mathbb{R}$ is defined only on a convex set C, then one can identify it with a convex function f satisfying the conditions of Def. 2.14 by defining

$$f(x) := \begin{cases} g(x) & \text{if } x \in C \\ \infty & \text{if } x \notin C. \end{cases}$$

Proposition 2.15. *A convex function is continuous on the interior of its effective domain.*

Note, however, that a convex function is *not* guaranteed to be continuous at the boundary of its domain. For example, the convex function in Figure 2.4 is discontinuous at the point x, though it is upper semicontinuous everywhere.

2.2.1 Operations on Convex Functions

As in the case of convex sets, there are a number of operations which, when applied to convex functions, preserve convexity and semicontinuity. The most relevant of these to the work presented here are the following:

Proposition 2.16 (Addition and scaling). *Given convex functions $f_i : \mathbb{R}^n \to \mathbb{R}$ and scalars $\lambda_i \in \mathbb{R}$, the function $\sum_{i \in \mathcal{I}} \lambda_i f_i$ is convex if each $\lambda_i \geq 0$, and strictly convex if at least one function f_i is strictly convex with $\lambda_i > 0$.*

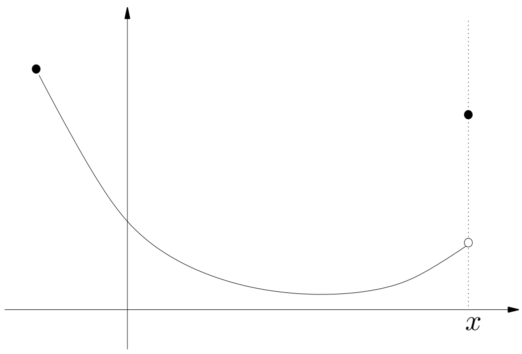

Figure 2.4: A discontinuous convex function.

Proposition 2.17 (Pointwise supremum).

i. The pointwise supremum of an arbitrary collection of convex functions is convex.

ii. The pointwise supremum of an arbitrary collection of lower semicontinuous functions is lower semicontinuous.

2.2.2 Support and Gauge Functions

Of particular interest in our development of robust control policies will be two convex functions defined in relation to a convex set C. These are the *support function* and the *gauge function*:

Definition 2.18 (Support and Gauge Functions). *Given a convex set $C \subseteq \mathbb{R}^n$, the* support function $\sigma_C : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is defined as:*

$$\sigma_C(x) := \sup_{y \in C}(x^\top y).$$

If $0 \in C$, the gauge function $\gamma_C : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is defined as:*

$$\gamma_C(x) := \inf \{\lambda \geq 0 \mid x \in \lambda C\}.$$

For a closed set C, the support and gauge functions have straightforward geometric interpretations; the set $\{y \mid x^\top y = \sigma_C(x)\}$ defines a plane tangent to C with normal vector x, while $\lambda = \gamma_C(x)$ is the smallest amount by which C can be scaled while guaranteeing that $x \in \lambda C$ (see Figure 2.5).

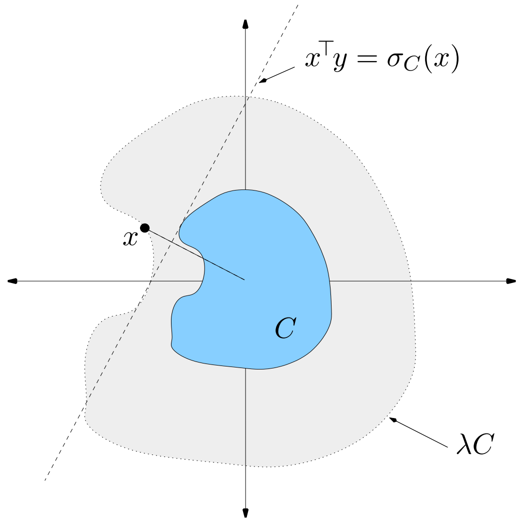

Figure 2.5: Support and Gauge Functions

The support and gauge functions of a set C have several properties that will be useful in subsequent sections; principal among these properties is their relation to one another with respect to the polar set C°:

Proposition 2.19 (Properties of Support and Gauge Functions). *If C is a closed and convex set with $0 \in C$, the following properties hold:*

i. $\sigma_C(\cdot) \geq 0$.

ii. $\sigma_C(\cdot) = \gamma_{C^\circ}(\cdot)$ *and* $\gamma_C(\cdot) = \sigma_{C^\circ}(\cdot)$.

iii. If C is also compact and symmetric (i.e. $x \in C$ implies $(-x) \in C$), then its gauge function γ_C corresponds to a norm. In particular, for the p–norm ball $\mathcal{B}_p$, $\gamma_{\mathcal{B}_p} = \|\cdot\|_p$ where $1 \leq p \leq \infty$.

2.3 Convex Optimization

A *convex optimization problem* is a minimization problem in the form

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \leq 0, \quad \forall i \in \{1, \ldots, p\} \\ & g_i(x) = 0, \quad \forall i \in \{1, \ldots, q\}, \end{aligned} \tag{2.5}$$

where each of the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and each of the functions $g_i : \mathbb{R}^n \to \mathbb{R}$ is affine. The function f_0 is referred to as the *cost* or *objective* function, while the remaining functions f_i and g_i are referred to as the problem *constraints*. Note that a variety of problems can be cast in the general framework of (2.5), e.g. the problem of maximizing a concave function can be recast as a convex optimization problem via a change of sign.

Occasionally it will be of interest to find a *feasible point* for the problem (2.5), i.e. one satisfying the constraints, without regard to optimality. The problem of finding such a point is easily written in the form (2.5) by defining an optimization problem with zero objective function:

$$\begin{array}{rll} \min\limits_{x} & 0 & \\ \text{subject to:} & f_i(x) \leq 0, & \forall i \in \{1,\ldots,p\} \\ & g_i(x) = 0, & \forall i \in \{1,\ldots,q\}. \end{array} \tag{2.6}$$

As a result, we will generally refer to the problem of finding a point that satisfies some set of convex constraints as a convex optimization problem, with the understanding that such a problem can be posed in the form (2.6).

In the remainder of this section we outline some of the most important classes of convex optimization problems.

2.3.1 Linear and Quadratic Programs

A *quadratic program* or *QP* is a problem in the form

$$\begin{array}{rll} \min\limits_{x} & c_0^\top x + \frac{1}{2}x^\top Q x & \\ \text{subject to:} & c_i^\top x \leq d_i, & \forall i \in \{1,\ldots,p\} \\ & a_i^\top x = b_i, & \forall i \in \{1,\ldots,q\}. \end{array} \tag{2.7}$$

The problem (2.7) is a convex optimization problem if the matrix $Q \succeq 0$, and we will generally assume that this is the case. If $Q = 0$, then the objective function in (2.7) is linear, and the problem is referred to as a *linear program* or *LP*.

Note that the problem of finding a point $x \in C$, where C is a polyhedral set defined as in (2.2), is a feasibility problem which can be cast as an LP.

2.3.2 Second-Order Cone Programs

A *second-order cone program* or *SOCP* is a convex optimization problem in the form

$$\begin{aligned} \min_{x} \quad & c_0^\top x \\ \text{subject to:} \quad & \begin{aligned} \|C_i x + d_i\|_2 \le a_i^\top x + b_i, \qquad & \forall i \in \{1,\dots,p\} \\ a_i^\top x = b_i, \qquad & \forall i \in \{1,\dots,q\}. \end{aligned} \end{aligned} \tag{2.8}$$

Note that if each of the matrices C_i and vectors d_i is zero, then (2.8) reduces to a linear program. More generally, *any* convex quadratic program can be written as a second-order cone program using appropriate variable transformations [LVBL98].

2.3.3 Linear Matrix Inequalities and Semidefinite Programs

A *linear matrix inequality* or *LMI* is a constraint in the form

$$F(x) := F_0 + \sum_{i=1}^{n} F_i x_i \preceq 0, \tag{2.9}$$

where each of the matrices F_i is symmetric and $x := (x_1, \dots, x_n)$. An LMI constraint in the form (2.9) is a convex constraint on x, i.e. $\{x \mid F(x) \preceq 0\}$ is a closed and convex set. The same result holds if the LMI constraint in (2.9) is strict, i.e. $F(x) \prec 0$, although in this case the set $\{x \mid F(x) \prec 0\}$ is open.

Note that inequalities involving matrix valued variables can be written in the standard form (2.9). For example, the problem of finding a solution to the discrete-time Lyapunov inequality

$$A^\top P A - P \prec 0 \tag{2.10}$$

can be written in the form (2.9) by defining an appropriate basis set $\{P_1, \dots P_m\}$ for the symmetric matrix variable $P \in \mathbb{R}^{n\times n}$, where $m = n(n+1)/2$, and defining $F_0 = 0$ and $F_i := A^\top P_i A - P_i$. Matrix inequalities such as (2.10) are therefore generally referred to as LMIs when it is clear from the context which matrices are intended as variables, with the understanding that they can be converted to the form (2.9) when necessary.

The following Lemma often proves useful in rewriting some matrix inequalities in LMI form:

Lemma 2.20 (Schur Complement). *If A, B and C are real matrices of compatible dimension with*

$$X := \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix},$$

then the following results hold:

i. $X \prec 0$ *if and only if* $C \prec 0$ *and* $(A - BC^{-1}B^\top) \prec 0$.

ii. *If* $C \prec 0$, *then* $X \preceq 0$ *if and only if* $A - BC^{-1}B^\top \preceq 0$.

iii. $X \preceq 0$ *if and only if* $C \preceq 0$, $(A - BC^\dagger B^\top) \preceq 0$ *and* $B(I - CC^\dagger) = 0$.

iv. X *is invertible if both* C *and* $(A - BC^{-1}B^\top)$ *are invertible.*

A *semidefinite program* or *SDP* is a problem in the form

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{subject to:} \quad & F_0 + \sum_{i=1}^{n} F_i x_i \preceq 0. \end{aligned} \tag{2.11}$$

Note that multiple LMI constraints are easily handled via appropriate redefinition of the matrices F_i. Both QPs and SOCPs can be considered subclasses of semidefinite programming problems, since both problem types can be written in the general form (2.11). However, it is generally better to solve a problem as an SOCP or QP if it is possible to do so, since stronger computational complexity guarantees are generally possible for QPs and SOCPs than for SDPs [LVBL98], and computational methods and software for these problems are more mature.

2.3.4 Generalized Inequalities and Conic Programs

Given a proper cone $K \subseteq \mathbb{R}^n$, we can define a *generalized inequality* as a partial ordering on $\mathbb{R}^n$ using K:

$$a \preceq_K b \quad \Leftrightarrow \quad b - a \in K \tag{2.12a}$$

$$a \prec_K b \quad \Leftrightarrow \quad b - a \in \operatorname{int}(K). \tag{2.12b}$$

Analogous expressions are used to define the inequalities $\succeq_K$ and $\succ_K$. Note that the familiar element-wise inequality $\leq$ on $\mathbb{R}^n$ is equivalent to (2.12) with K equal to the nonnegative orthant $\mathbb{R}^n_+$. The usual inequality for matrices $A \succeq B$ is also equivalent to (2.12) with K equal to the positive semidefinite cone.

The simplest form of convex optimization problem involving generalized inequalities is one with a single affine inequality:

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{subject to:} \quad & \begin{aligned} & Cx \preceq_K d \\ & Ax = b. \end{aligned} \end{aligned} \tag{2.13}$$

A problem in this form is called a *cone program.* The problem formulations for LPs, SOCPs and SDPs can all be treated as special cases of cone programs.

2.4 Parametric Minimization

Given a convex function $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$, we will often want to minimize the function $f(x, u)$ with respect to the variable u only. Such a problem is referred to as a *parametric minimization* problem. This section presents some of the characteristics of the function $\min_u f(x, u)$ with respect to x, as well as the properties of the minimizers of this function.

In the control applications to be presented in subsequent chapters, the variable x will generally represent the state of a dynamic system, and the variables u will represent some collection of parameters determining a control strategy for the system. Of particular interest are conditions guaranteeing that the function $\inf_u f(x, u)$ is convex and lower semicontinuous with respect to x, so that the resulting function can be employed as a Lyapunov function in establishing results on stability. Additionally, we are interested in conditions ensuring that the sets $\operatorname{argmin}_u f(x, u)$ are single-valued and continuous with respect to x, so that control laws defined by these minimizers can be guaranteed to have these properties.

We first require some preliminary definitions and results:

Definition 2.21 (Uniform Level Boundedness [RW98, Defn. 1.16]). *A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ taking values $f(x, u)$ is said to be level bounded in u locally uniformly in x if, for each $\bar{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, there exists a set $V \subseteq \mathbb{R}^n$ and a bounded set $B \subset \mathbb{R}^n$ such that $\bar{x} \in \operatorname{int}(V)$ and*

$$\{u \mid f(x, u) \leq \alpha\} \subseteq B, \quad \textit{for all } x \in V.$$

The conditions of Def. 2.21 are more general than is strictly necessary for our purposes. We will therefore employ the following result establishing sufficient conditions for a function to meet these requirements.

Proposition 2.22. *A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ taking values $f(x,u)$ is level bounded in u locally uniformly in x if there exists a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ defined as*

$$g(x,u) := \|Cx + Du\|^c ,$$

such that $g(x,u) \le f(x,u)$ for all x and u, where $\|\cdot\|$ is any norm, D is full column rank and $c > 0$.

Proof. We first show that $g(x,u)$ is level bounded in u locally uniformly in x. Choose any $\bar{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, and define $V = \{\bar{x} + \delta x \mid \|\delta x\| \le 1\}$ so that $\bar{x} \in \operatorname{int}(V)$. From Definition 2.21, it is sufficient to show that

$$\bigcup_{x \in V} \{u \mid \|Cx + Du\|^c \le \alpha\} \subseteq B$$

for some bounded set B. Restricting α and $\alpha^{\frac{1}{c}}$ to be positive without loss of generality,

$$\begin{aligned}
\bigcup_{x \in V} \{u \mid \|Cx + Du\|^c \le \alpha\} &= \bigcup_{x \in V} \left\{u \;\middle|\; \|Cx + Du\| \le \alpha^{\frac{1}{c}} \right\} \\
&\subseteq \bigcup_{x \in V} \left\{u \;\middle|\; \|Du\| \le \alpha^{\frac{1}{c}} + \|Cx\| \right\} \\
&\subseteq \left\{u \;\middle|\; \|Du\| \le \alpha^{\frac{1}{c}} + \|C\bar{x}\| + \sup_{\|\delta x\| \le 1} \|C\delta x\| \right\},
\end{aligned} \tag{2.14}$$

where the latter two expressions come from straightforward application of the properties of vector norms. The set in the right hand side of (2.14) is bounded since the matrix D is full column rank [HJ85, Thm. 5.3.2 & Cor. 5.4.8], establishing the result for $g(x,u)$. The result follows for $f(x,u)$ since the inequality $g(x,u) \le f(x,u)$ guarantees

$$\{u \mid f(x,u) \le \alpha\} \subseteq \{u \mid g(x,u) \le \alpha\}$$

for all $x \in V$. $\square$

Proposition 2.23 (Parametric Optimization). *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ be a convex, proper and lower semicontinuous function and define*

$$p(x) := \inf_u f(x,u), \qquad P(x) := \operatorname*{argmin}_u f(x,u).$$

Properties of p

i. The function p is convex on $\mathbb{R}^n$.

ii. The function p is also lower semicontinuous and proper on $\mathbb{R}^n$ if either;

(a) $f(x,u)$ is level bounded in u locally uniformly in x, or

(b) for some $\bar{x} \in \mathbb{R}^n$ the set $P(\bar{x})$ is nonempty and bounded.

Properties of P

If $f(x,u)$ is also level bounded in u locally uniformly in x then;

iii. For each $x \in \operatorname{dom}(p)$, the set $P(x)$ is nonempty, convex and compact. If $x \notin \operatorname{dom}(p)$, then $P(x) = \emptyset$.

iv. If in addition $f(x,u)$ is strictly convex in u, then P is single valued on $\operatorname{dom}(P)$ *and continuous on* $\operatorname{int}(\operatorname{dom}(P))$.

Proof. The proposition is a combination of various standard results in convex analysis. Convexity of f is sufficient to establish convexity of the function p and of the set $P(x)$ for each x [RW98, Prop. 2.22] in (i) and (iii). The remainder of the results in (iii) rely on f being lower semicontinuous and proper with $f(x,u)$ level bounded in u locally uniformly in x [RW98, Prop. 1.17]. The alternative results (iia) and (iib) come from [RW98, Prop. 1.17] and [RW98, Cor. 3.32] respectively. Part (iv) is from [RW98, Thm. 3.31] and [RW98, Cor. 7.43].

Some care is required when considering the continuity or convexity of functions that result from convex parametric minimization – recall that not all convex functions are continuous (cf. Figure 2.4). When minimizing a convex function over a subset of its variables, it is *not* the case that continuity will be preserved, even if the original function is strictly continuous on its effective domain. The following example illustrates this point:

Example 2.24. *Define the set* $C \subset \mathbb{R}^2 \times \mathbb{R}$ *as*

$$C := \left\{ (x, z) \;\middle|\; 0 \le z \le 1,\ (x_1 - z)^2 + x_2^2 \le 1 \right\}$$

and define the function $f : C \to \mathbb{R}$ *as* $f(x, z) = z$*. Then*

$$p(x) := \min_z f(x, z)$$

is lower semicontinuous everywhere on C *by virtue of Prop. 2.23, but is discontinuous at the point* $x = 0$*. See Figure 2.6.*

Note that the example given in Figure 2.4 is of an upper semicontinuous convex function, while Example 2.24 yields the lower semicontinuous convex function shown in Figure 2.6. In agreement with Prop. 2.15, both functions are continuous on the interior of their domains.

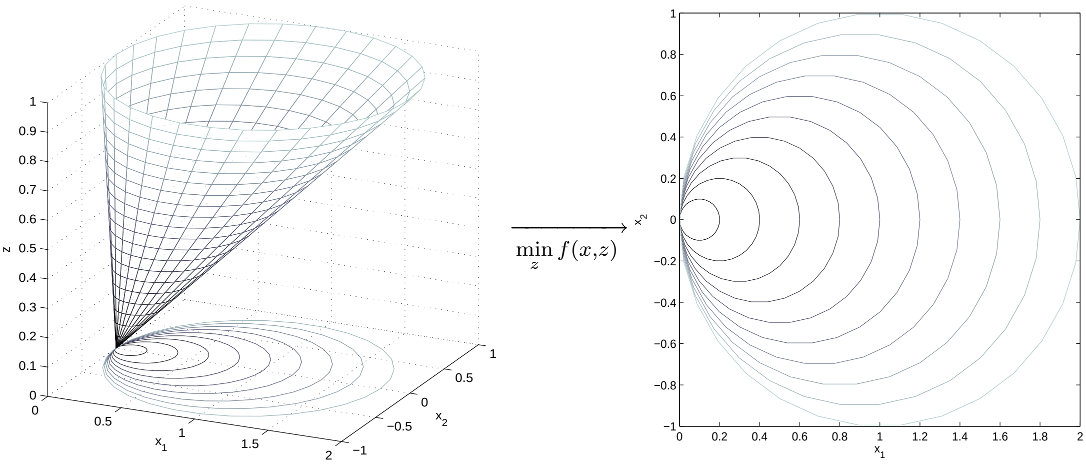

Figure 2.6: Loss of Continuity in Convex Parametric Minimization

3.1 Problem Definition

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + Gw, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^l$ is an external disturbance.

The current and future values of the disturbance are unknown and may change unpredictably from one time instant to the next, but are assumed to be contained in a known set W. The actual values of the state, input and disturbance at time instant k will be denoted by $x(k)$, $u(k)$ and $w(k)$, respectively.

Where it is clear from the context, x, u and w will be used to denote the current value of the state, input and disturbance (note that since the system is time-invariant, the current time can always be taken as zero). For the majority of this dissertation it will be assumed that, at each sample instant, a measurement of the state x is available, though this assumption will be relaxed in Chapter 8.

The system is subject to mixed constraints on the state and input, so that the system must satisfy

$$(x, u) \in Z, \tag{3.2}$$

where $Z \subset \mathbb{R}^n \times \mathbb{R}^m$. The constraints defining the set Z may arise from either hard physical constraints (e.g. actuator or other physical plant limitations) or from other design objectives based on safety or performance considerations. In either case, a design goal is to guarantee that the state and input of the closed-loop system remain in Z for all time and for all disturbance sequences generated from the set W.

Since the disturbance does not necessarily decay to zero, it may not be possible to drive the state of the system to the origin. Instead, the best one can hope for is to drive the state of the system to a target/terminal constraint set $X_f \subset \mathbb{R}^n$. In this chapter, it will be shown how, in conjunction with appropriately defined finite horizon control policies, the set X_f can be used as a target set in time-optimal control or as a terminal constraint in a receding horizon controller with guaranteed invariance properties.

We will make use of the following assumptions about this system throughout:

A3.1 (Standing Assumptions)

i. The pair (A, B) is stabilizable.

ii. The matrix G has full column rank.

iii. The state and input constraint set $Z \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is closed, convex, contains the origin in its interior and is bounded in the inputs, i.e. there exists a bounded set B such that $Z \subseteq \mathbb{R}^n \times B$.

iv. The terminal constraint set $X_f \subseteq \mathbb{R}^n$ is closed, convex and contains the origin in its interior.

v. The disturbance set W is compact and contains the origin in its interior.

Note that the assumption that G is full column rank and that the origin is in the interior of W are not unnecessarily restrictive; in cases where W contains the origin in its relative interior[1] and/or G is *not* full column rank, one may redefine W and G suitably such that the stated assumptions hold.

3.1.1 Notation

In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and

[1] The *relative interior* of a convex set C is the interior of C with respect to the smallest affine subset containing C. See, for example, [Roc70, Sect. 5].

disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{lN}$, respectively, as

$$\mathbf{x} := \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \ \mathbf{u} := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \ \mathbf{w} := \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} \tag{3.3}$$

where $x_0 =: x$ denotes the current measured value of the state and $x_{i+1} := Ax_i + Bu_i + Gw_i$, $i = 0, \ldots, N-1$ denotes the prediction of the state after i time instants.

Let the set $\mathcal{W} := W^N := W \times \cdots \times W$, so that $\mathbf{w} \in \mathcal{W}$, and define a closed and convex set $\mathcal{Z}$, appropriately constructed from Z and X_f, such that the constraints to be satisfied are equivalent to $(\mathbf{x}, \mathbf{u}) \in \mathcal{Z}$, i.e.

$$\mathcal{Z} := \left\{ (\mathbf{x}, \mathbf{u}) \;\middle|\; \begin{array}{c} (x_i, u_i) \in Z, \ \forall i \in \mathbb{Z}_{[0,N-1]} \\ x_N \in X_f \end{array} \right\}. \tag{3.4}$$

Finally, define the matrices $\mathbf{A} \in \mathbb{R}^{n(N+1)\times n}$ and $\mathbf{E} \in \mathbb{R}^{n(N+1)\times nN}$ as

$$\mathbf{A} := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{E} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ A & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I_n \end{bmatrix},$$

and the matrices $\mathcal{B} \in \mathbb{R}^{nN\times mN}$, $\mathcal{G} \in \mathbb{R}^{nN\times lN}$, $\mathbf{B} \in \mathbb{R}^{n(N+1)\times mN}$ and $\mathbf{G} \in \mathbb{R}^{n(N+1)\times lN}$ as

$$\mathcal{B} := I_N \otimes B, \quad \mathcal{G} := I_N \otimes G, \quad \mathbf{B} := \mathbf{E}\mathcal{B}, \quad \mathbf{G} := \mathbf{E}\mathcal{G},$$

respectively. Using these definitions in conjunction with the system dynamics (3.1), the state sequence $\mathbf{x}$ can then be written in vectorized form as

$$\begin{aligned} \mathbf{x} &= \mathbf{A}x + \mathbf{E}\mathcal{B}\mathbf{u} + \mathbf{E}\mathcal{G}\mathbf{w} \\ &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w}. \end{aligned}$$

3.2 A State Feedback Policy Parameterization

Finding an *arbitrary* finite horizon control policy that satisfies the constraints $\mathcal{Z}$ for all admissible disturbance sequences generated from $\mathcal{W}$ is extremely difficult in general. Current proposals for defining such policies generally require solution via robust dynamic programming [MRVK06] or very large scale optimization problems [SM98]. As a result, we will find it convenient to restrict the class of control policies considered to those that are *affine* in the sequence of states, i.e. those in the form[2]

$$u_i = g_i + \sum_{j=0}^{i} K_{i,j} x_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{3.5}$$

where each $K_{i,j} \in \mathbb{R}^{m \times n}$ and $g_i \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{K} \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{K} := \begin{bmatrix} K_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ K_{N-1,0} & \cdots & K_{N-1,N-1} & 0 \end{bmatrix}, \quad \mathbf{g} := \begin{bmatrix} g_0 \\ \vdots \\ g_{N-1}, \end{bmatrix}, \tag{3.6}$$

so that the input sequence (3.5) can be written in vectorized form as

$$\mathbf{u} = \mathbf{K}\mathbf{x} + \mathbf{g}. \tag{3.7}$$

For a given initial state x, we say that the pair $(\mathbf{K}, \mathbf{g})$ is admissible if the control policy (3.5) guarantees that for all allowable disturbance sequences of length N, the constraints Z are satisfied over the horizon $i = 0, \ldots, N-1$, and that the state is in the target set X_f at the end of the horizon. More precisely, the set of admissible $(\mathbf{K}, \mathbf{g})$ is defined as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{K}, \mathbf{g}) \;\middle|\; \begin{array}{r} (\mathbf{K}, \mathbf{g}) \text{ satisfies (3.6)}, \; x_0 = x \\ x_{i+1} = Ax_i + Bu_i + Gw_i \\ u_i = g_i + \sum_{j=0}^{i} K_{i,j} x_j \\ (x_i, u_i) \in Z, \; x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]}, \; \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}, \tag{3.8}$$

[2] Since the current state x will be assumed known, it is possible to set $K_{0,0} = 0$ without loss of generality. However, presentation of the results in this chapter is somewhat simplified if we do not impose this constraint.

or, in more compact form, as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{K}, \mathbf{g}) \;\middle|\; \begin{aligned} (\mathbf{K}, \mathbf{g}) &\text{ satisfies (3.6)} \\ \mathbf{x} &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} &= \mathbf{K}\mathbf{x} + \mathbf{g} \\ (\mathbf{x}, \mathbf{u}) &\in \mathcal{Z}, \ \forall \mathbf{w} \in \mathcal{W} \end{aligned} \right\}. \tag{3.9}$$

The set of initial states x for which an admissible control policy of the form (3.5) exists is defined as

$$X_N^{sf} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{sf}(x) \neq \emptyset \right\}. \tag{3.10}$$

It is critical to note that it may not be possible to select a single policy pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$ such that it is admissible for all $x \in X_N^{sf}$. Indeed, it is possible to find examples where there exists a pair $(x, \tilde{x}) \in X_N^{sf} \times X_N^{sf}$ such that $\Pi_N^{sf}(x) \bigcap \Pi_N^{sf}(\tilde{x}) = \emptyset$. For problems of non-trivial size, it is therefore necessary to calculate an admissible pair $(\mathbf{K}, \mathbf{g})$ on-line, given a measurement of the current state x.

Once an admissible control policy is computed for the current state, there are many ways in which it can be applied to the system; time-varying, time-optimal and receding horizon implementations are the most common, and are considered in detail in Section 3.6.

It is important to emphasize that, due to the dependence of (3.9) on the current state x, the implemented control policy will, in general, be a *nonlinear* function of the state x, even though it may have been defined in terms of the class of affine state feedback policies of the form (3.5).

Remark 3.1. *Note that the state feedback policy* (3.5) *subsumes the well-known class of "pre-stabilizing" control policies [Bem98, LK99, CRZ01], in which the control policy takes the form $u_i = c_i + Kx_i$, where K is* fixed. *In such schemes, the on-line computation is limited to finding an admissible perturbation sequence $\{c_i\}_{i=0}^{N-1}$. It can also be shown to subsume 'tube'-based schemes such as [MSR05] based on linear feedback.*

3.2.1 Nonconvexity in Affine State Feedback Policies

Finding an admissible policy pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$, given the current state x, has been believed to be a very difficult problem. This is due to the nonlinear relationship between $\mathbf{x}$ and $\mathbf{u}$ in (3.9), which results in the following property:

Proposition 3.2 (Nonconvexity). *For a given state $x \in X_N^{sf}$, the set of admissible affine state feedback control policies $\Pi_N^{sf}(x)$ is nonconvex, in general.*

The truth of this statement is easily verified by considering the following example:

Example 3.3 (Nonconvexity in affine state feedback policies). *Consider the SISO system $x^+ = x + u + w$ with initial state $x_0 = 0$, input constraint $|u| \leq 3$, bounded disturbances $|w| \leq 1$ and a planning horizon of $N = 3$. Consider a control policy of the form* (3.5) *with $\mathbf{g} = 0$ and $K_{2,1} = 0$, so that $u_0 = 0$ and*

$$u_1 = K_{1,1} w_0 \tag{3.11}$$

$$u_2 = [K_{2,2}(1 + K_{1,1})]\, w_0 + K_{2,2} w_1 \tag{3.12}$$

In order to satisfy the input constraints for all allowable disturbance sequences, the controls u_i must satisfy

$$|u_i| \leq 3,\ i = 1, 2,\ \forall \mathbf{w} \in \mathcal{W} \tag{3.13}$$

or, equivalently,

$$\max_{\mathbf{w} \in \mathcal{W}} |u_i| \leq 3,\ i = 1, 2. \tag{3.14}$$

Since the constraints on the components of $\mathbf{w}$ are independent, the input constraints are satisfied for all $\mathbf{w} \in \mathcal{W}$ if and only if

$$|K_{1,1}| \leq 3 \tag{3.15}$$

$$|K_{2,2}(1 + K_{1,1})| + |K_{2,2}| \leq 3. \tag{3.16}$$

It is straightforward to verify that the set of gains $(K_{1,1}, K_{2,2})$ which satisfy these constraints is nonconvex; the pairs $(-3, 1)$ and $(-1, 3)$ are acceptable, while the pair $(-2, 2)$ is not. The set of admissible values for $(K_{1,1}, K_{2,2})$, representing the intersection of the set $\Pi_N^{sf}(0)$ with the plane $\mathbf{g} = 0$, $K_{1,2} = 0$, is shown in Figure 3.1.

It is surprising to note that, though the set $\Pi_N^{sf}(x)$ may be nonconvex, the set X_N^{sf} is *always* convex. Proof of this is deferred until Section 3.5. Additionally, despite the fact that $\Pi_N^{sf}(x)$ may be nonconvex, we will show that one can still find an admissible $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$ by solving an equivalent convex optimization problem.

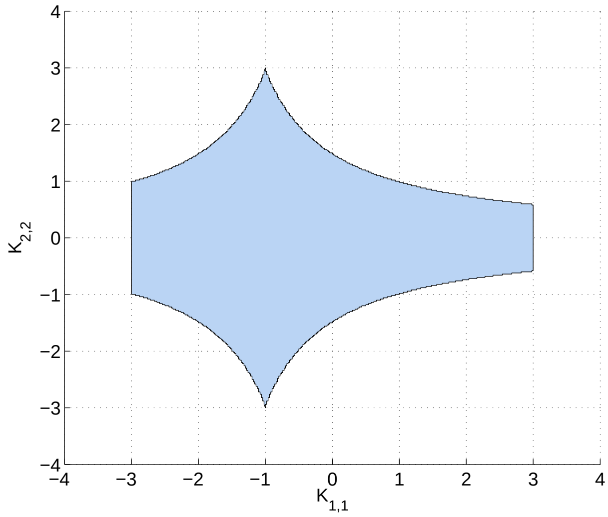

Figure 3.1: Nonconvexity of $\Pi_N^{sf}(0)$ in Example 3.3

3.3 A Disturbance Feedback Policy Parameterization

As a result of nonconvexity in the state feedback formulation of the previous section, we seek an alternative scheme for which the class of feedback policies is convex. One alternative to (3.5) is to parameterize the control policy as an affine function of the sequence of past *disturbances*, so that

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} G w_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{3.17}$$

where each $M_{i,j} \in \mathbb{R}^{m \times n}$ and $v_i \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance inputs Gw_j are easily calculated as the difference between the predicted and actual states at each step, i.e.

$$Gw_j = x_{j+1} - Ax_j - Bu_j, \quad \forall j \in \mathbb{Z}_{[0,N-1]}. \tag{3.18}$$

The above parameterization appears to have originally been suggested some time ago within the context of stochastic programs with recourse [GW74]. More recently, it has been revisited as a means for finding solutions to a class of robust optimization problems, called affinely adjustable robust counterpart (AARC) problems [BTGGN04, BBN06, Gus02], and robust model predictive control problems [L03a, vHB02].

For notational convenience, define the vector $\mathbf{v} \in \mathbb{R}^{mN}$ and the strictly block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} v_0 \\ \vdots \\ \vdots \\ v_{N-1} \end{bmatrix}, \tag{3.19}$$

so that the input sequence (3.17) can be written in vectorized form as

$$\mathbf{u} = \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v}. \tag{3.20}$$

In a manner similar to (3.8), define the set of admissible $(\mathbf{M}, \mathbf{v})$ as

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)}, \ x_0 = x \\ x_{i+1} = Ax_i + Bu_i + Gw_i \\ u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} G w_j \\ (x_i, u_i) \in Z, \ x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]}, \ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\} \tag{3.21}$$

or, in more compact form, as

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} = \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \\ (\mathbf{x}, \mathbf{u}) \in \mathcal{Z}, \ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{3.22}$$

Define the set of initial states x for which an admissible control policy of the form (3.17) exists as

$$X_N^{df} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{df}(x) \neq \emptyset \right\}. \tag{3.23}$$

Note that the state and disturbance feedback parameterizations (3.5) and (3.17) are qualitatively similar; in Section 3.5 we will show that they are actually *equivalent*. However, the two parameterizations have slightly different interpretations, and we will require *both* in order to establish various geometric and system-theoretic properties of receding horizon control laws based on these policies.

In the next section, we discuss the main benefit of adopting the parameterization (3.17); namely, that an admissible affine disturbance feedback policy can be found by solving a convex and tractable optimization problem.

3.4 Convexity and Closedness

In this section we establish convexity and closedness of the sets $\Pi_N^{df}(x)$ and X_N^{df}. These properties will make the disturbance feedback parameterization (3.17) an attractive alternative to the state feedback parameterization (3.5). Define the set

$$\mathcal{C}_N := \bigcap_{\mathbf{w}\in\mathcal{W}} \left\{ (x,\mathbf{M},\mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M},\mathbf{v}) &\text{ satisfies (3.19)} \\ \mathbf{x} &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} &= \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \\ (\mathbf{x},\mathbf{u}) &\in \mathcal{Z} \end{aligned} \right\}. \tag{3.24}$$

This set is closed and convex since it is the intersection of closed and convex sets. The set X_N^{df} can then be defined as a linear mapping of this set. However, the set $\mathcal{C}_N$ is not guaranteed to be compact when G is not full row rank or when the state and input constraints $\mathcal{Z}$ are not bounded in the state dimension, so care must be taken to ensure that closedness is preserved[3] when treating X_N^{df} as a linear mapping of $\mathcal{C}_N$.

Lemma 3.4. *Given a linear mapping $L : \mathcal{C}_N \to \mathbb{R}^s$, if $L(x,\mathbf{M},\mathbf{v}) \neq 0$ for every $x \neq 0$, then the set $L(\mathcal{C}_N)$ is closed and convex.*

Proof. A linear map of a convex set is always convex (Prop. 2.10). To ensure closedness, define the set $\mathcal{M}$ and its orthogonal complement $\mathcal{M}_\perp$ as

$$\mathcal{M} := \{\mathbf{M} \mid \mathbf{M} \text{ satisfies (3.19)},\; \mathbf{M}y = 0,\; \forall y \perp \mathcal{R}(\mathcal{G})\} \tag{3.25a}$$

$$\mathcal{M}_\perp := \{\mathbf{M} \mid \mathbf{M} \text{ satisfies (3.19)},\; \mathbf{M}y = 0,\; \forall y \in \mathcal{R}(\mathcal{G})\}. \tag{3.25b}$$

Both of these sets are subspaces, with $\mathcal{M} \cup \mathcal{M}_\perp$ equal to the set of all matrices satisfying (3.19). Define the set

$$\tilde{\mathcal{C}}_N := \mathcal{C}_N \cap (\mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^{mN}), \tag{3.26}$$

[3] Recall that, in general, a linear mapping of a closed but unbounded set is *not* guaranteed to be closed. See Figure 2.2.

such that $\mathcal{C}_N$ and $\tilde{\mathcal{C}}_N$ differ only according to their inclusion of the subspace $\mathcal{M}_\perp$. Since $\mathcal{M}_\perp$ is defined such that the nullspace of every element of $\mathcal{M}_\perp$ contains the set $\mathcal{G}\mathcal{W}$, it follows that $\mathcal{C}_N$ in (3.24) can alternatively be written as

$$\mathcal{C}_N = \tilde{\mathcal{C}}_N \oplus (\{0\} \times \mathcal{M}_\perp \times \{0\}). \tag{3.27}$$

Recalling that the state and input constraints Z are assumed bounded in the inputs and the disturbance set W is assumed to contain the origin in its interior in **A3.1**, the set $\mathcal{M}$ is also bounded, since $\max_{\mathbf{w}\in\mathcal{W}} \|\mathbf{M}\mathcal{G}\mathbf{w}\| > 0$ for any nonzero $\mathbf{M} \in \mathcal{M}$. The set $\tilde{\mathcal{C}}_N$ is therefore bounded in policies, i.e. there exist bounded sets $B_1 \subseteq \mathbb{R}^{mN\times nN}$ and $B_2 \subseteq \mathbb{R}^{mN}$ such that $\tilde{\mathcal{C}}_N \subseteq (\mathbb{R}^n \times B_1 \times B_2)$. Then

$$L(\mathcal{C}_N) = L\left(\tilde{\mathcal{C}}_N \oplus (\{0\} \times \mathcal{M}_\perp \times \{0\})\right) \tag{3.28}$$

$$= L\left(\tilde{\mathcal{C}}_N\right) \oplus L\Big(\{0\} \times \mathcal{M}_\perp \times \{0\}\Big), \tag{3.29}$$

where the latter relation follows since linear mappings are distributive with respect to set addition. The set $L(\tilde{\mathcal{C}}_N)$ is closed since $\tilde{\mathcal{C}}_N$ is compact in policies (Prop. 2.10(ii)), so that (3.29) is the sum of closed and orthogonal sets, therefore also closed (Prop. 2.9). $\square$

We can now state the main result of this section:

Theorem 3.5 (Convexity). *For every state $x \in X_N^{df}$, the set of admissible affine disturbance feedback policies $\Pi_N^{df}(x)$ is closed and convex. Furthermore, the set of states X_N^{df}, for which at least one admissible affine disturbance feedback policy exists, is also closed and convex.*

Proof. The set $\Pi_N^{df}(x)$ represents a planar 'slice' through $\mathcal{C}_N$, and can be written as

$$\Pi_N^{df}(x) := \bigcap_{\mathbf{w}\in\mathcal{W}} \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M},\mathbf{v}) &\text{ satisfies (3.19)} \\ \mathbf{x} &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} &= \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \\ (\mathbf{x},\mathbf{u}) &\in \mathcal{Z} \end{aligned} \right\}.$$

Like the set $\mathcal{C}_N$, the set $\Pi_N^{df}(x)$ is closed and convex since it is the intersection of closed and convex sets. The projection $\mathcal{C}_N \mapsto X_N^{df}$ is a linear mapping satisfying the conditions of Lem. 3.4, so X_N^{df} is also closed and convex. $\square$

Remark 3.6. *Convexity and closedness of the set $\mathcal{C}_N$ will be important considerations when establishing the properties of optimal finite horizon control policies in Chapters 4 and 5. There we will need to minimize a convex and continuous objective function $(x, \mathbf{M}, \mathbf{v}) \mapsto J_N(x, \mathbf{M}, \mathbf{v})$ over an effective domain restricted to $\mathcal{C}_N$. If $\mathcal{C}_N$ is closed, then this function is lower semicontinuous and the results of Section 2.4 will apply.*

3.4.1 Handling Nonconvex Disturbance Sets

Recall that in **A3.1**, the set W was assumed to be convex and compact, with $0 \in \operatorname{int} W$. We show here that if W (equivalently $\mathcal{W}$) is nonconvex, it can be replaced with its convex hull without loss of generality.

Proposition 3.7 (Convexification of $\mathcal{W}$). *The sets $\Pi_N^{df}(x)$ and X_N^{df}, defined in* (3.21) *and* (3.23) *respectively, are unchanged if $\mathcal{W}$ is replaced with its convex hull.*

Proof. For a given state x, define $\bar{\Pi}_N^{df}(x)$ to be the set of policies guaranteeing constraint satisfaction for all $\mathcal{W} \in \operatorname{conv} \mathcal{W}$, i.e.

$$\bar{\Pi}_N^{df}(x) := \bigcap_{\mathbf{w} \in \operatorname{conv} \mathcal{W}} \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) & \text{ satisfies (3.19)} \\ \mathbf{x} &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} &= \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \\ (\mathbf{x}, \mathbf{u}) &\in \mathcal{Z} \end{aligned} \right\}$$

Since $\mathcal{W} \subset \operatorname{conv} \mathcal{W}$, $\bar{\Pi}_N^{df}(x) \subset \Pi_N^{df}(x)$ follows trivially. We show here that $\bar{\Pi}_N^{df}(x) \supset \Pi_N^{df}(x)$ as well, so that $\bar{\Pi}_N^{df}(x) = \Pi_N^{df}(x)$.

From Carathéodory's Theorem (Thm. 2.7), every element $\bar{\mathbf{w}} \in \operatorname{conv} \mathcal{W} \subseteq \mathbb{R}^{lN}$ can be written as a convex combination of $(lN + 1)$ points (not necessarily all different) in $\mathcal{W}$. Denote these points $\mathbf{w}_i$, so that $\bar{\mathbf{w}} = \sum_{i=1}^{(lN+1)} \lambda_i \mathbf{w}_i$, where each $\lambda_i \geq 0$ and $\sum_{i=1}^{(lN+1)} \lambda_i = 1$. Consider any control policy pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$, and define $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) := \sum_{i=1}^{(lN+1)} \lambda_i (\mathbf{x}_i, \mathbf{u}_i)$, where

$$\begin{aligned} \mathbf{x}_i &:= \mathbf{A}x + \mathbf{B}\mathbf{v} + (\mathbf{B}\mathbf{M}\mathcal{G} + \mathbf{G})\, \mathbf{w}_i \\ \mathbf{u}_i &:= \mathbf{M}\mathcal{G}\mathbf{w}_i + \mathbf{v}. \end{aligned}$$

From the definition of $\Pi_N^{df}(x)$, every pair $(\mathbf{x}_i, \mathbf{u}_i) \in \mathcal{Z}$. Since the set $\mathcal{Z}$ is convex and $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is a convex combination of the points $(\mathbf{x}_i, \mathbf{u}_i)$, it follows that $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathcal{Z}$, so that

$(\mathbf{M}, \mathbf{v})$ is constraint admissible for every $\bar{\mathbf{w}} \in \operatorname{conv} \mathcal{W}$, i.e. $(\mathbf{M}, \mathbf{v}) \in \bar{\Pi}_N^{df}(x)$, and therefore $\bar{\Pi}_N^{df}(x) \supset \Pi_N^{df}(x)$. This proves the result for the set $\Pi_N^{df}(x)$; the result then follows for the set X_N^{df} directly from the definition (3.23). □

Note that the above result is not indicative of excessive conservatism of the control policy parameterization (3.17), since an equivalent result holds for control policies constructed via robust dynamic programming techniques [BR71].

A consequence of Proposition 3.7 is that a wide variety of disturbance sets W can be handled easily within the proposed framework, including nonconvex or disjoint sets, so long as W is compact and $0 \in \operatorname{int}(\operatorname{conv} W)$. We will therefore generally assume that W is a convex set satisfying **A3.1**, with the understanding that we can replace W with $\operatorname{conv} W$ as necessary.

Corollary 3.8 (Polyhedral Sets). *If the constraint sets Z and X_f are polyhedral and the disturbance set W is a polytope, then the sets $\mathcal{C}_N$ and X_N^{df} are polyhedral and $\Pi_N^{df}(x)$ is polyhedral for each $x \in X_N^{df}$.*

Proof. If the set W (equivalently $\mathcal{W}$) is a polytope, then the set of extreme disturbance sequences (i.e. those whose elements take values from the vertices of $\mathcal{W}$) is finite. Denote each such extreme sequence $\mathbf{w}_i$, where $i \in \mathbb{Z}_{[1,q]}$ for some finite integer q, so that $\mathcal{W} \equiv \operatorname{conv} \{\mathbf{w}_i\}_{i=1}^q$. The set $\mathcal{C}_N$ can be written as

$$\mathcal{C}_N = \bigcap_{\{\mathbf{w}_i\}_{i=1}^q} \left\{ (x, \mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w}_i \\ \mathbf{u} = \mathbf{M}\mathcal{G}\mathbf{w}_i + \mathbf{v} \\ (\mathbf{x}, \mathbf{u}) \in \mathcal{Z} \end{array} \right\}, \tag{3.30}$$

which is the intersection of a finite collection of polyhedral sets, and is thus also polyhedral (Prop. 2.8). An identical argument establishes that $\Pi_N^{df}(x)$ is polyhedral for each $x \in X_N^{df}$. The set X_N^{df} is polyhedral since, as in the proof of Thm. 3.5, it can be written as a linear mapping of $\mathcal{C}_N$, and any linear map of a polyhedral set is polyhedral (Prop. 2.10). □

The results of this section will be of fundamental importance throughout this dissertation. If $\mathcal{W}$ is convex and compact, then it is conceptually possible to compute a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ in a computationally tractable way, given the current state x. Methods for calculating policies in this class for particular characterizations of the sets $\mathcal{Z}$ and W will be addressed in Chapter 6.

3.5 Equivalence of Affine Policy Parameterizations

An important question is whether the disturbance feedback parameterization (3.17) is more or less conservative than the state feedback parameterization (3.5). In this section, we show that they are equivalent.

Theorem 3.9 (Equivalence). *The set of admissible states $X_N^{df} = X_N^{sf}$. Additionally, given any $x \in X_N^{sf}$, for any admissible $(\mathbf{K}, \mathbf{g})$ an admissible $(\mathbf{M}, \mathbf{v})$ can be found which yields the same state and input sequence for all allowable disturbance sequences, and vice-versa.*

Proof. The set equality is established by showing that both $X_N^{sf} \subseteq X_N^{df}$ and $X_N^{df} \subseteq X_N^{sf}$.

$X_N^{sf} \subseteq X_N^{df}$: By definition, for a given $x \in X_N^{sf}$, there exists a pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$ that satisfies the constraints $\mathcal{Z}$ for all disturbance sequences $\mathbf{w} \in \mathcal{W}$. For a given $\mathbf{w} \in \mathcal{W}$, the inputs and states of the system can be written as :

$$\begin{aligned} \mathbf{u} &= \mathbf{K}\mathbf{x} + \mathbf{g} && (3.31)\\ &= \mathbf{K}(\mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathcal{G}\mathbf{w}) + \mathbf{g} && (3.32) \end{aligned}$$

The matrix $(I - \mathbf{KB})$ is always non-singular, since $\mathbf{KB}$ is strictly lower triangular. The control sequence can then be rewritten as an affine function of the disturbance sequence $\mathbf{w}$:

$$\mathbf{u} = (I - \mathbf{KB})^{-1}(\mathbf{KA}x + \mathbf{g}) + (I - \mathbf{KB})^{-1}\mathbf{KE}\mathcal{G}\mathbf{w}, \tag{3.33}$$

and an admissible $(\mathbf{M}, \mathbf{v})$ constructed by choosing

$$\begin{aligned} \mathbf{M} &= (I - \mathbf{KB})^{-1}\mathbf{KE} && (3.34a)\\ \mathbf{v} &= (I - \mathbf{KB})^{-1}(\mathbf{KA}x + \mathbf{g}). && (3.34b) \end{aligned}$$

This choice of $(\mathbf{M}, \mathbf{v})$ gives exactly the same input sequence as the pair $(\mathbf{K}, \mathbf{g})$, so the state and input constraints $\mathcal{Z}$ are satisfied for all disturbance sequences $\mathbf{w} \in \mathcal{W}$. The constraint (3.19) that $\mathbf{M}$ be strictly block lower triangular is satisfied because $\mathbf{M}$ is chosen in (3.34) as a product of the block lower triangular matrices $(I - \mathbf{KB})^{-1}$ and $\mathbf{K}$ and the strictly block lower triangular matrix $\mathbf{E}$. Therefore, $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ and thus $x \in X_N^{sf} \Rightarrow x \in X_N^{df}$.

$X_N^{df} \subseteq X_N^{sf}$: By definition, for a given $x \in X_N^{df}$, there exists a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ that satisfies the constraints $\mathcal{Z}$ for all disturbance sequences $\mathbf{w} \in \mathcal{W}$. For a given $\mathbf{w} \in \mathcal{W}$, the inputs and states of the system can be written as :

$$\mathbf{u} = \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \tag{3.35}$$

$$\mathbf{x} = \mathbf{A}x + \mathbf{B}(\mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v}) + \mathbf{G}\mathbf{w} \tag{3.36}$$

Recalling that since full state feedback is assumed, one can recover the uncertain terms Gw_i using the relation

$$Gw_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{3.37}$$

which can be written in matrix form as

$$\mathcal{G}\mathbf{w} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & -A & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A & I \end{bmatrix} \mathbf{x} - \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} Ax - \mathcal{B}\mathbf{u}, \tag{3.38}$$

or more compactly as

$$\mathcal{G}\mathbf{w} = \mathbf{E}^{\dagger}\mathbf{x} - \mathcal{I}Ax - \mathcal{B}\mathbf{u}. \tag{3.39}$$

It is easy to verify that the matrices $\mathbf{E}^{\dagger}$ and $\mathcal{I}^{\top}$ are left inverses of $\mathbf{E}$ and $\mathbf{A}$ respectively, so that $\mathbf{E}^{\dagger}\mathbf{E} = I$ and $\mathcal{I}^{\top}\mathbf{A} = I$. The input sequence can then be rewritten as

$$\mathbf{u} = \mathbf{M}(\mathbf{E}^{\dagger}\mathbf{x} - \mathcal{I}Ax - \mathcal{B}\mathbf{u}) + \mathbf{v} \tag{3.40}$$

$$= (I + \mathbf{M}\mathcal{B})^{-1}(\mathbf{M}\mathbf{E}^{\dagger}\mathbf{x} - \mathbf{M}\mathcal{I}Ax + \mathbf{v}). \tag{3.41}$$

The matrix $(I + \mathbf{M}\mathcal{B})$ is non-singular because the product $\mathbf{M}\mathcal{B}$ is strictly lower triangular. An admissible $(\mathbf{K}, \mathbf{g})$ can then be constructed by choosing

$$\mathbf{K} = (I + \mathbf{M}\mathcal{B})^{-1}\mathbf{M}\mathbf{E}^{\dagger} \tag{3.42a}$$

$$\mathbf{g} = (I + \mathbf{M}\mathcal{B})^{-1}(\mathbf{v} - \mathbf{M}\mathcal{I}Ax). \tag{3.42b}$$

This choice of $(\mathbf{K}, \mathbf{g})$ gives exactly the same input sequence as the pair $(\mathbf{M}, \mathbf{v})$, so the state and input constraints $\mathcal{Z}$ are satisfied for all disturbance sequences $\mathbf{w} \in \mathcal{W}$. The constraint (3.6) that $\mathbf{K}$ be block lower triangular is satisfied because $\mathbf{K}$ is chosen in (3.42) as a product of block lower triangular matrices. Therefore, $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$ and thus $x \in X_N^{df} \Rightarrow x \in X_N^{sf}$. $\square$

Note that the proof of Theorem 3.9 is not predicated on any of the conditions of **A3.1**. It is instead a direct consequence of the linearity of the system (3.1) and of the state and disturbance feedback policies (3.5) and (3.17).

Recalling Theorem 3.5 leads next to the following result, which is surprising in light of the potential nonconvexity of the set $\Pi_N^{sf}(x)$:

Corollary 3.10 (Convexity of X_N^{sf}). *The set of states X_N^{sf}, for which an admissible affine state feedback policy of the form* (3.5) *exists, is a closed and convex set.*

Remark 3.11. *The results of Theorem 3.9 have appeared independently in [GKM06] and [BBN06], with the latter appearing in the more general context of output feedback. In Chapter 8, we present a generalization of the results in [BBN06] that incorporates nonzero initial state estimates and observer dynamics, and provide several geometric and invariance results paralleling those developed in this chapter.*

An important consequence of Theorem 3.9 is that it provides a tractable method for finding an affine state feedback policy $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$, a problem which has generally been considered intractable due to nonconvexity of the feasible set $\Pi_N^{sf}(x)$ (cf. Example 3.3). Since the set $\Pi_N^{df}(x)$ is convex, it is possible in principle to find a policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ using convex optimization methods; one can then find a policy $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$ via the transformation (3.42). Methods for calculating such policies for specific classes of disturbances and constraint sets are discussed in Chapter 6.

Theorem 3.9 will also play a central role in the development of Chapters 4 and 5, where we consider various system theoretic properties of receding horizon control laws derived from the parameterizations (3.5) (equivalently, (3.17)). For many of the results, proof will be significantly more straightforward using one or the other of the two parameterizations; Theorem 3.5 will allow us to move freely between them, choosing whichever parameterization is most natural or convenient in each context.

3.5.1 Relation to Pre-Stabilizing Control Policies

As a direct consequence of the equivalence result in Theorem 3.9, the class of affine feedback policies in the form $u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} G w_j$ subsumes the class of policies based on perturbations to a *fixed* and pre-stabilizing linear control law in the form $u_i = c_i + K x_i$ (cf. Remark 3.1). We therefore conclude that, if the region of attraction for the class of pre-stabilizing policies with fixed gain K is denoted X_N^K, then $X_N^K \subseteq X_N^{df}$ always.

However, we note that calculating a feasible policy pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ for a given horizon length N may require considerable computational effort; the number of decision variables in the parameter $\mathbf{M}$ increase quadratically with the horizon length N, while the number of decision variables c_i in a pre-stabilizing scheme increases only linearly with N. A sensible strategy might therefore be to employ a pre-stabilizing scheme with a longer horizon, in the hope that $X_{\tilde{N}}^K \supset X_N^{df}$ for some $\tilde{N} > N$, while still requiring less computational effort.

We next show that this is *not* always possible, and provide an example for which $X_{\tilde{N}}^K$ is a strict subset of X_N^{df} for some fixed N, regardless of how large we allow $\tilde{N}$ to be:

Example 3.12. *Consider the system*

$$x^+ = 2x + 2u + w,$$

subject to the following input and terminal constraints:

$$u \in \{u \mid |u| \leq 0.7\}$$
$$X_f = \{x \mid |x| \leq 0.5\}$$

and subject to bounded disturbances $|w| \leq 0.25$. We define a stabilizing controller $K = -1.25$. For increasing horizon length N, we consider the size of the set X_N^{df} for this system, as well as the size of the set of feasible initial conditions X_N^K when the control policy for the system is restricted to perturbations to the fixed *linear feedback law $u = Kx$, i.e. those parameterized as $u_i = c_i + Kx_i$. The sizes of these sets with increasing horizon length are shown in Figure 3.2, where it is clear that $X_{\tilde{N}}^K \subset X_4^{df}$ for every $\tilde{N} \in \mathbb{N}$.*

Note that in Example 3.12 the terminal set X_f is robust positively invariant for the closed-loop system $x^+ = (A + BK)x + w$. Terminal conditions of this type will play a central role in the development of robust receding horizon controls to be introduced in Section 3.6.

3.5.2 Relation to the Youla Parameter

The equivalence result of Thm. 3.9 is closely related to the well-known Youla parameterization from linear systems theory [YJB76][ZDG96, Ch. 12], where the problem of finding a stabilizing linear controller for an unconstrained linear system is convexified via a similar variable transformation. A useful method of comparison is to draw the sequence of inputs resulting from the application of a given control policy $(\mathbf{M}, \mathbf{v})$ in block diagram form as

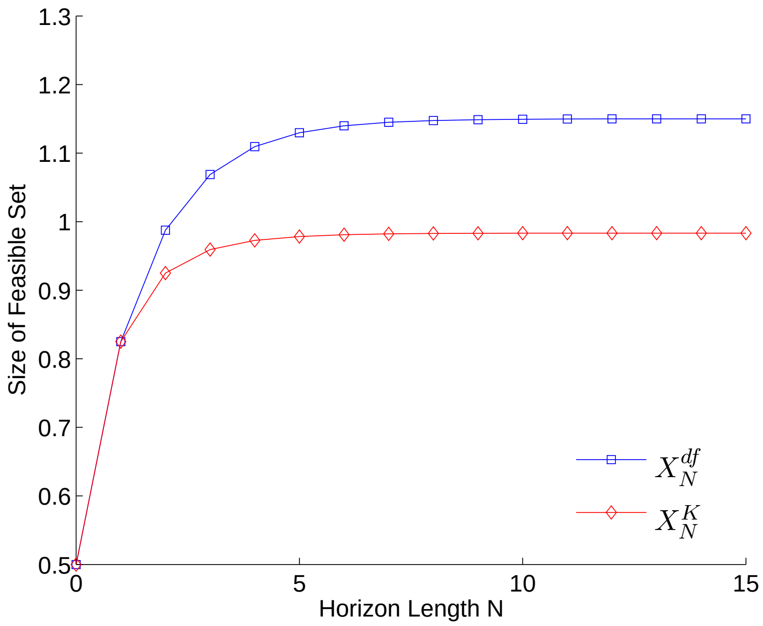

Figure 3.2: Sizes of X_N^{df} and X_N^K with increasing N

in Figure 3.3 – such a figure has a structure usually employed in *internal model control* (IMC) formulations for linear systems [GM82][Mac89, Ch.6] with stable plants, with the feedback parameter $\mathbf{M}$ taking the place of the Youla parameter (typically denoted Q). Note that it is not necessary to generalize this figure to the case of unstable plants, since the policy $(\mathbf{M}, \mathbf{v})$ is only defined over a finite horizon.

It should be emphasized, however, that the feedback scheme shown in Figure 3.3 is strictly notional – it will generally *not* be our intent to implement a calculated control policy $(\mathbf{M}, \mathbf{v})$ in this manner. Instead, we will usually calculate and implement policies $(\mathbf{M}, \mathbf{v})$ in a *receding horizon* fashion, resulting in *static* and *nonlinear* feedback control laws. In this case Figure 3.3 represents an internal model employed by the controller in considering possible future state trajectories resulting from a candidate control policy $(\mathbf{M}, \mathbf{v})$, and does *not* represent the actual closed-loop behavior of the system. In the next section, we consider the geometric and invariance properties of such receding horizon control laws.

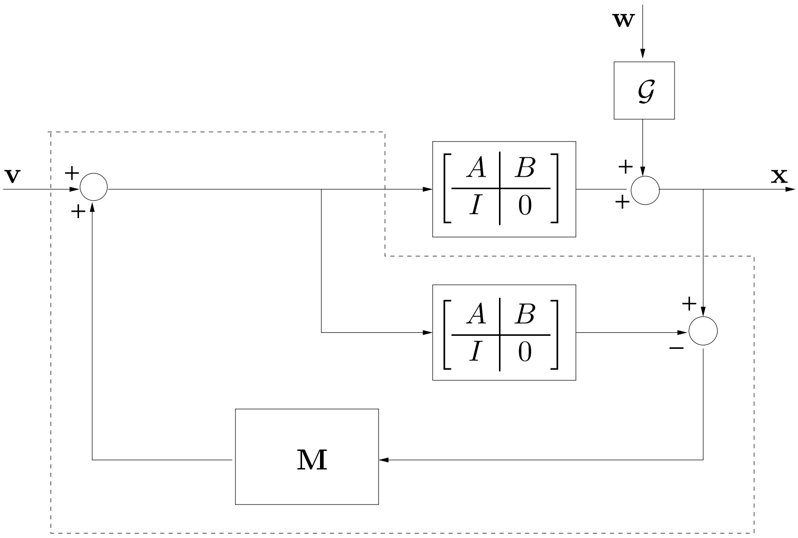

Figure 3.3: Internal Model Control

3.6 Geometric and Invariance Properties

It is well-known that the set of states for which an admissible open-loop input sequence exists (i.e. one with $\mathbf{K} = 0$ or $\mathbf{M} = 0$) may collapse to the empty set if the horizon is sufficiently large [SM98, Sect. F]. Furthermore, for time-varying, time-optimal or receding horizon control implementations of the affine control policies defined in this chapter, it may not be possible to guarantee constraint satisfaction for all time unless additional assumptions are made. In this section, we provide conditions under which these problems will not occur. The stability of receding horizon schemes based on these policies will be addressed in Chapters 4 and 5.

We first introduce the following standard assumption (cf. [MRRS00]):

A3.2 (Invariant Terminal Constraint) *A state feedback gain matrix K_f and terminal constraint set X_f have been chosen such that:*

i. The matrix $A + BK_f$ is Hurwitz.

ii. X_f is contained inside the set of states for which the constraints $(x, u) \in Z$ are satisfied under the control $u = K_f x$, i.e. $X_f \subseteq \{x \mid (x, K_f x) \in Z\}$.

iii. X_f is robust positively invariant for the closed-loop system $x^+ = (A + BK_f)x + Gw$, i.e. $(A + BK_f)x + Gw \in X_f$ for all $x \in X_f$ and all $w \in W$.

Under some additional, mild technical assumptions, one can compute a K_f and a polytopic X_f that satisfies **A3.2** when Z is a polytope and W is a polytope, an ellipsoid or the affine map of a p-norm ball. The reader is referred to [Bla99, KG98, LK99] and the references therein for details.

3.6.1 Monotonicity of X_N^{sf} and X_N^{df}

We are now in a position to give a sufficient condition under which one can guarantee that X_N^{sf} (equivalently, X_N^{df}) is nonempty and that the sets X_N^{sf} are non-decreasing (with respect to set inclusion) with horizon length N:

Proposition 3.13 (Nesting of X_N^{sf}). *If **A3.2** holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^{sf} \subseteq \cdots \subseteq X_{N-1}^{sf} \subseteq X_N^{sf} \subseteq X_{N+1}^{sf} \subseteq \cdots \tag{3.43}$$

Proof. The proof is by induction. Let $x \in X_N^{sf}$ and $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$. One can construct a pair $(\bar{\mathbf{K}}, \bar{\mathbf{g}}) \in X_{N+1}^{sf}$, where

$$\bar{\mathbf{K}} := \begin{bmatrix} \mathbf{K} & 0 & 0 \\ 0 & K_f & 0 \end{bmatrix} \quad \bar{\mathbf{g}} := \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix},$$

such that the final stage input will be $u_N = K_f x_N$. From the definition of $\Pi_N^{sf}(x)$, it follows that $x_N \in X_f$. If **A3.2** holds, then $(x_N, K_f x_N) \in Z$ and $x_{N+1} = Ax_N + Bu_N + Gw_N \in X_f$ for all $w_N \in W$. It then follows from the definition of $\Pi_{N+1}^{sf}(x)$ that $(\bar{\mathbf{K}}, \bar{\mathbf{g}}) \in \Pi_{N+1}^{sf}(x)$, hence $x \in X_{N+1}^{sf}$. The proof is completed by verifying, in a similar manner, that $X_f \subseteq X_1^{sf} \subseteq X_2^{sf}$. □

Remark 3.14. *For many examples, some of the inclusions in* (3.43) *are strict, rather than satisfied with equality. Note also that if $X_N^{sf} = X_{N+1}^{sf}$ for some N, then $X_i^{sf} = X_N^{sf}$ for all $i > N$.*

Recalling Theorem 3.9, the next result follows immediately:

Corollary 3.15 (Nesting of X_N^{df}). *If **A3.2** holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^{df} \subseteq \cdots \subseteq X_{N-1}^{df} \subseteq X_N^{df} \subseteq X_{N+1}^{df} \subseteq \cdots \tag{3.44}$$

Remark 3.16. *Corollary 3.15 should be compared with the equivalent result in [KM04b, Thm. 2]. The proof given here is more transparent, due to the application of Theorem 3.9 and Proposition 3.13.*

3.6.2 Time-varying Control Laws

We first consider what happens if one were to implement an admissible affine disturbance feedback policy in a time-varying fashion. Given any $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x(0))$, consider the following feedback policy which is time-varying over the prediction horizon, and static and linear thereafter:

$$u(k) = \begin{cases} v_k + \sum_{j=0}^{k-1} M_{k,j} G w(j) & \text{if } k \in \mathbb{Z}_{[0,N-1]} \\ K_f x(k) & \text{if } k \in \mathbb{Z}_{[N,\infty)}. \end{cases} \tag{3.45}$$

Recall that the realized disturbance inputs $Gw(j)$ can be recovered using the relation (3.18). Theorem 3.9 implies that we could also have defined an equivalent, time-varying affine state feedback policy $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}$, but we will generally choose to work with disturbance feedback policies due to the convenience of computation resulting from Theorem 3.5. The next result follows immediately:

Proposition 3.17 (Time-varying control). *Let* ***A3.2*** *hold, the initial state* $x(0) \in X_N^{df}$ *and* $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x(0))$*. For all allowable infinite disturbance sequences, the state of system* (3.1)*, in closed-loop with the feedback policy* (3.45)*, enters* X_f *in* N *steps or less and remains in* X_f *for all* $k \in \mathbb{Z}_{[N,\infty)}$*. Furthermore, the constraints* $(x(i), u(i)) \in Z$ *are satisfied for all time and for all allowable infinite disturbance sequences.*

3.6.3 Minimum-time Control Laws

We next derive some results for robust minimum-time control laws. Given a maximum horizon length $N_{\max}$ and the set $\mathcal{N} := \{1, \ldots, N_{\max}\}$, let

$$N^*(x) := \min_N \left\{ N \in \mathcal{N} \;\middle|\; \Pi_N^{sf}(x) \neq \emptyset \right\} \tag{3.46}$$

be the minimum horizon length for which an admissible state feedback policy of the form (3.5) exists.

Consider the *set-valued* map $\kappa_N : X_N^{sf} \to 2^{\mathbb{R}^m}$ (where $2^{\mathbb{R}^m}$ is the set of all subsets of $\mathbb{R}^m$), which is defined by considering only the first portion of an admissible state feedback parameter $(\mathbf{K}, \mathbf{g})$, i.e.

$$\kappa_N(x) := \left\{ u \;\middle|\; \exists (\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x) \text{ s.t. } u = g_0 + K_{0,0} x \right\}. \tag{3.47}$$

In order to define a minimum-time control law, we consider also the set-valued map $\kappa : \mathcal{X} \to 2^{\mathbb{R}^m}$, defined as

$$\kappa(x) := \begin{cases} \kappa_{N^*(x)}(x) & \text{if } x \notin X_f \\ K_f x & \text{if } x \in X_f \end{cases}, \tag{3.48}$$

where $\kappa_{N^*(x)}(x)$ is defined as in (3.47) with $N = N^*(x)$, and

$$\mathcal{X} := X_f \cup \left(\underset{N \in \mathcal{N}}{\cup} X_N^{sf} \right).$$

Let the robust time-optimal control law $\mu : \mathcal{X} \to \mathbb{R}^m$ be any selection from $\kappa(\cdot)$, i.e. $\mu(x) \in \kappa(x)$, for all $x \in \mathcal{X}$. Note that $\kappa(\cdot)$ is defined everywhere on $\mathcal{X}$ and that the state of the closed-loop system $x^+ = Ax + B\mu(x) + Gw$ will enter X_f in less than $N_{\max}$ steps if this is possible, even if **A3.2** does not hold. Furthermore, if $x \in X_N^{sf} \setminus X_{N-1}^{sf}$, then $Ax + B\mu(x) + Gw \in X_{N-1}^{sf}$ for all $w \in W$.

Proof of the following result is straightforward and closely parallels that of Proposition 3.13.

Proposition 3.18 (Minimum-time control). *If **A3.2** holds, then $\mathcal{X} = X_{N_{\max}}^{sf}$ and $\mathcal{X}$ is robust positively invariant for the closed-loop system $x^+ = Ax + B\mu(x) + Gw$, i.e. if $x \in \mathcal{X}$, then $x^+ \in \mathcal{X}$ for all $w \in W$. The state of the closed-loop system enters X_f in $N_{\max}$ steps or less and, once inside, remains inside for all time and all allowable infinite disturbance sequences. Furthermore, the constraints $(x(i), u(i)) \in Z$ are satisfied for all time and for all allowable infinite disturbance sequences if the initial state $x(0) \in \mathcal{X}$.*

Proof. The proof is straightforward, by showing that if some $(\mathbf{K}, \mathbf{g})$ is admissible at the current time instant, then a truncated version $(\hat{\mathbf{K}}, \hat{\mathbf{g}})$ is admissible at the next time instant. More precisely, given an initial state $x \in \mathcal{X}$ and feasible control policy $(\mathbf{K}, \mathbf{g}) \in \Pi_{N^*(x)}^{sf}(x)$, one can guarantee that if $x^+ = Ax + B(g_0 + K_{0,0}(x)) + Gw$, then $(\hat{\mathbf{K}}, \hat{\mathbf{g}}) \in \Pi_{(N^*(x)-1)}^{sf}(x^+)$ for all $w \in W$, where $\hat{\mathbf{K}} := \begin{bmatrix} 0 & I_{m(N-1)} \end{bmatrix} \mathbf{K} \begin{bmatrix} 0 \\ I_{nN} \end{bmatrix}$ and $\hat{\mathbf{g}} := \begin{bmatrix} 0 & I_{m(N-1)} \end{bmatrix} \mathbf{g}$. □

Remark 3.19. *Note that the control law defined above is not optimal in the sense of [BR71, GS71, Bla92, MS97] since X_N^{sf} is not, in general, equal to the set of states for which an* arbitrary, *nonlinear, time-varying state feedback control policy exists such that for all allowable disturbance sequences of length N, the state and input constraints Z are satisfied and the state arrives in X_f in exactly N steps.*

Due to the nonconvexity of $\Pi_N^{sf}(x)$, computing an admissible $(\mathbf{K}, \mathbf{g})$ in (3.47) at each time instant is seemingly problematic. However, by a straightforward application of Theorem 3.9, it follows that

$$\kappa_N(x) = \left\{ u \;\middle|\; \exists(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x) \text{ s.t. } u = v_0 \right\}, \tag{3.49}$$

where v_0 is the first component of the stacked vector $\mathbf{v}$. Hence, computation of an admissible control law in $\kappa_N(x)$ is possible using convex optimization methods, since the set $\Pi_N^{df}(x)$ is guaranteed to be convex from Theorem 3.5. Techniques for computing an admissible $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ for particular classes of disturbance set W and constraint sets Z will be discussed in Chapter 6.

3.6.4 Receding Horizon Control Laws

Finally, we consider what happens when the disturbance feedback parameterization (3.17) is used to design a receding horizon control (RHC) law. In RHC, an admissible feedback policy is computed at each time instant, but only the first component of the policy is applied. An important issue in RHC is whether one can ensure feasibility and constraint satisfaction for all time, despite the fact that a finite horizon is being used and only the first part of the policy is implemented at each sample instant [MRRS00].

Recalling the definition of the set-valued map $\kappa_N : X_N^{sf} \to 2^{\mathbb{R}^m}$ in (3.47) (equivalently, (3.49)), we define an admissible RHC law $\mu_N : X_N^{sf} \to \mathbb{R}^m$ as any selection from $\kappa_N(\cdot)$, i.e. $\mu_N(\cdot)$ has to satisfy

$$\mu_N(x) \in \kappa_N(x), \quad \forall x \in X_N^{sf}. \tag{3.50}$$

The resulting closed-loop system is then given by

$$x^+ = Ax + B\mu_N(x) + Gw. \tag{3.51}$$

Note that if the selection criteria in (3.50) is time-invariant, then the RHC law $\mu_N(\cdot)$ is also time-invariant and is, in general, a *nonlinear* function of the current state. The following result then follows using standard methods in receding horizon control [MRRS00], by employing the state feedback parameterization (3.5):

Proposition 3.20 (RHC). *If **A3.2** holds, then the set X_N^{sf} is robust positively invariant for the closed-loop system* (3.51)*, i.e. if $x \in X_N^{sf}$, then $Ax + B\mu_N(x) + Gw \in X_N^{sf}$ for all $w \in W$. Furthermore, the constraints $(x(i), u(i)) \in Z$ are satisfied for all time and for all allowable infinite disturbance sequences if the initial state $x(0) \in X_N^{sf}$.*

Proof. For any given state $x \in X_N^{sf}$ there exists a state feedback policy pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{sf}(x)$ by definition. Using arguments which closely parallel those in the proof of Proposition 3.13, one can show that if $u \in \kappa_N(x)$ and $w \in W$, then there exists a "shifted" policy pair $(\tilde{\mathbf{K}}, \tilde{\mathbf{g}}) \in \Pi_N^{sf}(Ax + Bu + Gw)$, hence $Ax + Bu + Gw \in X_N^{sf}$. $\square$

Remark 3.21. *The receding horizon control law $\mu_N(\cdot)$ should be contrasted with the control law* (3.45)*. Whereas* (3.45) *is a* time-varying *feedback policy that is dependent on present and past values of the system state, the RHC law $\mu_N(\cdot)$ is* time-invariant*, being defined as a function only of the current state, whenever the selection criteria is time-invariant. Note also that, unlike the time-optimal control policy discussed in Section 3.6.3, the RHC law $\mu_N(\cdot)$ does* not *guarantee that the system will reach the target set X_f in a finite amount of time.*

3.7 Conclusions

The state feedback RHC law $\mu_N(\cdot)$ will be the central focus of much of the remainder of this dissertation. We have thus far only provided conditions under which $\mu_N(\cdot)$ will guarantee constraint satisfaction for all time, where $\mu_N(x)$ was defined in (3.50) as any feasible selection from $\kappa_N(x)$. We have not yet specified criteria by which an optimal control law might be selected.

We address this problem in Chapters 4 and 5, where we present two different cost functions allowing us to discriminate between policies in the set $\Pi_N^{df}(x)$. Using an expected value cost (Chapter 4), or a min-max cost as in $\mathcal{H}_\infty$ control (Chapter 5), we will show that the resulting closed-loop system is input-to-state or ℓ_2 stable, respectively. In Chapters 6 and 7, we address the practical issue of calculating an optimal policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ with respect to these two cost functions for various classes of disturbance set W when the constraint sets Z and X_f are defined by linear inequalities.

Chapter 4. Expected Value Costs ($\mathcal{H}_2$ Control)

4.1 Introduction

In this chapter, we once again consider the discrete-time LTI system:

$$x^+ = Ax + Bu + Gw \tag{4.1}$$

$$z = C_z x + D_z u, \tag{4.2}$$

where the system (4.1) is identical to the system (3.1) introduced in Chapter 3. We treat the vector $z \in \mathbb{R}^q$ as a *controlled output* of (4.1), and will continue to assume that, at each sample instant, a measurement of the state x is available.

Recall that in Section 3.2 we introduced a class of robust control policies for the control of the system (4.1) over a finite horizon N. These policies modelled the input u_i at each time instant as an affine function of the state sequence $\{x_0, \ldots, x_i\}$, so that

$$u_i = g_i + \sum_{j=0}^{i} K_{i,j} x_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}. \tag{4.3}$$

Given a state/input constraint set $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ and target/terminal state constraint set $X_f \subset \mathbb{R}^n$, we defined $\Pi_N^{sf}(x)$ as the set of all control policies of the form (4.3) guaranteed to steer the system (4.1) to the set X_f after N time steps, while guaranteeing that at each time step $(x, u) \in Z$.

In Section 3.6.4, we defined a class of control laws based on a receding horizon implementation of finite horizon policies in the form (4.3). In particular, we established appropriate terminal conditions such that the system (4.1)–(4.2), in closed-loop with a control law in this class, could be guaranteed to satisfy the constraints $(x, u) \in Z$ over an infinite horizon. In this chapter and the next, we address the *stability* of the system (4.1)–(4.2) in closed-loop with such control laws.

To this end, we define a cost function $\Phi : \mathbb{R}^n \times \mathbb{R}^{mN} \times \mathbb{R}^{lN} \to \mathbb{R}_+$ that is quadratic in the controlled outputs, and seek a control policy in the form (4.3) that is optimal with respect to this cost function in some sense. We define

$$\Phi(x, \mathbf{u}, \mathbf{w}) := \|x_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_2^2 \,, \tag{4.4}$$

where $x_{i+1} := Ax_i + Bu_i + Gw_i$ and $z_i := C_z x_i + D_z u_i$ for all $i \in \mathbb{Z}_{[0,N-1]}$. In this chapter we will assume that, in addition to being contained inside the bounded set W, the disturbances w in (4.1) are independent and identically distributed with zero mean and known covariance, and will consequently define an optimal policy to be one that minimizes the expected value of (4.4) over policies in the form (4.3).

By employing the equivalent disturbance feedback parameterization introduced in Chapter 3, we will show that the problem of finding such an optimal policy can be posed as a convex optimization problem, and will further demonstrate that such optimal policies allow for the synthesis of a receding horizon control (RHC) law guaranteeing that the closed-loop system is input-to-state stable (ISS). In Chapter 5, we will instead employ a min-max cost based on (4.4) where the disturbances are negatively weighted as in $\mathcal{H}_\infty$ control, and show that the resulting closed-loop system has finite ℓ_2 gain.

Throughout this chapter, we will make the following assumptions relating to (4.1)–(4.4):

A4.1 (Standing Assumptions) *The following conditions hold:*

*i. The assumptions **A3.1** hold.*

ii. The pair (C_z, A) is detectable.

iii. The matrices C_z and D_z are full column rank, with $C_z^\top D_z = 0$.

iv. The matrix P is positive semidefinite.

Remark 4.1. *Note that if **A4.1**(iii) holds, then the cost function* (4.4) *can be rewritten in the standard quadratic form*

$$\Phi(x, \mathbf{u}, \mathbf{w}) = \|x_N\|_P^2 + \sum_{i=0}^{N-1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 \right),$$

where $C_z^\top C_z =: Q \succ 0$ and $D_z^\top D_z =: R \succ 0$. We use the cost function (4.4) *augmented with the condition **A4.1**(iii) primarily for consistency of notation with Chapter 5.*

4.1.1 Notation and Definitions

As in Chapter 3, we assume that the current and future values of the disturbance are unknown and may change from one time instant to the next, but are contained in a compact and convex set W containing the origin in its interior. We further assume for the purposes of this chapter that the disturbances are independent and identically distributed, with $\mathbb{E}[w] = 0$ and positive semidefinite covariance[1] matrix $C_w := \mathbb{E}\left[ww^\top\right] \in \mathbb{R}^{l\times l}$. Finally, we define the matrix $\mathbf{C}_w := I \otimes C_w$, so that $\mathbb{E}\left[\mathbf{w}\mathbf{w}^\top\right] = \mathbf{C}_w \in \mathbb{R}^{lN\times lN}$.

In addition to the stacked state, input and disturbance sequences $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{w}$ defined in Section 3.1.1, we define a stacked vector of controlled outputs $\mathbf{z} \in \mathbb{R}^{qN}$ as

$$\mathbf{z} := \mathrm{vec}(z_0, z_1, \ldots, z_{N-1}), \tag{4.5}$$

and will often find it convenient to write the controlled outputs $\mathbf{z}$ in vectorized form as

$$\mathbf{z} = \mathbf{C}_z\mathbf{x} + \mathbf{D}_z\mathbf{u},$$

where $\mathbf{C}_z := \begin{bmatrix} I_N \otimes C_z & 0 \end{bmatrix}$ and $\mathbf{D}_z := I_N \otimes D_z$. Recalling the definitions of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ from Section 3.1.1, and further defining

$$\tilde{A} := \begin{pmatrix} A^{N-1} & \cdots & A & I \end{pmatrix}, \tag{4.6}$$

$$\tilde{B} := \tilde{A}(I_N \otimes B), \quad \tilde{G} := \tilde{A}(I_N \otimes G), \tag{4.7}$$

we can write $\mathbf{z}$ and x_N as

$$\begin{aligned} \mathbf{z} &= \mathbf{C}_z\mathbf{A}x + (\mathbf{C}_z\mathbf{B} + \mathbf{D}_z)\mathbf{u} + \mathbf{C}_z\mathbf{G}\mathbf{w} \\ x_N &= A_N^N x + \tilde{B}\mathbf{u} + \tilde{G}\mathbf{w}, \end{aligned}$$

where $x_0 = x$ is the initial state. The cost function $\Phi(x, \mathbf{u}, \mathbf{w})$, defined in (4.4), can then be written as

$$\Phi(x, \mathbf{u}, \mathbf{w}) = \|H_x x + H_u\mathbf{u} + H_w\mathbf{w}\|_2^2, \tag{4.8}$$

[1]Though we assume that $0 \in \mathrm{int}\, W$, and will consider only control policies that are robust to *every* disturbance sequence drawing values from W, we do not exclude the case $C_w \succeq 0$, or even $C_w = 0$. The result is that we can define policies that are robust to large disturbance sets W while treating as vanishingly small the *probability* that such disturbances will actually occur in some (or all) directions.

where

$$H_x := \begin{pmatrix} \mathbf{C}_z\mathbf{A} \\ P^{\frac{1}{2}}A^N \end{pmatrix}, \ H_u := \begin{pmatrix} \mathbf{C}_z\mathbf{B} + \mathbf{D}_z \\ P^{\frac{1}{2}}\tilde{B} \end{pmatrix}, \text{ and } H_w := \begin{pmatrix} \mathbf{C}_z\mathbf{G} \\ P^{\frac{1}{2}}\tilde{G} \end{pmatrix}. \tag{4.9}$$

In subsequent sections we will find the following fact about H_u useful in establishing various results.

Lemma 4.2. *If D_z is full column rank, then H_u is full column rank.*

Proof. Obvious from the expansion

$$H_u = \begin{bmatrix} D_z & 0 & \cdots & \cdots & 0 \\ C_zB & D_z & 0 & \cdots & 0 \\ C_zAB & C_zB & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & D_z & 0 \\ C_zA^{N-2}B & C_zA^{N-3}B & \cdots & C_zB & D_z \\ P^{\frac{1}{2}}A^{N-1}B & P^{\frac{1}{2}}A^{N-2}B & \cdots & P^{\frac{1}{2}}AB & P^{\frac{1}{2}}B \end{bmatrix}.$$

□

4.2 An Expected Value Cost Function

In this chapter we define an optimal policy pair $(\mathbf{K}^*(x), \mathbf{g}^*(x)) \in \Pi_N^{sf}(x)$ to be one that minimizes the expected value of the cost function (4.8) over the set of feasible affine state feedback control policies $\Pi_N^{sf}(x)$. We thus define

$$V_N(x, \mathbf{K}, \mathbf{g}) := \mathbb{E}\left[\Phi(x, \bar{\mathbf{u}}, \mathbf{w})\right], \tag{4.10}$$

where $\bar{\mathbf{u}} := \mathbf{K}\bar{\mathbf{x}} + \mathbf{g}$ and $\bar{\mathbf{x}} := (I - \mathbf{BK})^{-1}(\mathbf{A}x + \mathbf{Bg} + \mathbf{Gw})$, and define an optimal policy pair as

$$(\mathbf{K}^*(x), \mathbf{g}^*(x)) := \underset{(\mathbf{K},\mathbf{g})\in\Pi_N^{sf}(x)}{\operatorname{argmin}} \ V_N(x, \mathbf{K}, \mathbf{g}). \tag{4.11}$$

We assume for the moment that the minimizer in (4.11) exists and is well-defined. We define the receding horizon control law $\mu_N : X_N^{sf} \to \mathbb{R}^m$ by the first part of the optimal affine state feedback control policy in (4.11), i.e.

$$\mu_N(x) := g_0^*(x) + K_{0,0}^*(x)x. \tag{4.12}$$

Note that the control law $\mu_N(\cdot)$ is time-invariant and is, in general, a nonlinear function of the current state. The system (4.1), in closed-loop with the controller $\mu_N(\cdot)$, becomes

$$x^+ = Ax + B\mu_N(x) + Gw. \tag{4.13}$$

We also define the value function $V_N^* : X_N^{sf} \to \mathbb{R}_+$ to be

$$V_N^*(x) := \min_{(\mathbf{K},\mathbf{g})\in\Pi_N^{sf}(x)} V_N(x,\mathbf{K},\mathbf{g}). \tag{4.14}$$

In the remainder of this chapter, conditions sufficient to guarantee the stability of the closed-loop system (4.13) under the receding horizon control law $\mu_N(\cdot)$ will be derived. We first establish some basic properties relating to the existence of the minimizer in (4.11), to the convexity of the value function $V_N^*(\cdot)$ and to the continuity of the receding horizon control law $\mu_N(\cdot)$; derivation of these results will be greatly facilitated by appealing to the convexity and equivalence results of Chapter 3.

4.2.1 Exploiting Equivalence to Compute the RHC Law

The difficulty with implementing the control law $\mu_N(\cdot)$ in (4.12) lies in the nonconvexity of the set of feasible policies $\Pi_N^{sf}(x)$ (cf. Example 3.3), in the nonconvexity of the function $V_N(x,\cdot,\cdot)$, and consequently in the nonconvexity of the optimization problem (4.14). We therefore exploit the alternative disturbance feedback policy parameterization (3.17), and define the analogous cost function

$$J_N(x,\mathbf{M},\mathbf{v}) := \mathbb{E}\left[\Phi(x,\hat{\mathbf{u}},\mathbf{w})\right], \tag{4.15}$$

where $\hat{\mathbf{u}} := \mathbf{M}\mathbf{w} + \mathbf{v}$. In this case we define an optimal policy as

$$(\mathbf{M}^*(x),\mathbf{v}^*(x)) := \operatorname*{argmin}_{(\mathbf{M},\mathbf{v})\in\Pi_N^{df}(x)} J_N(x,\mathbf{M},\mathbf{v}). \tag{4.16}$$

We again assume for the moment that the minimizer in (4.16) exists and is well-defined. Proof of the following result then follows by direct application of Theorem 3.9.

Proposition 4.3 (Computation of $\mu_N(x)$). *The minimum value of $J_N(x,\cdot,\cdot)$ taken over* *the set of admissible affine disturbance feedback parameters is* $V_N^*(x)$*, i.e.*

$$V_N^*(x) = \min_{(\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x)} J_N(x, \mathbf{M}, \mathbf{v}). \tag{4.17}$$

The RHC law $\mu_N(\cdot)$*, defined in* (4.12)*, is given by the first part of the optimal control sequence* $\mathbf{v}^*(\cdot)$*, i.e.*

$$\mu_N(x) = v_0^*(x) = g_0^*(x) + K_{0,0}^*(x)x, \quad \forall x \in X_N^{sf}. \tag{4.18}$$

In the remainder of this section we show that, given an initial state x, the optimization problem (4.17) is convex, and thus that the value of the RHC law $\mu_N(\cdot)$ can in principle be calculated using convex optimization techniques, since the set of feasible policies $(\mathbf{M}, \mathbf{v})$ is a convex set. Methods for characterizing this set, and for calculating such control laws for particular classes of constraint and disturbance sets, will be addressed in Chapter 6.

In Section 4.3 we will show that the value function $V_N^*(\cdot)$ defined in (4.14) is convex and lower semicontinuous everywhere, despite the fact that the function $(\mathbf{K}, \mathbf{g}) \mapsto V_N(x, \mathbf{K}, \mathbf{g})$ is generally nonconvex. This will enable us to prove that the closed-loop system (4.13) is input-to-state stable (ISS) under the control law $\mu_N(\cdot)$. For the particular case where the constraint sets Z and X_f and disturbance set W are polytopic, stronger results will be possible – we will show that the value function $V_N^*(\cdot)$ is Lipschitz continuous, providing tighter bounds on the ISS gain.

4.2.2 Convexity of the Cost Function

We first demonstrate that the function $J_N(x, \cdot, \cdot)$ is convex, so that the problem (4.17) can be posed as a convex optimization problem.

Proposition 4.4 (Convex Cost). *The function* $(x, \mathbf{M}, \mathbf{v}) \mapsto J_N(x, \mathbf{M}, \mathbf{v})$ *is convex.*

Proof. The function (4.15) can be rewritten as

$$J_N(x, \mathbf{M}, \mathbf{v}) = \mathbb{E}\left[\|H_x x + H_u(\mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v}) + H_w \mathbf{w}\|_2^2\right].$$

Since $\mathbb{E}[\mathbf{w}] = 0$ and $\mathbf{w}$ is independent of both $\mathbf{v}$ and $\mathbf{M}$, this simplifies to

$$J_N(x, \mathbf{M}, \mathbf{v}) = \|H_x x + H_u \mathbf{v}\|_2^2 + \mathbb{E}\left[\|(H_u \mathbf{M}\mathcal{G} + H_w)\,\mathbf{w}\|_2^2\,.\right]$$

This can be further simplified by noting that $\mathbb{E}\left[\mathbf{w}^\top X \mathbf{w}\right] = \mathrm{tr}(\mathbf{C}_w^{\frac{1}{2}} X \mathbf{C}_w^{\frac{1}{2}})$ for any X, so that[2]

$$J_N(x, \mathbf{M}, \mathbf{v}) = \|H_x x + H_u \mathbf{v}\|_2^2 + \mathrm{tr}\left(\mathbf{C}_w^{\frac{1}{2}} (H_u \mathbf{M}\mathcal{G} + H_w)^\top (H_u \mathbf{M}\mathcal{G} + H_w) \mathbf{C}_w^{\frac{1}{2}}\right), \tag{4.19}$$

which is convex, since it is a convex function of vector and matrix norms. □

Since the function $J_N(x, \cdot, \cdot)$ is convex and is to be minimized over the convex set $\Pi_N^{df}(x)$, the optimization problem (4.17) is solvable in principle using standard methods from convex optimization. However, since the minimization is to be performed over the potentially unbounded set $\Pi_N^{df}(x)$, it is not immediately obvious that a minimizer exists in (4.16); we address this issue in the next section.

In Chapter 6, methods for actually solving (4.17) for certain classes of constraints and disturbance sets will be examined. For example, it will be shown that, when the constraint set Z is polytopic, then the optimization problem (4.17) is a quadratic program (QP) if the disturbance set W is polytopic, or a second-order cone program (SOCP) if W is ellipsoidal or 2–norm bounded.

4.3 Preliminary Results

We wish to find conditions under which the closed-loop system (4.13) is input-to-state stable (ISS). In order to do this, we first develop some results related to the convexity of the value function $V_N^*(\cdot)$ in (4.14), and to input-to-state stability for systems with convex Lyapunov functions. Proofs for all of the results in this section are presented in the appendix to the chapter.

[2] Note that since $\mathbf{w}$ is assumed zero mean, it follows that

$$\mathbb{E}\left[\mathbf{w}^\top (X\mathbf{w})\right] = \mathrm{tr}\left(\mathbb{E}\left[(X\mathbf{w})\mathbf{w}^\top\right]\right) = \mathrm{tr}\left(X\mathbb{E}\left[\mathbf{w}\mathbf{w}^\top\right]\right) = \mathrm{tr}\,(X\mathbf{C}_w) = \mathrm{tr}(\mathbf{C}_w^{\frac{1}{2}} X \mathbf{C}_w^{\frac{1}{2}}).$$

4.3.1 Continuity and Convexity

We first demonstrate that the value function $V_N^*(\cdot)$ in (4.14) is convex and continuous on the interior of its domain; this property will prove useful in our subsequent proof of stability for the closed-loop system (4.13). Note that the proof presented here requires *only* convexity of the state and input constraints, and does *not* make the usual assumption (as in [BMDP02, Bor03]) that the constraint sets Z and X_f and disturbance set W are polyhedral, leading to a piecewise quadratic value function. We instead exploit several results from variational analysis to establish convexity of $V_N^*(\cdot)$ directly; situations where both the constraint and disturbance sets are polyhedral will be treated as a special case.

Proposition 4.5 (Properties of $V_N^*(\cdot)$ and $\mu_N(\cdot)$). *If X_N^{sf} has nonempty interior, then the receding horizon control law $\mu_N(\cdot)$ is unique on X_N^{sf} and continuous on $\mathrm{int}(X_N^{sf})$. The value function $V_N^*(\cdot)$ is convex on X_N^{sf}, continuous on $\mathrm{int}(X_N^{sf})$ and lower semicontinuous everywhere on X_N^{sf}.*

Corollary 4.6. *If X_N^{sf} has nonempty interior, $\mathcal{Z}$ is polyhedral and W is polytopic, then the receding horizon control law $\mu_N(\cdot)$ is piecewise affine and the value function $V_N^*(\cdot)$ is piecewise quadratic on X_N^{sf}.*

Corollary 4.7. *The function $J_N(x,\cdot,\cdot)$ attains its minimum on the set $\Pi_N^{df}(x)$.*

Note that in conjunction with Thm. 3.9, Cor. 4.7 implies that $V_N(x,\cdot,\cdot)$ also attains its minimum on the set $\Pi_N^{sf}(x)$ in (4.17), so that the optimal policy sets in (4.11) and (4.16) are both well defined.

4.3.2 Input-to-State Stability

We next develop a generic result on the input-to-state stability of systems with convex value functions. We can then exploit the convexity of the value function $V_N^*(\cdot)$ to provide conditions under which the closed-loop system (4.13) is input-to-state stable.

Consider a nonlinear, time-invariant, discrete-time system of the form

$$x^+ = f(x,\omega), \tag{4.20}$$

where $x \in \mathbb{R}^n$ is the state and $\omega \in \mathbb{R}^p$ is a disturbance that takes on values in a compact set $\Omega \subset \mathbb{R}^p$ containing the origin. It is assumed that the state is measured at each time instant, that $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is continuous at the origin and that $f(0,0) = 0$. Given an initial state x and a disturbance sequence $\omega(\cdot)$, where $\omega(k) \in \Omega$ for all $k \in \mathbb{Z}_{[0,\infty)}$, let the solution to (4.20) at time k be denoted by $\phi(k, x, \omega(\cdot))$. For systems of this type, a useful notion of stability is input-to-state stability [JW01, Kha02, Son00].

Definition 4.8 ($\mathcal{K}$ and $\mathcal{K}_\infty$ functions). *A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a* $\mathcal{K}$-function *if it is strictly increasing and $\gamma(0) = 0$. It is a* $\mathcal{K}_\infty$-function *if, in addition, $\gamma(s) \to \infty$ as $s \to \infty$.*

Definition 4.9 ($\mathcal{KL}$ functions). *A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{KL}$-function if, for all $k \geq 0$, the function $\beta(\cdot, k)$ is a* $\mathcal{K}$-function *and, for each $s \geq 0$, $\beta(s, \cdot)$ is decreasing, with $\beta(s, k) \to 0$ as $k \to \infty$.*

Definition 4.10 (Input-to-State Stability). *The system* (4.20) *is* input-to-state stable (ISS) *in $\mathcal{X} \subseteq \mathbb{R}^n$ if there exist a $\mathcal{KL}$-function $\beta(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that for all initial states $x \in \mathcal{X}$ and disturbance sequences $\omega(\cdot)$, where $\omega(k) \in \Omega$ for all $k \in \mathbb{Z}_{[0,\infty)}$, the solution of the system satisfies $\phi(k, x, \omega(\cdot)) \in \mathcal{X}$ and*

$$\|\phi(k, x, \omega(\cdot))\| \leq \beta(\|x\|, k) + \gamma\left(\sup\left\{\|\omega(\tau)\| \;\middle|\; \tau \in \mathbb{Z}_{[0,k-1]}\right\}\right) \tag{4.21}$$

for all $k \in \mathbb{N}$.

Input-to-state stability implies that the origin is an asymptotically stable point for the undisturbed system $x^+ = f(x, 0)$ with region of attraction $\mathcal{X}$, and also that all state trajectories are bounded for all bounded disturbance sequences. Furthermore, every trajectory $\phi(k, x, \omega(\cdot)) \to 0$ if $\omega(k) \to 0$ as $k \to \infty$.

In order to be self-contained, we also recall the following useful result from [JW01, Lem 3.5]:

Lemma 4.11 (ISS-Lyapunov function). *The system* (4.20) *is ISS in $\mathcal{X} \subseteq \mathbb{R}^n$ if the following conditions are satisfied:*

i. $\mathcal{X}$ contains the origin in its interior and is robust positively invariant for (4.20), *i.e. $f(x, \omega) \in \mathcal{X}$ for all $x \in \mathcal{X}$ and all $\omega \in \Omega$.*

ii. There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$, a $\mathcal{K}$-function $\sigma(\cdot)$, and a function $V : \mathcal{X} \to \mathbb{R}_+$ such that for all $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{4.22a}$$

$$V(f(x, \omega)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|\omega\|). \tag{4.22b}$$

Remark 4.12. *A function $V(\cdot)$ that satisfies the conditions in Lemma 4.11 is called an* ISS-Lyapunov function*. Note that although it is assumed in [JW01], continuity of the function $V(\cdot)$ is* not *required in the proof of Lemma 4.11, nor do we require that $f(\cdot)$ be continuous everywhere on $\mathcal{X}$; recall that, in general, convex functions are* not *guaranteed to be continuous everywhere.*

In order to establish that the closed-loop system (4.13) is ISS, we will make use of the following result building on Lemma 4.11:

Proposition 4.13 (Convex Lyapunov function for undisturbed system).
Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a compact robust positively invariant set for (4.20) *containing the origin in its interior. Furthermore, let there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$, and a convex function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{X}$,*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{4.23a}$$

$$V(f(x,0)) - V(x) \leq -\alpha_3(\|x\|). \tag{4.23b}$$

The function $V(\cdot)$ is an ISS-Lyapunov function and the system (4.20) *is ISS in $\mathcal{X}$ if $f(\cdot)$ can be written as*

$$f(x,\omega) := g(x) + \omega, \tag{4.23c}$$

where $g(\cdot)$ is continuous at the origin with $g(0) = 0$, and Ω is compact and convex, containing the origin in its relative interior.

Remark 4.14. *Note that Proposition 4.13 requires only that $V(\cdot)$ be convex on $\mathcal{X}$, and does* not *require continuity of the function $V(\cdot)$ everywhere on (and, in particular, on the boundary of) its domain. This allows application of the result to a broad class of systems with arbitrary convex constraints, since in these cases one can often only find functions, such as the value function* (4.17)*, which are convex and lower semicontinuous.*

We next derive a result on input-to-state stability for systems with Lipschitz continuous Lyapunov functions. This result will allow for a less conservative estimate of the ISS-gain function $\sigma(\cdot)$ in (4.22b) than that found in the proof of Prop. 4.13; this will prove useful in providing an alternative proof of input-to-state stability in the case that the state and input constraint set Z and disturbance set W are polytopic.

Proposition 4.15 (Lipschitz Lyapunov function for undisturbed system).
Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a robust positively invariant set for (4.20) *containing the origin in its interior. Furthermore, let there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a function*

$V : \mathcal{X} \to \mathbb{R}_+$ that is Lipschitz continuous on $\mathcal{X}$ such that for all $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{4.24a}$$

$$V(f(x,0)) - V(x) \leq -\alpha_3(\|x\|). \tag{4.24b}$$

The function $V(\cdot)$ is an ISS-Lyapunov function and the origin is ISS for the system (4.20) *with region of attraction $\mathcal{X}$ if either of the following conditions are satisfied:*

i. $f : \mathcal{X} \times \Omega \to \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{X} \times \Omega$.

ii. $f(x, \omega) := g(x) + \omega$, where $g : \mathcal{X} \to \mathbb{R}^n$ is continuous on $\mathcal{X}$.

4.4 Input-to-State Stability of Receding Horizon Control Laws

Given the results of the previous sections, we can now provide conditions under which the closed-loop system (4.13) is guaranteed to be ISS. We first make the following assumption:

A4.2 (Terminal Cost and Constraint)

i. A state feedback gain matrix K_f and terminal constraint set X_f have been chosen such that

(a) The matrix $A + BK_f$ is Hurwitz.

(b) $X_f \subseteq \{x \mid (x, K_f x) \in Z\}$.

(c) $(A + BK_f)x + Gw \in X_f$, for all $x \in X_f$ and all $w \in W$.

ii. The feedback matrix K_f and terminal cost matrix P are derived from the solution to the discrete algebraic Riccati equation:

$$\begin{aligned} P &:= C_z^\top C_z + A^\top PA - A^\top PB(D_z^\top D_z + B^\top PB)^{-1}B^\top PA \\ K_f &:= -(D_z^\top D_z + B^\top PB)^{-1}B^\top PA. \end{aligned}$$

iii. The set of state and input constraints Z is compact and contains the origin in its interior.

Remark 4.16. *Note that the conditions in **A4.2(i)** are identical to those in **A3.2**. If **A4.2**(ii) holds, then the terminal cost function $F(x) := x^\top Px$ is a Lyapunov function in* *the terminal set X_f for the undisturbed closed-loop system $x^+ = (A + BK_f)x$, in the sense that*

$$F((A + BK_f)x) - F(x) \leq -x^\top(C_z^\top C_z + K_f^\top D_z^\top D_z K_f)x, \quad \forall x \in X_f, \tag{4.25}$$

which also guarantees the Hurwitz condition in ***A4.2****(i). No cross terms $C_z^\top D_z$ appear in* ***A4.2****(ii) since these are assumed zero in* ***A4.1****. The assumption about compactness of Z in* ***A4.2****(iii) is more restrictive than the standing assumption in* ***A4.1*** *(equivalently* ***A3.1****) that the closed set Z is bounded only in the inputs; this compactness is required in the proofs of stability that follow.*

Remark 4.17. *In the absence of constraints, the control policy $u = K_f x$ minimizes* both *the expected value of $\Phi(x, \mathbf{u}, \mathbf{w})$ (assuming $\mathbb{E}[\mathbf{w}] = 0$),* and *the value of the deterministic or certainty-equivalent cost one would compute by setting $\mathbf{w} = \{0\}$ [vW81] (i.e. by calculating $\Phi(x, \mathbf{u}, 0)$). It should be noted that this certainty equivalence property does* not *hold in the general constrained case considered here; i.e.*

$$V_N^*(x) = \underset{(\mathbf{M},\mathbf{v})\in\Pi_N^{df}(x)}{\operatorname{argmin}} \mathbb{E}[\Phi_N(x, \bar{\mathbf{u}}, \mathbf{w})] \neq \underset{(\mathbf{M},\mathbf{v})\in\Pi_N^{df}(x)}{\operatorname{argmin}} \Phi(x, \mathbb{E}[\bar{\mathbf{u}}], 0) = \underset{(\mathbf{M},\mathbf{v})\in\Pi_N^{df}(x)}{\operatorname{argmin}} \Phi(x, \mathbf{v}, 0),$$

where $\bar{\mathbf{u}} = \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v}$. However, it is still true that $v_0^*(x) = K_f x$ for all $x \in X_f$, since in this case ***A4.2****(i) guarantees that the optimal* unconstrained *state feedback gain K_f is also constraint admissible, so that the RHC law matches the behavior of an unconstrained LQR or $\mathcal{H}_2$ control law.*

The next result shows that the receding horizon control law $\mu_N(\cdot)$ stabilizes the *undisturbed* system $x^+ = Ax + B\mu_N(x)$. This result is required in the proof of input-to-state stability for the disturbed system (4.13).

Lemma 4.18 (Properties of $\mu_N(\cdot)$ and $V_N^*(0)$). *If* ***A4.2*** *holds, then the following conditions hold:*

i. The RHC law $\mu_N(\cdot)$ satisfies $\mu_N(0) = 0$.

ii. There exists a positive constant k_1 such that $V_N^*(Ax + B\mu_N(x)) - V_N^*(x) \leq -k_1 \|x\|_2^2$.

iii. There exist positive constants k_2 and k_3 such that $k_2 \|x\|_2^2 \leq (V_N^*(x) - V_N^*(0)) \leq k_3 \|x\|_2^2$.

iv. The undisturbed closed-loop system $x^+ = Ax + B\mu_N(x)$ is exponentially stable in X_N^{sf}.

Proof. See the appendix.

Combining this result with that of Proposition 4.13, which relates to ISS stability for systems with convex Lyapunov functions, leads to the main result of this chapter:

Theorem 4.19 (ISS for RHC). *If **A4.2** holds, then the closed-loop system* (4.13) *is ISS in X_N^{sf}. Furthermore, the input and state constraints $(x(i), u(i)) \in Z$ are satisfied for all time and for all allowable disturbance sequences if the initial state $x(0) \in X_N^{sf}$.*

Proof. For the system of interest, define $V(\cdot) = V_N^*(\cdot) - V_N^*(0)$, and let $\Omega := GW$ and $f(x, w) := Ax + B\mu_N(x) + Gw$. If **A4.2** holds, then the set X_N^{sf} is robust positively invariant for system (4.13), with $0 \in \operatorname{int} X_N^{sf}$ (Prop. 3.13). The set X_N^{sf} is closed (Thm. 3.5) and bounded because Z is assumed compact in **A4.1**(iii), hence X_N^{sf} is compact.

From Prop. 4.5 the function $V_N^*(\cdot)$ is convex and continuous on $\operatorname{int}(X_N^{sf})$. The remainder of the proof follows by direct application of the results in Lem. 4.18 and Prop. 4.13. □

The above result leads immediately to the following modified version of Theorem 4.19 with a less conservative bound on the ISS-gain function $\sigma(\cdot)$:

Theorem 4.20 (ISS for RHC with Polytopic Constraints). *Let the sets W, Z and X_f be polytopes. If **A4.2** holds, then the closed-loop system* (4.13) *is ISS in X_N^{sf}. Furthermore, the input and state constraints $(x(i), u(i)) \in Z$ are satisfied for all time and for all allowable disturbance sequences if the initial state $x(0) \in X_N^{sf}$. The ISS-gain satisfies $\sigma(\cdot) \leq L_V \|(\cdot)\|$, where L_V is the Lipschitz constant of the ISS Lyapunov function $V_N^*(\cdot)$.*

Proof. Since Z is assumed compact, the function V_N^* is piecewise quadratic (Prop. 4.6) with compact domain, hence Lipschitz continuous. The remainder of the proof is identical to that of Theorem 4.19, with the use of Cor. 4.6 and Prop. 4.15 in place of Prop. 4.5 and Prop. 4.13, respectively. The bound on the ISS-gain function $\sigma(\cdot)$ is immediate from the proof of Prop. 4.15(ii). □

Remark 4.21. *Upon examination of the proof of Lemma 4.18, it is straightforward to show that if $C_w = 0$ then all of the results of this section still hold if the Riccati condition **A4.2**(ii) on the matrices K_f and P is replaced with a more relaxed requirement of the form* (4.25).

4.4.1 Non-quadratic costs

The choice of a quadratic function in (4.4) is primarily due to the convenience it affords in dealing with the expected value operation in (4.15), and in particular to the ease with which convexity can be established in Prop. 4.4. If one instead defines the cost as a function of the *nominal* or *disturbance free* state and input trajectory, then a larger variety of cost functions can be handled conveniently.

Define the nominal states $\hat{x}_i$ and nominal inputs $\hat{u}_i$ to be the expected values of the states x_i and inputs u_i respectively. Note that since $\mathbb{E}[w_i] = 0$ by assumption, if the inputs u_i are chosen according to a disturbance feedback policy $(\mathbf{M}, \mathbf{v})$ of the form (3.17), then

$$\hat{\mathbf{x}} := \text{vec}(\hat{x}_0, \ldots, \hat{x}_{N-1}, \hat{x}_N) = \mathbf{A}x + \mathbf{B}\mathbf{v}. \tag{4.26}$$

$$\hat{\mathbf{u}} := \text{vec}(\hat{u}_0, \ldots, \hat{u}_{N-1}) = \mathbf{v}. \tag{4.27}$$

Define a pair of proper, convex lower semicontinuous functions $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}$, and a (potentially nonlinear) terminal control law $\kappa_f : X_f \to \mathbb{R}^m$. Then the following result can be proven via straightforward modification of the results presented thus far in this chapter:

Theorem 4.22. *Suppose that the following conditions hold in place of those in* ***A4.2****:*

i. $\ell(x, u)$ *is strictly convex in* u.

ii. There exist positive constants k_1, k_2, k_3, k_4 *and* c *such that*

(a) $k_1 \|(x, u)\|^c \leq \ell(x, u) \leq x_2 \|(x, u)\|^c$

(b) $k_3 \|x\|^c \leq F(x) \leq k_4 \|x\|^c$,

iii. For all $x \in X_f$, *the terminal set* X_f, *control law* $\kappa_f(\cdot)$ *and cost* $F(\cdot)$ *satisfy*

(a) $X_f \subseteq \{x \mid (x, \kappa_f(x)) \in Z\}$

(b) $F(Ax + B\kappa_f(x)) - F(x) \leq -\ell(x, u)$

(c) $Ax + B\kappa_f(x) + Gw \in X_f$ *for all* $w \in W$.

iv. The set Z *is compact.*

If the definitions (4.4) *and* (4.15) *are replaced with*

$$\Phi(x, \mathbf{u}, \mathbf{w}) := F(x_N) + \sum_{i=0}^{N-1} \ell(x_i, u_i) \tag{4.28}$$

$$J_N(x, \mathbf{M}, \mathbf{v}) := \Phi(x, \mathbb{E}[\mathbf{u}]) = \Phi(x, \mathbf{v}, 0) \tag{4.29}$$

respectively, then all of the results in Prop 4.4, Prop. 4.5 and Thm. 4.19 still hold.

4.5 Conclusions

By exploiting the results in Chapter 3 in the calculation of optimal receding horizon control laws, we have shown that input-to-state stability of the closed-loop system can be established for problems with general convex state and input constraints using the expected value of a quadratic cost, given appropriate terminal conditions.

The keys to these results are proving the existence of minimizers and convexity of the value function in the underlying optimal control problem using results from variational analysis, as well as providing conditions under which input-to-state stability may be established using convex Lyapunov functions. In the particular case that all the constraint sets are polytopic, stronger bounds on the ISS gain of the system were established.

In the next chapter we consider an alternative to the expected value cost function considered here, and employ a min-max cost where the disturbances are negatively weighted as in $\mathcal{H}_\infty$ control. This will allow us to formulate RHC laws with guaranteed bounds on their ℓ_2 gain.

We have not yet commented in detail on how one could actually compute the solution to any of the convex problems posed thus far. We rectify this in Chapters 6 and 7, where computational methods will be presented.

4.A Proofs

Proof of Proposition 4.5

Proof. We exploit the equivalence results of Prop. 4.3 and work with the convex problems (4.16) and (4.17). Define the function $f : \mathbb{R}^n \times \mathbb{R}^{mN \times nN} \times \mathbb{R}^{mN} \to \bar{\mathbb{R}}$ as

$$f(x, \mathbf{M}, \mathbf{v}) := \begin{cases} J_N(x, \mathbf{M}, \mathbf{v}) & \text{if } (x, \mathbf{M}, \mathbf{v}) \in \mathcal{C}_N, \\ \infty & \text{otherwise,} \end{cases} \tag{4.30}$$

where $\mathcal{C}_N$ is defined as in (3.24) and is nonempty since X_N^{df} is assumed nonempty (recall that $X_N^{sf} = X_N^{df}$ from Thm. 3.9). Recall from (4.19) that

$$J_N(x, \mathbf{M}, \mathbf{v}) = \|H_x x + H_u \mathbf{v}\|_2^2 + \operatorname{tr}\left(\mathbf{C}_w^{\frac{1}{2}}(H_u \mathbf{M}\mathcal{G} + H_w)^\top (H_u \mathbf{M}\mathcal{G} + H_w)\mathbf{C}_w^{\frac{1}{2}}\right), \tag{4.31}$$

and from (4.16) and (4.17) that

$$V_N^*(x) = \min_{(\mathbf{M},\mathbf{v})} f(x, \mathbf{M}, \mathbf{v}), \quad (\mathbf{M}^*(x), \mathbf{v}^*(x)) = \operatorname*{argmin}_{(\mathbf{M},\mathbf{v})} f(x, \mathbf{M}, \mathbf{v}). \tag{4.32}$$

The central difficulty is that the function $f(x, \mathbf{M}, \mathbf{v})$ is not uniformly level bounded and the set $\mathcal{C}_N$ is unbounded in the policy parameter $\mathbf{M}$, so that it is not possible to apply the results of Prop. 2.23 directly to (4.32). We therefore define $\mathcal{V}_N \subseteq \mathbb{R}^n \times \mathbb{R}^{mN \times lN} \times \mathbb{R}^{mN}$

$$\mathcal{V}_N := \left\{ (x, \mathbf{Y}, \mathbf{v}) \;\middle|\; \mathbf{Y} = \mathbf{M}\mathcal{G}\mathbf{C}_w^{\frac{1}{2}},\ (x, \mathbf{M}, \mathbf{v}) \in \mathcal{C}_N \right\}. \tag{4.33}$$

Since the mapping $\mathcal{C}_N \mapsto \mathcal{V}_N$ is a linear transformation satisfying the conditions of Lem. 3.4, the set $\mathcal{V}_N$ is nonempty, convex and closed. Define

$$\tilde{f}(x, \mathbf{M}, \mathbf{v}) := \begin{cases} \tilde{J}_N(x, \mathbf{Y}, \mathbf{v}) & \text{if } (x, \mathbf{Y}, \mathbf{v}) \in \mathcal{V}_N, \\ \infty & \text{otherwise,} \end{cases} \tag{4.34}$$

where

$$\tilde{J}_N(x, \mathbf{Y}, \mathbf{v}) := \|H_x x + H_u \mathbf{v}\|_2^2 + \operatorname{tr}\left((H_u \mathbf{Y} + H_w \mathbf{C}_w^{\frac{1}{2}})^\top (H_u \mathbf{Y} + H_w \mathbf{C}_w^{\frac{1}{2}})\right), \tag{4.35}$$

so that the function (4.34) is convex, lower semicontinuous and proper. Since H_u is full column rank (Lem. 4.1.1), $\tilde{f}(x, \mathbf{Y}, \mathbf{v})$ is strictly convex in $(\mathbf{Y}, \mathbf{v})$ and level bounded in $(\mathbf{Y}, \mathbf{v})$ locally uniformly in x (Prop. 2.22). Rewriting (4.32) as

$$V_N^*(x) = \min_{(\mathbf{Y},\mathbf{v})} \tilde{f}(x, \mathbf{Y}, \mathbf{v}), \quad (\mathbf{Y}^*(x), \mathbf{v}^*(x)) = \operatorname*{argmin}_{(\mathbf{Y},\mathbf{v})} \tilde{f}(x, \mathbf{Y}, \mathbf{v}), \tag{4.36}$$

it follows that V_N^* is convex and lower semicontinuous on X_N^{df} (Prop. 2.23(ii)) and strictly continuous on $\operatorname{int} X_N^{sf}$ (Prop. 2.15). The optimal feedback policy parameter $\mathbf{v}^*(x)$, defined in (4.16) is single-valued on X_N^{sf} and continuous on $\operatorname{int} X_N^{sf}$ (Prop. 2.23(iv)). The uniqueness and continuity properties of $\mu_N(\cdot) = v_0^*(\cdot)$ then follow directly. □

Proof of Corollary 4.6

Proof. If Z, X_f and W are polyhedral, then the set $\mathcal{C}_N$ is polyhedral (Cor. 3.8). The set $\mathcal{V}_N$ in (4.33) is therefore also polyhedral (Prop. 2.10(iii)), so that the optimization problem (4.36) is a strictly convex quadratic program in $(\mathbf{Y}, \mathbf{v})$. By applying the results in [BMDP02], it follows that $V_N^*(\cdot)$ is piecewise quadratic and $\mathbf{v}^*(\cdot)$ (hence $\mu_N(\cdot)$) is piecewise affine on X_N^{sf}. □

Proof of Corollary 4.7

Proof. Obvious from the proof of Prop. 4.5. □

Proof of Proposition 4.13

Proof. We assume that the condition $\Omega \neq \{0\}$ holds, otherwise the proof is trivial. It is sufficient to show that there exists a constant γ such that

$$V(f(x, \omega)) - V(f(x, 0)) \leq \gamma \|\omega\| \tag{4.37}$$

for all $x \in \mathcal{X}$ and all $\omega \in \Omega$. It then follows that

$$\begin{aligned} V(f(x, \omega)) - V(x) &= V(f(x, 0)) - V(x) + V(f(x, \omega)) - V(f(x, 0)) && (4.38)\\ &\leq -\alpha_3(\|x\|) + \gamma \|\omega\|, && (4.39) \end{aligned}$$

and the conditions of Lemma 4.11 are satisfied with $\sigma(s) := \gamma \|s\|$.

When the disturbance set Ω is compact and contains the origin in its (relative) interior, there exists a constant $\rho > 0$ such that

$$\rho := \max\left\{\epsilon \;\middle|\; (\mathcal{B}_\epsilon \bigcap \operatorname{lin}\Omega) \subseteq \Omega\right\}, \tag{4.40}$$

where $\mathcal{B}_\epsilon := \{x \mid \|x\| \leq \epsilon\}$; this is the size of the smallest vector[3] on the (relative) boundary of Ω. Since the set $\mathcal{X}$ is assumed compact, (4.23a) implies that $V(\cdot)$ is upper bounded by a constant b and lower bounded by 0. Since the set $\mathcal{X}$ is robust positively invariant, it follows that

$$g(x) \in \mathcal{X} \sim \Omega, \quad \forall x \in \mathcal{X} \tag{4.41}$$

where $\mathcal{X} \sim \Omega$ denotes the Pontryagin difference, i.e.

$$\mathcal{X} \sim \Omega := \{x \in \mathbb{R}^n \mid x + \omega \in \mathcal{X}, \forall \omega \in \Omega\}. \tag{4.42}$$

Finding a suitable γ in (4.37) is equivalent to finding one that satisfies

$$V(\tilde{x} + \omega) - V(\tilde{x}) \leq \gamma \|\omega\|, \quad \forall \tilde{x} \in \mathcal{X} \sim \Omega,\ \forall \omega \in \Omega. \tag{4.43}$$

Since Ω is assumed convex and compact with $0 \in \operatorname{int}(\Omega)$, for any given $\omega \in \Omega$ there exists a $\tilde{\omega}$ on the (relative) boundary of Ω such that $\omega = \tau\tilde{\omega}$, with $0 \leq \tau \leq 1$ and $\tau = \|\omega\| / \|\tilde{\omega}\| \leq \|\omega\| / \rho$. Since $\mathcal{X}$ is robust positively invariant, $\tilde{x} + \tilde{\omega} \in \mathcal{X}$ for all $\tilde{x} \in \mathcal{X} \sim \Omega$. Since V is convex, it follows that $V(\tilde{x} + \omega) \leq (1 - \tau)V(\tilde{x}) + \tau V(\tilde{x} + \tilde{\omega})$, or

$$V(\tilde{x} + \omega) - V(\tilde{x}) \leq \tau(V(\tilde{x} + \tilde{\omega}) - V(\tilde{x})) \leq (b/\rho)\|\omega\|. \tag{4.44}$$

The proof is completed by selecting $\gamma := b/\rho$. □

Proof of Proposition 4.15

Proof. Let L_V be the Lipschitz constant of $V(\cdot)$.

i. Since $\|V(f(x,\omega)) - V(f(x,0))\| \leq L_V\|f(x,\omega) - f(x,0)\| \leq L_V L_f\|\omega\|$, where L_f is the Lipschitz constant of $f(\cdot)$, it follows that $V(f(x,\omega)) - V(x) = V(f(x,0)) - V(x) + V(f(x,\omega)) - V(f(x,0)) \leq -\alpha_3(\|x\|) + L_V L_f\|\omega\|$. The proof is completed by letting $\sigma(s) := L_V L_f s$ in Lemma 4.11.

ii. Note that $\|V(f(x,\omega)) - V(f(x,0))\| \leq L_V\|\omega\|$. The proof is completed as for (i), but by letting $\sigma(s) := L_V s$ in Lemma 4.11. □

[3] Note that when Ω has a nonempty interior, this simplifies to $\rho = \max\{\epsilon \mid \mathcal{B}_\epsilon \subseteq \Omega\}$.

Proof of Lemma 4.18

Proof. If the matrices K_f and P satisfy the Riccati condition in **A4.2**(ii), then it can be shown [ÅW97, Sec. 11.2] that the cost function $\Phi(x, \mathbf{u}, \mathbf{w})$ defined in (4.4) can be rewritten as

$$\begin{aligned}\Phi(x, \mathbf{u}, \mathbf{w}) = x^\top P x + &\sum_{i=0}^{N-1} (u_i - K_f x_i)^\top (B^\top P B + D_z^\top D_z)(u_i - K_f x_i) + \\ &\sum_{i=0}^{N-1} \Big(w_i^\top G^\top P (A x_i + B u_i) + (A x_i + B u_i)^\top P G w_i \Big) + \sum_{i=0}^{N-1} w_i^\top G^\top P G w_i. \end{aligned} \tag{4.45}$$

Since the disturbances w_i are assumed independent and zero mean, the expected value of this function is

$$\mathbb{E}\left[\Phi(x, \mathbf{u}, \mathbf{w})\right] = x^\top P x + \mathbb{E}\left[\sum_{i=0}^{N-1} \|(u_i - K_f x_i)\|^2_{(B^\top P B + D_z^\top D_z)} + \sum_{i=0}^{N-1} w_i^\top G^\top P G w_i \right] \tag{4.46a}$$

$$= x^\top P x + \mathbb{E}\left[\sum_{i=0}^{N-1} \|(u_i - K_f x_i)\|^2_{(B^\top P B + D_z^\top D_z)} \right] + \mathrm{tr}\left(\mathbf{C}_w^{\frac{1}{2}} \mathcal{G}^\top P \mathcal{G} \mathbf{C}_w^{\frac{1}{2}} \right). \tag{4.46b}$$

Recalling the definition of $V_N(x, \mathbf{K}, \mathbf{g})$ in (4.10), if **A4.2**(i) holds then

$$(I_N \otimes K_f, 0) = \underset{(\mathbf{K},\mathbf{g}) \in \Pi_N^{sf}(x)}{\operatorname{argmin}} \; V_N(x, \mathbf{K}, \mathbf{g}), \text{ for all } x \in X_f, \tag{4.47}$$

and the RHC law $\mu_N(x) = K_f x$. This proves part (i), since $0 \in X_f$ by assumption. To prove part (ii), suppose that for some state $x \in X_N^{sf}$ the policy $(\mathbf{K}^*(x), \mathbf{g}^*(x))$ is optimal, so that $V_N^*(x) = V_N(x, \mathbf{K}^*(x), \mathbf{g}^*(x))$ and $\mu_N(x) = K_{0,0}^*(x)x + g_0^*(x)$. By definition,

$$V_N(x, \mathbf{K}^*(x), \mathbf{g}^*(x)) = \mathbb{E}\left[\Phi(x, \mathbf{u}^*(x), \mathbf{w})\right],$$

where $\mathbf{u}^*(x) := (I - \mathbf{K}^*(x)\mathbf{B})^{-1}(\mathbf{K}^*(x)\mathbf{A}x + \mathbf{G}\mathbf{w}) + \mathbf{g}$ and $u_0^*(x) = \mu_N(x)$. From examination of (4.46), it follows that

$$\begin{aligned} J_N(x, \tilde{\mathbf{K}}(x^+), \tilde{\mathbf{g}}(x^+)) = \mathbb{E}\left[\Phi\left(x, \mathbf{u}^*(x), \mathbf{w}\right) \Big| w_0 = 0 \right] + \|Ax + B\mu_N(x)\|_P^2 \\ - \|x\|_P^2 - \|\mu_N(x) - K_f x\|^2_{(B^\top P B + D_z^\top D_z)}. \end{aligned}$$

Recalling the definitions of K_f and P in **A4.2**,

$$\begin{aligned}
&\|Ax+B\mu_N(x)\|_P^2-\|x\|_P^2 - \|\mu_N(x)-K_f x\|^2_{(B^\top PB+D_z^\top D_z)} \\
&\qquad = -x^\top\left[P-A^\top PA+K_f^\top(B^\top PB+D_z^\top D_z)K_f\right]x-\mu_N(x)^\top[B^\top PB+D_z^\top D_z]\mu_N(x) \\
&\qquad = -\|C_z x\|_2^2-\|\mu_N(x)\|^2_{(B^\top PB+D_z^\top D_z)}\,,
\end{aligned}$$

so that

$$J_N(x,\tilde{\mathbf{K}}(x^+),\tilde{\mathbf{g}}(x^+)) = \mathbb{E}\left[\Phi\left(x,\mathbf{u}^*(x),\mathbf{w}\right)\middle| w_0=0\right] - \|C_z x\|_2^2 - \|\mu_N(x)\|^2_{(B^\top PB+D_z^\top D_z)}\,.$$

Since $\Phi(x,\mathbf{u}^*(x),\mathbf{w})$ is a convex quadratic function of $\mathbf{u}^*(x)$ and the vector $\mathbf{u}^*(x)$ is an affine function of the independent, zero-mean disturbances $\{w_0,\dots,w_{N-1}\}$, it follows that

$$\mathbb{E}\left[\Phi\left(x,\mathbf{u}^*(x),\mathbf{w}\right)\middle| w_0=0\right] \le \mathbb{E}\left[\Phi\left(x,\mathbf{u}^*(x),\mathbf{w}\right)\right] = V_N^*(x), \tag{4.48}$$

so that

$$V_N^*(Ax+B\mu_N(x)) \le V_N(x^+,\tilde{\mathbf{K}}(x^+),\tilde{\mathbf{g}}(x^+)) \le V_N^*(x) - \|C_z x\|_2^2 - \|\mu_N(x)\|^2_{(B^\top PB+D_z^\top D_z)}\,.$$

For a matrix Q, define $\bar{\sigma}(Q)$ and $\underline{\sigma}(Q)$ as the maximum and minimum singular values, respectively. Selecting $k_1 = \underline{\sigma}(C_z^\top C_z) > 0$ proves part (ii), since C_z is assumed full column rank in **A4.1**. To prove part (iii), note that $V_N^*(x) \ge \|C_z x\|_2^2$, so that the lower bound can be obtained by selecting $k_2 = \underline{\sigma}(C_z^\top C_z)$. To obtain an upper bound, define

$$\begin{aligned}
u_b &:= \max\{\|u\| \mid \exists x,\ (x,u)\in Z\} \\
x_b &:= \max\{\|x\| \mid \exists u,\ (x,u)\in Z\} \\
\rho &:= \max\left\{\alpha \;\middle|\; \alpha\mathcal{B}_2^n \subseteq X_N^{sf}\right\},
\end{aligned}$$

all of which are finite and positive since each of Z and X_f is assumed to be compact and to contain the origin in its interior. An upper bound for the function $V_N(\cdot)$ is then

$$V_b := N\left(\bar{\sigma}(C_z^\top C_z)x_b^2 + \bar{\sigma}(D_z^\top D_z)u_b^2\right) + \bar{\sigma}(P)x_b^2. \qquad \square$$

Noting from (4.46) and (4.47) that the convex function $(V_N^*(\cdot) - V_N^*(0)) = \|\cdot\|_P^2$ on X_f, it follows that $V_N^*(x) - V_N^*(0) \le \left(\max[\bar{\sigma}(P), V_b/\rho^2]\right)\|x\|_2^2$, for all $x \in X_N^{sf}$. Since $\rho \le x_b$ by construction, selecting $k_2 := \max[\bar{\sigma}(P), V_b/\rho^2] = V_b/\rho^2$ proves (iii). Part (iv) follows from (ii) and (iii) using standard proofs of exponential stability [Kha02, Cor. 3.4].

Chapter 5. Min-Max Costs ($\mathcal{H}_\infty$ Control)

5.1 Introduction

In this chapter, we again consider the problem of finding an optimal control policy for the system

$$x^+ = Ax + Bu + Gw \tag{5.1}$$

$$z = C_z x + D_z u, \tag{5.2}$$

over a finite horizon N, while satisfying a set of mixed constraints $(x, u) \in Z$ for every possible disturbance sequence drawing values from a compact set W. As in Chapter 3, the class of policies considered will be restricted to those modelling the input u_i at each time step as an affine function of the prior disturbance (equivalently, prior state) sequence $\{w_0, \ldots, w_i\}$, so that

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} G w_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}. \tag{5.3}$$

In Chapter 4, an optimal policy was defined as one of the form (5.3) that minimized the expected value of a certain quadratic cost function. It was shown that such an optimal policy could be calculated in principle via convex optimization techniques, and that the system (5.1) could be rendered input-to-state stable (ISS) in closed-loop with a receding horizon control law synthesized from policies that were optimal in this sense. When such a closed-loop system operates far from its state and inputs constraints, its behavior matches that of a system subject to an optimal controller in the LQR or $\mathcal{H}_2$ sense.

In this chapter, we employ a different notion of optimality. Given a positive scalar γ and an initial state x, we wish to determine whether there exists a constraint admissible feedback policy of the form (5.3) and a nonnegative scalar $\beta(x)$ such that the following finite horizon ℓ_2 gain property, routinely encountered in the literature on finite horizon $\mathcal{H}_\infty$ control [HM80, GL95, JB95, Kha02], holds for all allowable disturbance sequences over a finite horizon:

$$\|x_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_2^2 \leq \beta(x) + \sum_{i=0}^{N-1} \gamma^2 \|w_i\|_2^2, \tag{5.4}$$

where $x_0 := x$, $x_{i+1} := Ax_i + Bu_i + Gw_i$ and $z_i := C_z x_i + D_z u_i$ for all $i \in \mathbb{Z}_{[0,N-1]}$. We also wish to find conditions under which receding horizon control laws synthesized from (5.3) can guarantee a finite gain condition similar to (5.4) over an *infinite* horizon, while simultaneously satisfying the system constraints for all time. To this end, we define a cost function $\Phi : \mathbb{R}^n \times \mathbb{R}^{mN} \times \mathbb{R}^{lN} \to \mathbb{R}$ that is negatively weighted in the disturbances

$$\Phi(x, \mathbf{u}, \mathbf{w}) := \|x_N\|_P^2 + \sum_{i=0}^{N-1} \left(\|z_i\|_2^2 - \gamma^2 \|w_i\|_2^2 \right), \tag{5.5}$$

and will consider a policy to be optimal if it minimizes the maximum value (over all allowable disturbances) of (5.5) over policies of the form (5.3)[1]. Throughout this chapter, we make use of the following standing assumptions relating to (5.1)–(5.4):

A5.1 (Standing Assumptions) *The following conditions hold:*

*i. The assumptions **A3.1** hold.*

ii. The pair (C_z, A) is detectable and (C_z, A, B_z) has no zeros on the unit circle.

iii. The matrix D_z is full column rank.

iv. The matrix P is positive semidefinite.

We will also make use of much of the notation introduced in Sections 3.1.1 and 4.1.1.

Remark 5.1. *Note that unlike the conditions in **A4.1** used throughout Chapter 4, we do not make any assumption about the rank of the matrix C_z, and allow $C_z^\top D_z \neq 0$.*

[1] Note that (5.5) differs from the related cost function defined in (4.4) for the expected value case, since it includes a negatively weighted disturbance component.

5.2 A Min-Max Cost Function

In this chapter we work directly with the feedback policy parameterization (5.3), bearing in mind that an equivalence condition similar to Prop. 4.3 can easily be found for the state feedback policy parameterization presented in Chapter 3. We define the finite horizon quadratic cost function

$$J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}) := \Phi(x, \hat{\mathbf{u}}, \mathbf{w}), \tag{5.6}$$

where $\hat{\mathbf{u}} = \mathbf{M}\mathbf{w} + \mathbf{v}$, and consider a zero-sum game of the form:

$$\min_{(\mathbf{M},\mathbf{v})} \max_{\mathbf{w}\in\mathcal{W}} J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}). \tag{5.7}$$

We first define a set of constraints on the feedback policy $(\mathbf{M}, \mathbf{v})$ and the gain γ such that this zero-sum game can be guaranteed to be convex-concave. This will ensure the existence of a saddle point solution in pure policies (see [BB91, Sect. 2.1]), and that (5.7) is solvable via convex optimization techniques. We can then combine these constraints with those presented in Chapter 3 to ensure robust constraint satisfaction, and define the set of policies that is *both* constraint admissible *and* such that the problem (5.7) is convex-concave. To this end, we consider the maximization part of (5.7) in isolation, and define

$$J_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) := \max_{\mathbf{w}\in\mathcal{W}} J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}). \tag{5.8}$$

Recalling the definitions of Section 4.1.1, (5.6) can be written as

$$J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}) = \|H_x x + H_u(\mathbf{v} + \mathbf{M}\mathcal{G}\mathbf{w}) + H_w\mathbf{w}\|_2^2 - \gamma^2 \|\mathbf{w}\|_2^2, \tag{5.9}$$

so that the following condition, guaranteeing that the minimization part of (5.7) is *convex*, holds:

Proposition 5.2. *For each fixed γ, the function $(x, \mathbf{M}, \mathbf{v}) \mapsto J_N^*(x, \gamma, \mathbf{M}, \mathbf{v})$ is convex, lower semicontinuous, proper and bounded below by zero.*

Proof. For any fixed $\mathbf{w}' \in \mathcal{W}$, the function $(x, \mathbf{M}, \mathbf{v}) \mapsto J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}')$ is convex and continuous. Convexity and lower semicontinuity follow since the pointwise supremum of lower semicontinuous and convex functions is lower semicontinuous and convex (Prop. 2.17). The lower bound is obvious from inspection of (5.9) since $0 \in \mathcal{W}$ by assumption. Since $\mathcal{W}$ is assumed compact, the function is also finite for every $(\mathbf{M}, \mathbf{v})$; hence proper. $\square$

We next impose a condition on γ such that the maximization problem in (5.8) is *concave*, so that our eventual min-max policy optimization problem will be convex-concave:

Proposition 5.3. *For any given* $\mathbf{M}$*, there exists a* $\gamma \geq 0$ *such that the following linear matrix inequality (LMI) holds:*

$$\begin{pmatrix} -\gamma I & (H_u\mathbf{M}\mathcal{G} + H_w) \\ (H_u\mathbf{M}\mathcal{G} + H_w)^\top & -\gamma I \end{pmatrix} \preceq 0. \tag{5.10}$$

Furthermore, if (5.10) *is satisfied, then the function* $\mathbf{w} \mapsto J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w})$ *is concave.*

Proof. From inspection of (5.9), it is obvious that in order for $\mathbf{w} \mapsto J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w})$ to be concave, γ and $\mathbf{M}$ must satisfy

$$(H_u\mathbf{M}\mathcal{G} + H_w)^\top(H_u\mathbf{M}\mathcal{G} + H_w) - \gamma^2 I \preceq 0, \tag{5.11}$$

and that it is always possible to ensure satisfaction of (5.11) by choosing γ large enough. We need only to show that (5.10) is equivalent to (5.11). We first consider $\gamma > 0$, and multiply (5.11) by $1/\gamma$ to get the equivalent condition

$$(H_u\mathbf{M}\mathcal{G} + H_w)^\top(\gamma^{-1} I)(H_u\mathbf{M}\mathcal{G} + H_w) - \gamma I \preceq 0, \tag{5.12}$$

which is itself equivalent to (5.10) from the Schur complement Lemma 2.20. For the case $\gamma = 0$ the condition (5.11) is only satisfied if $(H_u\mathbf{M}\mathcal{G} + H_w) = 0$. This is also the case for (5.10) since any matrix of the form $\left[\begin{smallmatrix} 0 & X \\ X & 0 \end{smallmatrix}\right]$ is indefinite for every $X \neq 0$. $\square$

Remark 5.4. *If* (5.11) *holds, then the optimization problem* (5.8) *requires the maximization of a concave function over a convex set or, equivalently, the minimization of a convex function over a convex set. In particular, if W is a polytope, then* (5.8) *is equivalent to a tractable quadratic program (QP) and if W is an ellipsoid or the affine map of a 2-norm ball, then* (5.8) *can be written as a tractable second-order cone program (SOCP) [LVBL98]. We consider these problems in more detail in Chapter 6.*

5.2.1 Notation and Definitions

As a result of Prop. 5.3, we will consider a policy of the form (5.3) to be feasible (with respect to a given γ) if, given an initial state x, it is both constraint admissible and satisfies the LMI condition (5.10). To this end, for a given state x and positive scalar γ, we define the set of policies satisfying these conditions as

$$\Pi_N^{\gamma}(x,\gamma) := \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; (\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x),\, (\gamma,\mathbf{M}) \text{ satisfies } (5.10) \right\}, \tag{5.13}$$

and the set of states for which a policy of the form (5.13) exists as

$$X_N^{\gamma}(\gamma) := \left\{ x \;\middle|\; \Pi_N^{\gamma}(x,\gamma) \neq \emptyset \right\}. \tag{5.14}$$

Note that in these definitions, the superscript γ is used to indicate that an LMI constraint of the form (5.10) has been used as a part of the set definition, while the argument γ is a variable used to indicate the degree of tightness of this constraint.

Inclusion of these LMI constraints imposes tighter requirements on admissibility for the sets $\Pi_N^{\gamma}(x,\gamma)$ and $X_N^{\gamma}(\gamma)$ than for the related sets $\Pi_N^{df}(x)$ and X_N^{df} defined in Chapter 3, i.e. for any state x and positive scalar γ, the set inclusions $X_N^{\gamma} \subseteq X_N^{df}$ and $\Pi_N^{\gamma}(x,\gamma) \subseteq \Pi_N^{df}(x,\gamma)$ hold. It is also easily shown that $X_N^{\gamma}(\gamma_1) \subseteq X_N^{\gamma}(\gamma_2)$ for any $\gamma_1 \leq \gamma_2$. Also, since the state and input constraint set Z is assumed bounded in the inputs, there exists some $\gamma < \infty$ such that $X_N^{\gamma}(\gamma) = X_N^{df}$.

For a given initial state $x \in X_N^{df}$, we are interested in computing the smallest positive value of γ for which one can ensure that the finite horizon ℓ_2 gain property (5.4) holds when the control inputs are chosen in accordance with a control policy in the form (5.3). For this purpose, define the function $\gamma_N^* : X_N^{df} \to \mathbb{R}_+$ as

$$\gamma_N^*(x) := \inf \left\{ \gamma \;\middle|\; \exists (\mathbf{M},\mathbf{v}) \in \Pi_N^{\gamma}(x,\gamma) \right\}. \tag{5.15}$$

For a given initial state x and $\gamma \geq \gamma_N^*(x)$, we define an optimal policy pair (with respect to γ) as

$$(\mathbf{M}^*(x,\gamma), \mathbf{v}^*(x,\gamma)) := \underset{(\mathbf{M},\mathbf{v}) \in \Pi_N^{\gamma}(x,\gamma)}{\operatorname{argmin}} \; J_N^*(x,\gamma,\mathbf{M},\mathbf{v}), \tag{5.16}$$

and assume that such a minimizer can be partitioned in a manner identical to that in (3.19), so that the k^{th} element of $\mathbf{v}^*(x,\gamma)$ is denoted $v_k^*(x,\gamma)$, and the $(i,j)^{\text{th}}$ submatrix of $\mathbf{M}^*(x,\gamma)$ is denoted $M_{i,j}^*(x,\gamma)$. We also define the value function $V_N^* : X_N^{df} \times \mathbb{R}_+ \to \mathbb{R}_+$ as

$$V_N^*(x,\gamma) := \min_{(\mathbf{M},\mathbf{v}) \in \Pi_N^{\gamma}(x,\gamma)} J_N^*(x,\gamma,\mathbf{M},\mathbf{v}). \tag{5.17}$$

We assume for the moment that the minimizer in (5.16) exists and is well-defined.

5.2.2 Finite Horizon Control Laws

Using the results presented thus far, we can derive conditions under which a finite horizon gain condition of the form (5.4) can be guaranteed:

Theorem 5.5. *For a given initial state $x(0) \in X_N^{df}$ and $\gamma \geq \gamma_N^*(x(0))$, consider implementing the following time-varying control policy on the system* (5.1)*:*

$$u(k) = v_k^*(x(0),\gamma) + \sum_{j=0}^{k-1} M_{k,j}^*(x(0),\gamma)\,(x(j+1) - Ax(j) - Bu(j)) \tag{5.18}$$

for all $k \in \{0,1,\ldots,N-1\}$. For all disturbance sequences $\{w(k)\}_{k=0}^{N-1}$ drawing values from W, we have that $(x(k),u(k)) \in Z$ for $k \in \{0,1,\ldots,N-1\}$, $x(N) \in X_f$ and the following ℓ_2 gain property holds:

$$\|x(N)\|_P^2 + \sum_{k=0}^{N-1} \|z(k)\|_2^2 \leq V_N^*(x(0),\gamma) + \sum_{k=0}^{N-1} \gamma^2 \|w(k)\|_2^2. \tag{5.19}$$

5.3 Infinite Horizon ℓ_2 Gain Minimization

We next consider the *infinite* horizon case, and seek a control law that satisfies

$$\sum_{k=0}^{\infty} \|z(k)\|_2^2 \leq \beta(x(0)) + \sum_{k=0}^{\infty} \gamma^2 \|w(k)\|_2^2 \tag{5.20}$$

for all disturbance sequences satisfying $\sum_{k=0}^{\infty} \|w(k)\|_2^2 < \infty$. We construct our control law by exploiting the parameterization introduced in (5.3) and implementing the control in a receding horizon fashion. We define the receding horizon control law $\mu_N : X_N^{df} \times \mathbb{R}_+ \to \mathbb{R}^m$ by the first part of the optimal affine feedback control policy in (5.16), i.e.

$$\mu_N(x,\gamma) := v_0^*(x,\gamma). \tag{5.21}$$

Note that this control law is *nonlinear* in general. For a fixed γ, the closed-loop system dynamics become

$$x(k+1) = Ax(k) + B\mu_N(x(k),\gamma) + Gw(k) \tag{5.22a}$$
$$z(k) = C_z x(k) + D_z \mu_N(x(k),\gamma). \tag{5.22b}$$

We first establish various desirable properties of the control law $\mu_N(\cdot,\gamma)$ and value function $V_N^*(\cdot,\gamma)$ for each γ, as well as of the minimum gain function $\gamma_N^*(\cdot)$. We also provide a set of conditions under which the set $X_N^\gamma(\gamma)$ is robust positively invariant for the closed-loop system (5.22); these properties will be used in the next section to provide conditions under which the system (5.22) satisfies the ℓ_2 gain condition (5.20) while simultaneously satisfying the system constraints Z for all time. Proofs for all of the results in this section are presented in the appendix to the chapter.

5.3.1 Continuity and Convexity

In order to establish various properties of the functions $\gamma_N^*(\cdot)$, $V_N^*(\cdot,\gamma)$ and $\mu_N(\cdot,\gamma)$, we will find it useful to define the set

$$\mathcal{C}_N^\gamma := \{(x,\gamma,\mathbf{M},\mathbf{v}) \mid (x,\mathbf{M},\mathbf{v}) \in \mathcal{C}_N,\ (\gamma,\mathbf{M}) \text{ satisfies } (5.10)\}. \tag{5.23}$$

Note that given this set definition, it is obvious that the optimization problem defining $\gamma_N^*(x)$ in (5.15) is convex for each x.

Using arguments similar to those in the proof of Lemma 3.4, we can state the following results related to the convexity and closedness of the sets defined thus far:

Lemma 5.6 (Convexity and Closedness).

i. The set $\mathcal{C}_N^\gamma$ is closed and convex.

ii. For each $x \in X_N^{df}$ and $\gamma \geq 0$, the sets $\Pi_N^\gamma(x,\gamma)$ and $X_N^\gamma(\gamma)$ are closed and convex.

iii. The set $\left\{(x,\gamma) \;\middle|\; \exists(\mathbf{M},\mathbf{v}),\ (x,\gamma,\mathbf{M},\mathbf{v}) \in \mathcal{C}_N^\gamma\right\}$ is closed and convex.

Using these results, we can establish the following:

Proposition 5.7 (Properties of $\gamma_N^*(\cdot)$). *If X_N^{df} has nonempty interior, then the minimum gain function $\gamma_N^*(\cdot)$ is convex on X_N^{df}, continuous on $\operatorname{int}(X_N^{df})$ and lower semicontinuous everywhere on X_N^{df}.*

Proposition 5.8 (Properties of $V_N^*(\cdot,\gamma)$ and $\mu_N(\cdot,\gamma)$). *For a fixed $\gamma \geq 0$, if $X_N^\gamma(\gamma)$ has nonempty interior then the receding horizon control law $\mu_N(\cdot,\gamma)$ is unique on $X_N^\gamma(\gamma)$ and continuous on $\operatorname{int}(X_N^\gamma(\gamma))$. The value function $V_N^*(\cdot,\gamma)$ is convex on $X_N^\gamma(\gamma)$, continuous on $\operatorname{int}(X_N^\gamma(\gamma))$ and lower semicontinuous everywhere on $X_N^\gamma(\gamma)$.*

Corollary 5.9. *For any $x \in X_N^{df}$ and any $\gamma \geq \gamma_N^*(x)$, the function $J_N^*(x,\gamma,\cdot,\cdot)$ attains its minimum on the set $\Pi_N^\gamma(x,\gamma)$.*

5.3.2 Geometric and Invariance Properties

We next consider whether the optimization problem (5.16) required to implement the receding horizon control law $\mu_N(\cdot,\gamma)$ can be solved for all time for the closed-loop system (5.22). To make such an invariance guarantee, we require the following assumption on the terminal set X_f and cost P:

A5.2 (Terminal Cost and Constraint)

i. A state feedback gain matrix K_f and terminal constraint set X_f have been chosen such that

(a) The matrix $A + BK_f$ is Hurwitz.

(b) $X_f \subseteq \{x \mid (x, K_f x) \in Z\}$.

(c) $(A + BK_f)x + Gw \in X_f$, for all $x \in X_f$ and all $w \in W$.

ii. The feedback matrix K_f and terminal cost matrix P satisfy the solution to an unconstrained *$\mathcal{H}_\infty$ state feedback control problem [BB91, GL95] with ℓ_2 gain γ_f:*

$$P := C_z^\top C_z + A^\top \bar{P} A - (A^\top \bar{P} B + C_z^\top D_z)(D_z^\top D_z + B^\top \bar{P} B)^{-1}(B^\top \bar{P} A + D_z^\top C_z) \tag{5.24a}$$

$$K_f := -(D_z^\top D_z + B^\top \bar{P} B)^{-1}(B^\top \bar{P} A + D_z^\top C_z) \tag{5.24b}$$

where $\bar{P} := P + PG(\gamma_f^2 I - G^\top P G)^{-1} G^\top P$ and $\gamma_f^2 I - G^\top P G \succ 0$.

Remark 5.10. *Note that the conditions in **A5.2**(i) are identical to those in **A3.2** and that, unlike the assumptions required for input-to-state stability in **A4.2**, there is no compactness requirement on the constraint set Z. If the state $x \in X_f$ and gain $\gamma = \gamma_f$, then the RHC law $\mu_N(x) = K_f x$ matches the behavior of an unconstrained $\mathcal{H}_\infty$ control law.*

We first show that the sets $X_N^\gamma(\gamma)$ are non-decreasing (with respect to set inclusion) for a given γ:

Proposition 5.11. *If **A5.2** holds with $\gamma_f \leq \gamma$, then the following set inclusion property holds:*

$$X_f \subseteq X_1^\gamma(\gamma) \subseteq \cdots \subseteq X_{N-1}^\gamma(\gamma) \subseteq X_N(\gamma) \subseteq X_{N+1}^\gamma(\gamma) \cdots$$

Corollary 5.12. *If **A5.2** holds, then the minimum gain function $\gamma_N^*(\cdot)$ satisfies $\gamma_{N+1}^*(x) \leq \max\{\gamma_f, \gamma_N^*(x)\}$ for all $x \in X_N^{df}$.*

Remark 5.13. *The significance of Corollary 5.12 is that, unlike existing $\mathcal{H}_\infty$ receding horizon control schemes such as [Rao00, Chap. 9], it will allow us to guarantee that the achievable infinite horizon ℓ_2 gain for the closed-loop system* (5.22) *is non-increasing with increasing horizon length.*

Using these results we can also demonstrate that, for any $\gamma \geq \gamma_f$, each of the sets $X_N^\gamma(\gamma)$ is robust positively invariant for the closed-loop system (5.22):

Proposition 5.14. *If **A5.2** holds with $\gamma_f \leq \gamma$, then, for each $N \in \{1, 2, \dots\}$, the set $X_N^\gamma(\gamma)$ is robust positively invariant for the closed-loop system* (5.22a)*, i.e. if $x \in X_N^\gamma(\gamma)$, then $x^+ = Ax + B\mu_N(x,\gamma) + Gw \in X_N^\gamma(\gamma)$ for all $w \in W$.*

5.3.3 Finite ℓ_2 Gain in Receding Horizon Control

We can now state the main result of this chapter, which allows us to place an upper bound on the ℓ_2 gain of the closed-loop system (5.22) under the proposed RHC law $\mu_N(\cdot,\gamma)$:

Theorem 5.15. *If **A5.2** holds with $\max\{\gamma_f, \gamma_N^*(x(0))\} \leq \gamma$ then, for the closed-loop system* (5.22)*, the ℓ_2 gain from the disturbance w to the costed/controlled variable z is less than γ. Furthermore, the constraints Z are satisfied for all time if the initial state $x(0) \in X_N^\gamma(\gamma)$.*

Proof. Define

$$\Delta\varphi(x,u,w) := \varphi(Ax + Bu + Gw) - \varphi(x), \tag{5.25}$$

and functions

$$V_f(x,u,w) := \|x\|_P^2 \tag{5.26}$$
$$\ell(x,u,w) := (\|z\|_2^2 - \gamma^2 \|w\|_2^2) \tag{5.27}$$
$$\ell_f(x,u,w) := (\|z\|_2^2 - \gamma_f^2 \|w\|_2^2), \tag{5.28}$$

where $z = C_z x + D_z u$. If **A5.2** holds for the terminal cost matrix P and controller K_f, then a standard result in unconstrained linear $\mathcal{H}_\infty$ control is that

$$\max_{w \in \mathbb{R}^l} \left[\Delta V_f + \ell_f\right](x, K_f x, w) = 0. \tag{5.29}$$

Since $\gamma_f \leq \gamma$ implies $\ell(x,u,w) \leq \ell_f(x,u,w)$ for all (x,u,w),

$$[\Delta V_f + \ell]\,(x,u,w) \leq [\Delta V_f + \ell_f]\,(x,u,w), \text{ for all } (x,u,w), \tag{5.30}$$

so that

$$\max_{w\in W} [\Delta V_f + \ell]\,(x, K_f x, w) \leq 0 \tag{5.31}$$

for any x. Exploiting the equivalence between the policy parameterization (5.3) and affine state feedback policies (Thm. 3.9) and the assumed robust invariance of the terminal set X_f in **A5.2**, given a state $x^+ = Ax + B\mu_N(x,\gamma) + Gw$, one can construct a feasible control policy $(\tilde{\mathbf{M}}, \tilde{\mathbf{v}}) \in \Pi_N^\gamma(x^+, \gamma)$ such that the terminal control input is $u_N = K_f x_N$. Application of (5.31) then guarantees that

$$[\Delta V_\gamma + \ell](x, \mu_N(x,\gamma), w) \leq 0 \tag{5.32}$$

for all $x \in X_N^\gamma(\gamma)$ and all $w \in W$, where $V_\gamma(\cdot) := V_N^*(\cdot,\gamma)$ — note that this is a standard result from the literature on predictive control [MRRS00]. For any integer q and disturbance sequence $\{w(k)\}_{i=0}^{q-1}$ taking values in W, it then follows from (5.32) that

$$V_N^*(x(q),\gamma) \leq V_N^*(x(0),\gamma) - \sum_{k=0}^{q-1} \ell\,(x(k), \mu_N(x(k),\gamma), w(k)) \tag{5.33}$$

where $x(k+1) = Ax(k) + B\mu_N(x(k),\gamma) + Gw(k)$ for all $k \in \mathbb{Z}_{[0,q-1]}$. From the lower bound established in Prop. 5.2, it follows that

$$V_N^*(x,\gamma) = \inf_{(\mathbf{M},\mathbf{v})} \left\{ J_N^*(x,\gamma,\mathbf{M},\mathbf{v}) \;\middle|\; (\mathbf{M},\mathbf{v}) \in \Pi_N^\gamma(x,\gamma) \right\} \geq 0, \tag{5.34}$$

so that, for any integer $q \geq 0$, (5.33) implies

$$\sum_{k=0}^{q-1} \|z(k)\|_2^2 \leq V_N^*(x(0),\gamma) + \gamma^2 \sum_{k=0}^{q-1} \|w(k)\|_2^2, \tag{5.35}$$

and the claim about finite gain follows. Constraint satisfaction is then guaranteed by Prop. 5.14. □

Remark 5.16. *For a given initial state* $x(0)$*, it follows immediately that the achievable closed-loop gain* γ *in Theorem 5.15 is a non-increasing function of the horizon length* N*, since Corollary 5.12 ensures that* $\max\{\gamma_f, \gamma_{N+1}^*(x(0))\} \leq \max\{\gamma_f, \gamma_N^*(x(0))\}$.

5.4 Conclusions

By imposing additional convex constraints on the class of robust control policies proposed in Chapter 3, we have shown that one can formulate a robust RHC law with guaranteed bounds on the ℓ_2 gain of the closed-loop system, while simultaneously guaranteeing feasibility and constraint satisfaction for all time.

The proposed control law requires the solution, at each time instant, of a convex-concave min-max problem, making it a suitable candidate for on-line implementation. In Chapter 6, we present methods for actually solving this problem for particular classes of constraints and disturbances.

5.A Proofs

Proof of Lemma 5.6

Proof. The set of all $(\gamma, \mathbf{M})$ satisfying the LMI condition (5.10) is closed and convex [BEFB94], so $\mathcal{C}_N^\gamma$ is convex since it is the intersection of closed and convex sets, proving (i). A similar argument establishes convexity of the set $\Pi_N^\gamma(x,\gamma)$ in (ii).

For the remaining results, we define the sets

$$\mathcal{M} := \{\mathbf{M} \mid \mathbf{M} \text{ satisfies (3.19)},\ \mathbf{M}y = 0,\ \forall y \perp \mathcal{R}(\mathcal{G})\} \tag{5.36}$$

$$\mathcal{M}_\perp := \{\mathbf{M} \mid \mathbf{M} \text{ satisfies (3.19)},\ \mathbf{M}y = 0,\ \forall y \in \mathcal{R}(\mathcal{G})\} \tag{5.37}$$

$$\tilde{\mathcal{C}}_N^\gamma := \mathcal{C}_N^\gamma \cap (\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}^{mN}). \tag{5.38}$$

Using arguments identical to those in the proof of Lemma 3.5, it is easily shown that

$$\mathcal{C}_N^\gamma = \tilde{\mathcal{C}}_N^\gamma \oplus (\{0\} \times \{0\} \times \mathcal{M}_\perp \times \{0\}), \tag{5.39}$$

and that the set $\tilde{\mathcal{C}}_N^\gamma$ is bounded in policies, i.e. there exist bounded sets $B_1 \subseteq \mathbb{R}^{mN \times nN}$ and $B_2 \subseteq \mathbb{R}^{mN}$ such that $\tilde{\mathcal{C}}_N^\gamma \subseteq (\mathbb{R}^n \times \mathbb{R}_+ \times B_1 \times B_2)$. Define a linear mapping L such that $L(x, \gamma, \mathbf{M}, \mathbf{v}) := x$, so that for a given $\gamma \geq 0$,

$$X_N^\gamma(\gamma) = L\left(\mathcal{C}_N^\gamma \bigcap (\mathbb{R}^n \times \{\gamma\} \times \mathbb{R}^{mN \times nN} \times \mathbb{R}^{mN})\right) \tag{5.40}$$

$$= L\left(\tilde{\mathcal{C}}_N^\gamma \bigcap (\mathbb{R}^n \times \{\gamma\} \times \mathbb{R}^{mN \times nN} \times \mathbb{R}^{mN})\right) \oplus L(\{0\} \times \{0\} \times \mathcal{M}_\perp \times \{0\}). \tag{5.41}$$

The left-hand term in the summation (5.41) is closed and convex since it is compact in both policies and the parameter gamma (Prop. 2.10), so that (5.41) is the sum of closed and orthogonal sets, therefore also closed (Prop. 2.9). This completes the proof for (ii).

Proof of convexity and closedness of the set defined in (iii) is similar that of $X_N^\gamma(\gamma)$ above, but with the linear map L defined as $L(x, \gamma, \mathbf{M}, \mathbf{v}) := (x, \gamma)$. $\square$

Proof of Proposition 5.7

Proof. The epigraph of $\gamma_N^*(\cdot)$ is the closed and convex set defined in Lem. 5.6(iii), so that $\gamma_N^*(\cdot)$ is convex ([RW98, Prop. 2.4]) and lower semicontinuous ([RW98, Thm. 1.6]) on X_N^{df}. Strict continuity of $\gamma_N^*(\cdot)$ on $\mathrm{int}(X_N^{df})$ follows from its convexity on X_N^{df} (Prop. 2.15). $\square$

Proof of Proposition 5.8

Proof. Recalling the definition of $\tilde{\mathcal{C}}_N^\gamma$ in (5.38), for a fixed γ define

$$f(x, \mathbf{M}, \mathbf{v}) := \begin{cases} J_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) & \text{if } (x, \gamma, \mathbf{M}, \mathbf{v}) \in \tilde{\mathcal{C}}_N^\gamma, \\ \infty & \text{otherwise,} \end{cases} \tag{5.42}$$

where

$$J_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) = \max_{\mathbf{w} \in \mathcal{W}} \left(\|H_x x + H_u(\mathbf{v} + \mathbf{M}\mathcal{G}\mathbf{w}) + H_w \mathbf{w}\|_2^2 - \gamma^2 \|\mathbf{w}\|_2^2 \right). \tag{5.43}$$

Since the control law $\mu_N(\cdot, \gamma)$ is defined by the first part of the optimal policy parameter $\mathbf{v}^*(x, \gamma)$ only, we partition $\mathbf{v}$ into two components and define

$$\mathbf{v}_{[1,N]} := \text{vec}(v_1, \cdots, v_{N-1}),$$

so that $\mathbf{v} = \left[\begin{smallmatrix} v_0 \\ \mathbf{v}_{[1,N]} \end{smallmatrix}\right]$. The functions $V_N^*(\cdot, \gamma)$ and $\mu_N(\cdot, \gamma)$ can then be written as

$$V_N^*(x, \gamma) = \min_{v_0} p(x, v_0), \quad \mu_N(x, \gamma) = \operatorname*{argmin}_{v_0} p(x, v_0), \tag{5.44}$$

where

$$p(x, v_0) := \min_{(\mathbf{M}, \mathbf{v}_{[1,n]})} f(x, \mathbf{M}, \mathbf{v}). \tag{5.45}$$

All of the results then follow from Prop. 2.23 if (5.45) can be shown to be proper, convex, lower semicontinuous, strictly convex in v_0 and level bounded in v_0 locally uniformly in x. By defining

$$\bar{\mathbf{D}}_z := \begin{pmatrix} 0 & 0 \\ 0 & (I_{N-1} \otimes D_z) \end{pmatrix} \text{ and } H_u := \begin{pmatrix} \mathbf{C}_z \mathbf{B} + \bar{\mathbf{D}}_z \\ P^{\frac{1}{2}} \tilde{B} \end{pmatrix},$$

the convex function (5.43) can be rewritten as

$$J_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) = \|D_z v_0\|_2^2 + \bar{J}_N^*(x, \gamma, \mathbf{M}, \mathbf{v}), \tag{5.46}$$

where

$$\bar{J}_N^*(x, \mathbf{M}, \mathbf{v}, \gamma) := \max_{\mathbf{w} \in \mathcal{W}} \left(\left\| H_x x + \bar{H}_u \mathbf{v} + H_u \mathbf{M}\mathcal{G}\mathbf{w} + H_w \mathbf{w} \right\|_2^2 - \gamma^2 \|\mathbf{w}\|_2^2 \right). \tag{5.47}$$

The function $(x, \mathbf{M}, \mathbf{v}_{[1,N]}) \mapsto \bar{J}_N^*(x, \gamma, \mathbf{M}, \mathbf{v})$ can be shown to be convex, lower semicontinuous, proper and bounded below by zero using arguments identical to those in the proof of Prop. 5.2. We then define

$$\bar{f}(x, \mathbf{M}, \mathbf{v}) := \begin{cases} \bar{J}_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) & \text{if } (x, \gamma, \mathbf{M}, \mathbf{v}) \in \tilde{\mathcal{C}}_N^\gamma, \\ \infty & \text{otherwise}, \end{cases} \tag{5.48}$$

$$\bar{p}(x, v_0) := \min_{(\mathbf{M}, \mathbf{v}_{[1,n]})} \bar{f}(x, \mathbf{M}, \mathbf{v}), \tag{5.49}$$

so that (5.45) can be written as

$$p(x, v_0) := \|D_z v_0\|_2^2 + \bar{p}(x, v_0). \tag{5.50}$$

Recall from the proof of Lemma 5.6 that the set $\tilde{C}_N^\gamma$ is compact in policies. Since $\bar{f}(x, \mathbf{M}, \mathbf{v})$ is convex, lower semicontinuous and proper and $\tilde{C}_N^\gamma$ is nonempty by assumption, the function $(x, v_0) \mapsto \bar{p}(x, v_0)$ is convex, lower semicontinuous and proper (Prop. 2.23(iib)). Since $\|D_z v_0\|_2^2$ is convex and continuous in v_0, it follows that $(x, v_0) \mapsto p(x, v_0)$ is also convex, lower semicontinuous and proper. It also follows that $p(x, v_0) \geq \|Dv_0\|_2^2$ since (5.48) is bounded below by zero, and that (p, v_0) is strictly convex in v_0 since D_z is assumed full column rank and the sum of convex and strictly convex functions is strictly convex. The result then follows from straightforward application of Prop. 2.22 and Prop. 2.23. □

Proof of Corollary 5.9

Proof. Obvious from the proof of Prop. 5.8. □

Proof of Proposition 5.11

Proof. The proof is by induction and is similar to the proof of Prop. 3.13, except that satisfaction of the LMI condition (5.10) must also be ensured. Note that all of the matrix definitions required for the proof can be found in Section 4.1.1.

For any $x \in X_N^\gamma(\gamma)$, there exists a policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^\gamma(x, \gamma) \subseteq \Pi_N^{df}(x)$ by definition. If **A5.2** holds, then it is possible to construct a pair $(\mathbf{M}^+, \mathbf{v}^+) \in \Pi_{N+1}^{df}(x)$ such that the final input is $u_N := K_f x_N$ (Cor. 3.15), by defining

$$\mathbf{M}^+ := \begin{pmatrix} \mathbf{M} & 0 \\ \tilde{M} & 0 \end{pmatrix}, \quad \mathbf{v}^+ := \begin{pmatrix} \mathbf{v} \\ \tilde{v} \end{pmatrix}, \tag{5.51a}$$

where

$$\begin{aligned}\tilde{M} &:= K_f(\tilde{A} + \tilde{B}\mathbf{M}) \\ \tilde{v} &:= K_f(A^N x + \tilde{B}\mathbf{v}).\end{aligned} \tag{5.51b}$$

We need only show that this choice of policy also satisfies $(\mathbf{M}^+, \mathbf{v}^+) \in \Pi^\gamma_{N+1}(x,\gamma)$, i.e. that $(\gamma, \mathbf{M}^+)$ satisfies the LMI condition (5.10) or, equivalently, the QMI condition (5.11). We therefore define

$$H_N := \gamma^2 I - (H_u \mathbf{M}\mathcal{G} + H_w)^\top (H_u \mathbf{M}\mathcal{G} + H_w), \tag{5.52}$$

so that (5.11) is satisfied if $H_N \succeq 0$, and will show that the policy $(\mathbf{M}^+, \mathbf{v}^+)$ satisfies

$$H_{N+1} := \gamma^2 I - (H_u^+ \mathbf{M}^+ \mathcal{G}^+ + H_w^+)^\top (H_u^+ \mathbf{M}^+ \mathcal{G}^+ + H_w^+) \succeq 0, \tag{5.53}$$

where

$$H_u^+ := \begin{pmatrix} \mathbf{C}_z\mathbf{B} + \mathbf{D}_z & 0 \\ C_z\tilde{B} & D_z \\ P^{\frac{1}{2}}A\tilde{B} & P^{\frac{1}{2}}B \end{pmatrix}, \quad H_w^+ := \begin{pmatrix} \mathbf{C}_z\mathbf{G} & 0 \\ C_z\tilde{G} & 0 \\ P^{\frac{1}{2}}A\tilde{G} & P^{\frac{1}{2}}G \end{pmatrix}, \text{ and } \mathcal{G}^+ := \begin{pmatrix} \mathcal{G} & 0 \\ 0 & G \end{pmatrix}. \tag{5.54}$$

Defining $Y := (\tilde{G} + \tilde{B}\mathbf{M}\mathcal{G})$, substituting (5.54) and (5.51) into (5.53) and collecting terms yields

$$H_{N+1} = \begin{pmatrix} H_N + X & -Y^\top (A + BK_f)^\top PG \\ -G^\top P(A+BK_f)Y & \gamma^2 I - G^\top PG \end{pmatrix}, \tag{5.55}$$

where

$$X := Y^\top \Big(P - (C_z + D_z K_f)^\top (C_z + D_z K_f) - (A + BK_f)^\top P (A + BK_f) \Big) Y. \tag{5.56}$$

Claim 5.17. *The following matrix identity holds:*

$$P - (C_z + D_z K_f)^\top (C_z + D_z K_f) = (A + BK_f)^\top \bar{P} (A + BK_f), \tag{5.57}$$

*when P, $\bar{P}$ and K_f are defined as in **A5.2**.*

Proof. Rewrite $\bar{P}$ in (5.24a) as

$$\bar{P} = C_z^\top C_z + A^\top \bar{P} A + (A^\top \bar{P} B + C_z^\top D_z) K_f, \tag{5.58}$$

and manipulate (5.24b) to get

$$K_f^\top(D_z^\top D_z + B^\top \bar{P} B)K_f + K_f^\top(B^\top \bar{P} A + D_z^\top C_z) = 0. \tag{5.59}$$

Adding (5.59) to the right hand side of (5.58) and collecting terms gives the desired result.□

Application of the result in Claim 5.17 to (5.56) gives

$$X = Y^\top\Big((A + BK_f)^\top(\bar{P} - P)(A + BK_f)\Big)Y \tag{5.60}$$
$$= Y^\top\Big((A + BK_f)^\top PG(\gamma_f^2 I - G^\top PG)^{-1}G^\top P(A + BK_f)\Big)Y, \tag{5.61}$$

where (5.61) results from the definition of $\bar{P}$ in **A5.2**. Applying the Schur complement Lemma 2.20 to (5.55), the QMI condition $H_{N+1} \succeq 0$ is equivalent to

$$H_N + X - Y^\top(A + BK_f)^\top PG(\gamma^2 I - G^\top PG)^{-1}G^\top P(A + BK_f)Y \succeq 0, \tag{5.62}$$

whenever $\gamma \geq \gamma_f$, since $(\gamma_f^2 I - G^\top PG) \succ 0$ by assumption in **A5.2**. Recalling that $H_N \succeq 0$ by construction, we can substitute (5.61) into (5.62) and conclude that $H_{N+1} \succeq 0$ if

$$PG(\gamma_f^2 I - G^\top PG)^{-1}G^\top P \succeq PG(\gamma^2 I - G^\top PG)^{-1}G^\top P, \tag{5.63}$$

which, once again applying **A5.2**, always holds since $\gamma \geq \gamma_f$ by assumption. We conclude that $(\mathbf{M}^+, \mathbf{v}^+)$ satisfies the LMI condition (5.10), so that $(\mathbf{M}^+, \mathbf{v}^+) \in \Pi_{N+1}^\gamma(x, \gamma)$ and $x \in X_{N+1}^\gamma(\gamma)$. The proof is completed by verifying, in a similar manner, that $X_f \subseteq X_1^\gamma(\gamma)$.□

Proof of Proposition 5.14

Proof. If **A5.2** holds then, from Prop. 5.11, there exists a policy

$$(\mathbf{M}^+, \mathbf{v}^+) \in \Pi_{N+1}^\gamma(x, \gamma) \subseteq \Pi_{N+1}^{df}(x)$$

such that the first component of $\mathbf{v}^+$ matches $\mu_N(x, \gamma)$. Using arguments identical to those in the proof of Prop. 3.20, a shifted pair $(\bar{\mathbf{M}}, \bar{\mathbf{v}}) \in \Pi_N^{df}(Ax + B\mu(x, \gamma) + Gw)$ can be constructed by partitioning $(\mathbf{M}^+, \mathbf{v}^+)$ into

$$\mathbf{M}^+ =: \begin{pmatrix} 0 & 0 \\ \bar{M} & \bar{\mathbf{M}} \end{pmatrix}, \quad \mathbf{v}^+ =: \begin{pmatrix} \bar{v} \\ \bar{\mathbf{v}} \end{pmatrix}. \tag{5.64}$$

Recalling the definitions in (5.54), $(\gamma, \mathbf{M}^+)$ satisfies

$$\begin{pmatrix} -\gamma I & (H_u^+ \mathbf{M}^+ \mathcal{G}^+ + H_w^+) \\ (H_u^+ \mathbf{M}^+ \mathcal{G}^+ + H_w^+)^\top & -\gamma I \end{pmatrix} \preceq 0. \tag{5.65}$$

Partitioning $\mathbf{M}^+$ as in (5.64) and rewriting the matrices in (5.54) as

$$H_u^+ = \begin{pmatrix} D_z & 0 \\ \mathbf{C}_z \mathbf{A} B & \mathbf{C}_z \mathbf{B} + \mathbf{D}_z \\ P^{\frac{1}{2}} A^N B & P^{\frac{1}{2}} \tilde{B} \end{pmatrix}, \; H_w^+ = \begin{pmatrix} 0 & 0 \\ C_z \mathbf{A} G & \mathbf{C}_z \mathbf{G} \\ P^{\frac{1}{2}} A^N G & P^{\frac{1}{2}} \tilde{G} \end{pmatrix}, \text{ and } \mathcal{G}^+ = \begin{pmatrix} G & 0 \\ 0 & \mathcal{G} \end{pmatrix}, \tag{5.66}$$

the inequality (5.65) can be written as

$$\begin{pmatrix} -\gamma I & 0 & 0 & 0 \\ 0 & -\gamma I & \bar{Y} & (H_u \bar{\mathbf{M}} \mathcal{G} + H_w) \\ 0 & \bar{Y}^\top & -\gamma I & 0 \\ 0 & (H_u \bar{\mathbf{M}} \mathcal{G} + H_w)^\top & 0 & -\gamma I \end{pmatrix} \preceq 0, \tag{5.67}$$

where

$$\bar{Y} := \begin{pmatrix} (\mathbf{C}_z \mathbf{B} + \mathbf{D}_z) \bar{M} G + \mathbf{C}_z \mathbf{A} G \\ P^{\frac{1}{2}} (A^N + \tilde{B} \bar{M}) G \end{pmatrix}. \tag{5.68}$$

Rearranging rows and columns of (5.67) to get

$$\begin{pmatrix} -\gamma I & 0 & 0 & 0 \\ 0 & -\gamma I & \bar{Y}^\top & 0 \\ 0 & \bar{Y} & -\gamma I & (H_u \bar{\mathbf{M}} \mathcal{G} + H_w) \\ 0 & 0 & (H_u \bar{\mathbf{M}} \mathcal{G} + H_w)^\top & -\gamma I \end{pmatrix} \preceq 0, \tag{5.69}$$

it follows that $(\gamma, \bar{\mathbf{M}})$ satisfies the LMI condition (5.10), since any principal submatrix of a negative semidefinite matrix is negative semidefinite [HJ91, Thm. 7.1.2]. □

Chapter 6. Computational Methods

6.1 Introduction

In Chapter 3 it was shown that

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) &\text{ satisfies (3.19)} \\ \mathbf{x} &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} &= \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \\ (\mathbf{x}, \mathbf{u}) &\in \mathcal{Z},\ \forall \mathbf{w} \in \mathcal{W} \end{aligned} \right\}, \tag{6.1}$$

the set of admissible finite horizon affine feedback policies from a given state x, is a convex set when the constraint set $\mathcal{Z}$ is convex.

In Chapters 4 and 5, different cost functions were used to help discriminate between policies $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$. In both chapters, conditions were presented that ensured certain theoretical properties relating to the stability and continuity of receding horizon control laws synthesized from such optimal policies. It was also shown that the problem of selecting a policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ that is optimal with respect to either of these cost functions is a convex problem. We have thus far stated that such an optimization problem is convex in principle only, since the difficulty of actually solving such a problem is determined by the simplicity with which the set (6.1) can be characterized.

In this chapter, we address the *computational* issues relating to these theoretical results. We consider the finite horizon problem of determining a policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ (either feasible or optimal in some sense) given an initial state x. The main motivation is to show that such finite horizon problems can be solved using standard convex optimization methods in those cases in which the universal quantifier in (6.1) can be easily eliminated, so that the receding horizon control laws described in Chapters 4 and 5 can be implemented efficiently.

6.1.1 Definitions and Notation

We continue to use much of the notation introduced in Sections 3.1.1 and 4.1.1, and will employ the basic set of assumptions **A3.1**. We will restrict our attention to the particular case where the state and input constraint sets Z and X_f are polytopic, so that

$$Z := \{(x, u) \mid C_c x + D_c u \leq b\} \tag{6.2}$$

$$X_f := \{x \mid Y_c x \leq z\}, \tag{6.3}$$

for some matrices $C_c \in \mathbb{R}^{s \times n}$, $D_c \in \mathbb{R}^{s \times m}$ and $Y_c \in \mathbb{R}^{s_f \times n}$, and vectors $b \in \mathbb{R}^s$ and $z \in \mathbb{R}^{s_f}$. The constraint set $\mathcal{Z}$ can then be defined as a set of $t := sN + s_f$ linear inequalities in the form

$$\mathcal{Z} = \left\{ (\mathbf{x}, \mathbf{u}) \;\middle|\; \begin{array}{r} \mathbf{C}_c \mathbf{x} + \mathbf{D}_c \mathbf{u} \leq \mathbf{b} \\ Y_c x_N \leq z \end{array} \right\}, \tag{6.4}$$

where

$$\mathbf{C}_c := \begin{bmatrix} I_N \otimes C_c & 0 \end{bmatrix},\ \mathbf{D}_c := I_N \otimes D_c \ \text{ and } \mathbf{b} := \mathbf{1}_N \otimes b. \tag{6.5}$$

Note, however, that all of the results in this chapter will hold when the set $\mathcal{Z}$ in (6.4) is defined for *any* matrices $\mathbf{C}_c \in \mathbb{R}^{t \times nN}$ and $\mathbf{D}_c \in \mathbb{R}^{t \times mN}$ and vector $f \in \mathbb{R}^t$, and not just for those with the particular structure of (6.5). This includes problems with time-varying or rate-limited constraints on the states and inputs, although theoretical properties of control laws derived for systems with such constraints will not be considered.

If we further define

$$F_x := \begin{pmatrix} \mathbf{C}_c \mathbf{A} \\ Y_c A^N \end{pmatrix},\ F_u := \begin{pmatrix} \mathbf{C}_c \mathbf{B} + \mathbf{D}_c \\ Y_c \tilde{B} \end{pmatrix},\ F_w := \begin{pmatrix} \mathbf{C}_c \mathbf{G} \\ Y_c \tilde{G} \end{pmatrix},\ \text{and } \mathbf{f} := \begin{pmatrix} \mathbf{b} \\ z \end{pmatrix}, \tag{6.6}$$

then the set of feasible policies $\Pi_N^{df}(x)$, defined in (6.1), can be written as

$$\Pi_N^{df}(x) := \bigcap_{\mathbf{w} \in \mathcal{W}} \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ F_x x + F_u \mathbf{v} + (F_u \mathbf{M} \mathcal{G} + F_w)\mathbf{w} \leq f \end{array} \right\}, \tag{6.7}$$

or equivalently, as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ F_x x + F_u \mathbf{v} + \max_{\mathbf{w} \in \mathcal{W}} (F_u \mathbf{M} \mathcal{G} + F_w)\mathbf{w} \leq f \end{array} \right\}. \tag{6.8}$$

The maximization $\max_{\mathbf{w}\in\mathcal{W}}(F_u\mathbf{M}\mathcal{G}+F_w)\mathbf{w}$ in (6.8) is to be interpreted row-wise; note that the maximum value in each row is achieved since the set $\mathcal{W}$ is assumed compact. This row-wise maximization is equivalent to evaluating the support function $\sigma_{\mathcal{W}}(\cdot)$ of the set $\mathcal{W}$ on each column of the matrix $(F_u\mathbf{M}\mathcal{G}+F_w)^\top$, so that Π_N^{df} can be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (3.19)} \\ (F_x x + F_u\mathbf{v})_i + \sigma_{\mathcal{W}}\left((F_u\mathbf{M}\mathcal{G}+F_w)_i^\top \right) \leq (f)_i \\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}, \tag{6.9}$$

where $(F)_i$ represents the i^{th} row of a matrix F, and $(f)_i$ the i^{th} element of a vector f. As in the case for general convex constraints $\mathcal{Z}$, the set of feasible control policies $\Pi_N^{df}(x)$ in (6.9) is convex (Thm. 3.5). This is reflected in the fact that $\sigma_{\mathcal{W}}(\cdot)$ is a convex function, so that each of the constraints defining the set $\Pi_N^{df}(x)$ in (6.9) is convex.

Note that since $0 \in \operatorname{int}\mathcal{W}$ by assumption, the support function $\sigma_{\mathcal{W}}(x)$ is positive for any nonzero x (Prop. 2.19). Therefore each of the terms $\sigma_{\mathcal{W}}\left((F_u\mathbf{M}\mathcal{G}+F_w)_i^\top\right)$ in (6.9) can be viewed as a reduction or contraction of the corresponding right-hand side constraint term $(f)_i$, i.e. each of the constraints in (6.9) can be written as

$$(F_x x + F_u\mathbf{v})_i \leq (f)_i - c_i, \tag{6.10}$$

where

$$c_i := \sigma_{\mathcal{W}}\left((F_u\mathbf{M}\mathcal{G}+F_w)_i^\top \right) \geq 0. \tag{6.11}$$

The degree to which each of these constraints is contracted is then dictated by the particular choice of the feedback parameter $\mathbf{M}$.

6.1.2 Non-polytopic state and input constraints

Before proceeding, we briefly touch on the reasons for dealing exclusively with polytopic state and input constraint sets $\mathcal{Z}$, while allowing the disturbance set $\mathcal{W}$ to be any compact set. Suppose that $\mathcal{Z}$ were characterized by the intersection of feasible regions for t inequality constraints defined by functions $f_i : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \to \mathbb{R}$:

$$\mathcal{Z} = \left\{ (\mathbf{x},\mathbf{u}) \;\middle|\; f_i(\mathbf{x},\mathbf{u}) \leq 0,\ \forall i \in \mathbb{Z}_{[1,t]} \right\}, \tag{6.12}$$

so that the set $\Pi_N^{df}(x)$ could be written as

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M},\mathbf{v}) &\text{ satisfies (3.19)} \\ \mathbf{x} &= \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} &= \mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v} \\ \sup_{\mathbf{w}\in\mathcal{W}} f_i(\mathbf{x},\mathbf{u}) &\le 0,\ \forall i \in \mathbb{Z}_{[1,t]} \end{aligned} \right\}.$$

Defining the functions $g_i : \mathbb{R}^{mN\times nN} \times \mathbb{R}^{mN} \to \mathbb{R}$ as

$$g_i(\mathbf{M},\mathbf{v}) := \sup_{\mathbf{w}\in\mathcal{W}} f_i\big([\mathbf{A}x + \mathbf{B}(\mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v}) + \mathbf{G}\mathbf{w}], [\mathbf{M}\mathcal{G}\mathbf{w} + \mathbf{v}]\big), \tag{6.13}$$

we can conclude that if each function f_i is convex, then each of the functions g_i is also convex, since they are defined as the pointwise suprema of convex functions. Convexity of these functions then ensures that the set of feasible policies $\Pi_N^{df}(x)$ is convex as well[1].

On the other hand, for each $i \in \mathbb{Z}_{[1,t]}$ define the function $h_i : \mathbb{R}^{lN} \to \mathbb{R}$ as

$$h_i(\mathbf{w}) := f_i\left([\mathbf{A}x + \mathbf{B}(\bar{\mathbf{M}}\mathcal{G}\mathbf{w} + \bar{\mathbf{v}}) + \mathbf{G}\mathbf{w}], [\bar{\mathbf{M}}\mathcal{G}\mathbf{w} + \bar{\mathbf{v}}]\right),$$

where $(\bar{\mathbf{M}},\bar{\mathbf{v}})$ is some fixed control policy. In order to evaluate whether that policy is constraint admissible, i.e. whether $(\bar{\mathbf{M}},\bar{\mathbf{v}}) \in \Pi_N^{df}(x)$, it would be necessary to solve each of the optimization problems

$$\max_{\mathbf{w}\in\mathcal{W}} h_i(\mathbf{w}), \quad \forall i \in \mathbb{Z}_{[1,t]}. \tag{6.14}$$

If any of the functions f_i is convex (so that h_i is also convex), then the associated problem in (6.14) requires the *maximization* of a convex function over a convex set, a very difficult proposition in general. However, if each of the functions f_i is *concave*, then the optimization problems (6.14) are all convex.

For the particular case where the state and input constraints are polytopic, each of the functions f_i is affine, and therefore *both* convex *and* concave. This makes each of the functions g_i in (6.13) convex, as well as making each of the optimization problems (6.14) convex. Taken together these conditions considerably simplify the problem of finding a pair $(\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x)$.

[1]Note, however, that convexity of the functions f_i is *not* a necessary condition. For example the sets $\{(\mathbf{M},\mathbf{v}) \mid g_i(\mathbf{M},\mathbf{v}) \le 0\}$ could each be nonconvex, but with convex intersection.

6.2 Computation of Admissible Policies

We will find it helpful to convert the set of policies defined by (6.9) into a more computationally attractive form, by rewriting $\Pi_N^{df}(x)$ in terms of the gauge function of the polar set $\mathcal{W}^\circ$; this will allow us to remove the maximization implicit in the support function representation (6.9). Recalling from Lemma 2.19 that $\sigma_{\mathcal{W}}(\cdot) = \gamma_{\mathcal{W}^\circ}(\cdot)$, the set $\Pi_N^{df}(x)$ can be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies } (3.19) \\ (F_x x + F_u \mathbf{v})_i + \gamma_{\mathcal{W}^\circ}\left((F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \right) \leq (f)_i \\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}, \tag{6.15}$$

or, recalling the definition of the gauge function on page 16, as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies } (3.19) \\ \exists c_i \geq 0,\ (F_x x + F_u \mathbf{v})_i + c_i \leq (f)_i \\ (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \in c_i \mathcal{W}^\circ,\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}. \tag{6.16}$$

This alternative representation of $\Pi_N^{df}(x)$ will be considerably more convenient to work with when the polar set $\mathcal{W}^\circ$ is easily characterized. In the remainder of this section, we consider several cases in which a simple characterization of $\mathcal{W}^\circ$ is available, and for which computation of a feasible pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ is possible using one of the standard convex optimization methods discussed in Section 2.3.

6.2.1 Conic Disturbance Sets

We first consider the case where the disturbance set $\mathcal{W}$ is defined by an affine conic inequality. Suppose that $\mathcal{W}$ is defined as

$$\mathcal{W} := \{ \mathbf{w} \mid S\mathbf{w} \preceq_K h \}, \tag{6.17}$$

where K is a convex cone. We can establish the following result about its polar set:

Proposition 6.1. *Suppose that $\mathcal{W}$ is defined as in* (6.17) *with $0 \in \operatorname{int} \mathcal{W}$. The scaled polar set $c\mathcal{W}^\circ$ is*

$$c\mathcal{W}^\circ = \left\{ S^\top \mathbf{z} \;\middle|\; \exists \mathbf{z} \succeq_K 0,\ h^\top \mathbf{z} \leq c \right\}. \tag{6.18}$$

Proof. Recall that the polar set $\mathcal{W}^\circ$ is defined as

$$\mathcal{W}^\circ := \left\{ y \;\middle|\; \max\left\{ y^\top \mathbf{w} \mid \mathbf{w} \in \mathcal{W} \right\} \le 1 \right\}, \tag{6.19}$$

where 'sup' can be replaced with 'max' in the usual polar set definition since $\mathcal{W}$ is assumed compact, so that the maximum is always attained. Exploiting the dual of (6.19), we have that

$$\max_{\mathbf{w}} \left\{ y^\top \mathbf{w} \mid S\mathbf{w} \preceq_K h \right\} \le \min_{\mathbf{z}} \left\{ h^\top \mathbf{z} \;\middle|\; y = S^\top \mathbf{z},\ \mathbf{z} \succeq_{K^*} 0 \right\}, \tag{6.20}$$

where K^* is the dual cone of K. Since $0 \in \mathcal{W}$ is a strictly feasible point in (6.17), strong duality holds and the relation (6.20) is satisfied with equality [BV04, Sec. 5.9]. The polar set $\mathcal{W}^\circ$ can then be written as

$$\mathcal{W}^\circ = \left\{ S^\top \mathbf{z} \;\middle|\; \exists \mathbf{z} \succeq_{K^*} 0,\ h^\top \mathbf{z} \le 1 \right\}. \tag{6.21}$$

The relation (6.18) follows by noting that $y \in c\mathcal{W}^\circ$ is equivalent to $\frac{1}{c} y \in \mathcal{W}^\circ$. $\square$

Remark 6.2. *In certain cases, the relationship between a set and its polar admits a straightforward geometric interpretation. For example, suppose that* $\mathcal{W} := \{\mathbf{w} \mid S\mathbf{w} \le \mathbf{1}\}$*, so that* $\mathcal{W}$ *is defined by the intersection of half-planes whose normal vectors are the rows of the matrix* S*. Then from* (6.21)*, its polar set is*

$$\mathcal{W}^\circ = \left\{ S^\top \mathbf{z} \;\middle|\; \exists \mathbf{z} \ge 0,\ \mathbf{1}^\top \mathbf{z} \le 1 \right\},$$

which is a polytope whose vertices are defined by the rows of S*.*

Remark 6.3. *In defining the scaled polar set* (6.18)*, we have made the implicit assumption that the convex cone* K *is a subset of an inner product space where the inner product* $\langle x, y \rangle$ *is defined by the familiar vector product* $x^\top y$*. If* K *is a subset of a finite-dimensional space equipped with a different inner product (e.g. if* K *is the positive semidefinite cone with* $\langle X, Y \rangle \equiv \mathrm{tr}(X^\top Y)$*), one can more generally write*

$$\begin{aligned} \mathcal{W} &= \{\mathbf{w} \mid S\mathbf{w} \preceq_K h\} \\ c\mathcal{W}^\circ &= \left\{ S^* \mathbf{z} \mid \exists \mathbf{z} \succeq_{K^*} 0,\ \langle \mathbf{z}, h \rangle \le c \right\}, \end{aligned}$$

where S^* *is the adjoint of the linear operator* S*. We will make the assumption throughout that this degree of generality is not required, since all of the specific characterizations for* $\mathcal{W}$ *defined as in* (6.17) *that will be considered in detail will have* $\langle x, y \rangle \equiv x^\top y$ *and* $S^* \equiv S^\top$*.*

By direct substitution of (6.18) into (6.16), the set $\Pi_N^{df}(x)$ can then be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) &\text{ satisfies (3.19)} \\ \exists \mathbf{z}_i \succeq_{K^*} 0,\ &\forall i \in \mathbb{Z}_{[1,t]} \\ (F_x x + F_u \mathbf{v})_i + h^\top \mathbf{z}_i &\leq (f)_i \\ (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top &= S^\top \mathbf{z}_i \end{aligned} \right\}. \tag{6.22}$$

It is therefore possible, given an initial state x, to calculate a constraint admissible policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ by solving a single convex optimization problem defined by affine generalized inequalities when the disturbance set $\mathcal{W}$ is defined as in (6.17).

Remark 6.4. *It is of course possible to derive the expression* (6.22) *without recourse to the polar set* $\mathcal{W}^\circ$*, since one can exploit the duality relation* (6.20) *directly to eliminate the support function in* (6.9)*; this is the approach used by the author in [GKM06], and the methods are entirely equivalent. However, we will generally prefer the method presented here, since the polar set is a useful intermediate device that affords a more geometric view of the problem, and is slightly more flexible.*

In the remainder of this section, we discuss several special cases of (6.17) for the most common classes of disturbance set.

6.2.2 Polytopic Disturbance Sets

Suppose that $\mathcal{W}$ is a polytope containing the origin in its interior. In this case the disturbance set can be written as

$$\mathcal{W} = \{\mathbf{w} \mid S\mathbf{w} \leq h\}, \tag{6.23}$$

where $S \in \mathbb{R}^{a \times lN}$ and $h \in \mathbb{R}^a$; this set description is equivalent to (6.17) with K defined as the positive orthant in $\mathbb{R}^a$. In the most general case, this set description includes disturbances with both time-varying constraints and rate limits, although typically one would assume that $\mathcal{W} = W^N$, with the disturbance set W time-invariant[2]. Note that both 1– and ∞–norm bounded sets are special cases of this disturbance class, though these will be dealt with separately in Section 6.2.3. Recalling (6.21), the polar of $\mathcal{W}$ is

$$\mathcal{W}^\circ = \left\{ S^\top \mathbf{z} \;\middle|\; \mathbf{z} \geq 0,\ h^\top \mathbf{z} \leq 1 \right\},$$

[2] In the time-invariant case, if $W := \left\{ w \in \mathbb{R}^l \;\middle|\; \bar{S} w \leq \bar{h} \right\}$ then $\mathcal{W}$ can be written as in (6.23), with $S := I_N \otimes \bar{S}$ and $h := \mathbf{1}_N \otimes \bar{h}$.

so that the set of feasible policies (6.22) becomes

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (3.19)} \\ \exists \mathbf{z}_i \geq 0,\ \forall i \in \mathbb{Z}_{[1,t]} \\ (F_x x + F_u \mathbf{v})_i + h^\top \mathbf{z}_i \leq (f)_i \\ (F_u \mathbf{M}\mathcal{G} + F_w)_i^\top = S^\top \mathbf{z}_i \end{array} \right\}. \tag{6.24}$$

By combining the vectors $\mathbf{z}_i$ into a matrix $\mathbf{Z} := \begin{bmatrix} \mathbf{z}_1 & \dots & \mathbf{z}_N \end{bmatrix}^\top \in \mathbb{R}^{t\times a}$, the set $\Pi_N^{df}(x)$ can be expressed in terms of linear element-wise matrix inequalities

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (3.19)}, \exists \mathbf{Z} \geq 0 \\ F_x x + F_u \mathbf{v} + \mathbf{Z}h \leq f \\ \mathbf{Z}S = (F_u \mathbf{M}\mathcal{G} + F_w) \end{array} \right\}. \tag{6.25}$$

In this case it is possible to find a policy $(\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x)$ via the solution of a single linear program in a polynomial number of decision variables and constraints.

Remark 6.5. *The matrix products in* (6.25) *can be written in a vectorized form compatible with standard convex optimization software packages. Using the Kronecker product identity* $\mathrm{vec}(ABC) = (C^\top \otimes A)\mathrm{vec}(B)$, (6.25) *is equivalent to*

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (3.19)},\ \exists \mathbf{Z} \geq 0 \\ F_x x + F_u \mathbf{v} + (h^\top \otimes I_t)\mathrm{vec}(\mathbf{Z}) \leq f \\ (S^\top \otimes I_t)\mathrm{vec}(\mathbf{Z}) = (\mathcal{G}^\top \otimes F_u)\mathrm{vec}(\mathbf{M}) + \mathrm{vec}(F_w) \end{array} \right\}. \tag{6.26}$$

6.2.3 Norm Bounded Disturbance Sets

Suppose that the disturbance set $\mathcal{W}$ is the p–norm unit ball in $\mathbb{R}^{lN}$, so that it can be written as[3]

$$\mathcal{W} := \mathcal{B}_p^{lN} = \left\{ \mathbf{w} \in \mathbb{R}^{lN} \;\middle|\; \|\mathbf{w}\|_p \leq 1 \right\}. \tag{6.27}$$

We consider first the case where $p \in (1,\infty)/\{2\}$ – the particular cases $p \in \{1,2,\infty\}$ will be addressed separately in subsequent sections, where simpler results than those presented

[3] It is easy to apply the methods of this section to the case where the disturbance set W is p–norm bounded, i.e where $W = \mathcal{B}_p^l$, and $\mathcal{W} = W^N$. We work with the case $\mathcal{W} = \mathcal{B}_p^{lN}$ primarily for simplicity of notation.

here will be obtained. Recalling Prop. 2.12(vi), the polar of the set $\mathcal{W}$ in this case is

$$\mathcal{W}^\circ = \mathcal{B}_q^{lN} = \left\{ \mathbf{w} \in \mathbb{R}^{lN} \;\middle|\; \|\mathbf{w}\|_q \leq 1 \right\},$$

where $p^{-1} + q^{-1} = 1$. Substituting into (6.16), the set $\Pi_N^{df}(x)$ can be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ (F_x x + F_u \mathbf{v})_i + \left\| (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \right\|_q \leq (f)_i \\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}. \tag{6.28}$$

Remark 6.6. *This result can also be obtained by defining the convex cone*

$$K_p := \left\{ \left[\begin{smallmatrix} \mathbf{w} \\ t \end{smallmatrix}\right] \;\middle|\; \|\mathbf{w}\|_p \leq t \right\}, \tag{6.29}$$

so that the norm bounded disturbance set is defined as in (6.17)*, i.e.*

$$\mathcal{W} = \left\{ \mathbf{w} \;\middle|\; \left[\begin{smallmatrix} -I \\ 0 \end{smallmatrix}\right] \mathbf{w} \preceq_{K_p} \left[\begin{smallmatrix} 0 \\ \mathbf{1} \end{smallmatrix}\right] \right\}. \tag{6.30}$$

The result then follows by substitution into (6.22)*, where $K_p^* = K_q$.*

A feasible policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ can be found in this case via the solution of a single convex optimization problem. However, this optimization problem may still require considerable computational effort to solve relative to one based on one of the simpler classes of convex optimization problems described in Section 2.3. Fortunately, for control applications the cases $p \in \{1, 2, \infty\}$ are by far the most common, and the convex optimization problems arising in these cases have the added benefit of belonging to such simpler problem classes. We address each of these cases in turn in the remainder of this section.

2–norm Bounded Disturbance Sets

For the particular case $p = 2$, the set $\Pi_N^{df}(x)$ can be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ (F_x x + F_u \mathbf{v})_i + \left\| (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \right\|_2 \leq (f)_i \\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}, \tag{6.31}$$

so that a feasible policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ can be found by solving a second-order cone program, for which more specialized solution methods may be available [LVBL98]. Note that this result is easily extended to the case where the disturbance set $\mathcal{W}$ is ellipsoidal through judicious redefinition of the matrix $\mathcal{G}$.

Remark 6.7. *If the disturbance is Gaussian with a known mean and covariance, then one cannot guarantee that $\Pi_N^{df}(x)$ is nonempty since the disturbance sequence is no longer bounded. However, this problem can be circumvented by requiring that the constraints only hold with pre-specified probabilities. Suppose that a design goal is to guarantee that the i^{th} constraint in (6.8) holds only with probability η_i, i.e.*

$$\mathcal{P}\left[(F_x x + F_u \mathbf{v})_i + (F_u \mathbf{M}\mathcal{G} + F_w)_i \mathbf{w} \leq (f)_i\right] \geq \eta_i,\ \forall i \in \mathbb{Z}_{[1,t]}. \tag{6.32}$$

If the disturbance sequence $\mathbf{w}$ is Guassian with zero mean and covariance $\mathbf{C}_w$, then by straightforward application of the results in [BV04, pp. 157–8], these probabilistic constraints can be converted into the equivalent set of hard constraints

$$(F_x x + F_u \mathbf{v})_i + \varphi^{-1}(\eta_i)\left\| \mathbf{C}_w^{\frac{1}{2}}(F_u \mathbf{M}\mathcal{G} + F_w)_i^\top \right\|_2 \leq (f)_i,\ \forall i \in \mathbb{Z}_{[1,t]}, \tag{6.33}$$

where

$$\phi(z) := \frac{1}{2\pi}\int_{-\infty}^{z} e^{-t^2/2}\, dt \tag{6.34}$$

is the cumulative distribution function for a zero mean Guassian random variable with unit variance. If each of the probability thresholds $\eta_i \geq 0.5$, then each of the constraints in (6.33) is a second-order cone constraint, since $\phi^{-1}(\eta_i) \geq 0$ for any $\eta_i \geq 0.5$. In this case an admissible affine disturbance feedback policy can once again be found by solving a single, tractable SOCP.

1– and ∞–norm Bounded Disturbance Sets

If the disturbance set $\mathcal{W}$ is 1– or ∞–norm bounded, then it is possible to find a feasible policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ by solving a single linear program. This is evident from the fact that each of the sets $\mathcal{B}_1^{lN}$ and $\mathcal{B}_\infty^{lN}$ are polytopes, so that the results of Section 6.2.2 can be directly applied. In this section we take an alternative approach, and specialize the general results for p-norm bounded disturbances to the cases $p \in \{1, \infty\}$.

1–**Norm Bounds:** For the case $\mathcal{W} = \mathcal{B}_1^{lN}$, we set $q = \infty$ in (6.28) and rewrite $\Pi_N^{df}(x)$ as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ (F_x x + F_u \mathbf{v})_i + \left\| (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \right\|_\infty \leq (f)_i \\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}. \tag{6.35}$$

The constraints in this problem can be transformed to a set of purely linear constraints through the introduction of a set of scalar slack variables q_i bounding the absolute values of the vectors $(F_u \mathbf{M} \mathcal{G} + F_w)_i^\top$, so that

$$|(F_u \mathbf{M} \mathcal{G} + F_w)_i^\top| \leq \mathbf{1} q_i, \; \forall i \in \mathbb{Z}_{[1,t]}.$$

The set $\Pi_N^{df}(x)$ can then be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ \exists (q_1, \ldots, q_t) \geq 0, \; \forall i \in \mathbb{Z}_{[1,t]} \\ (F_x x + F_u \mathbf{v})_i + q_i \leq (f)_i \\ -\mathbf{1} q_i \leq (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \leq \mathbf{1} q_i \end{array} \right\}, \tag{6.36}$$

or, defining the vector $\mathbf{q} := [q_1, \cdots, q_t]^\top$, as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)}, \; \exists \mathbf{q} \\ F_x x + F_u \mathbf{v} + \mathbf{q} \leq f \\ -\mathbf{q} \mathbf{1}^\top \leq F_u \mathbf{M} \mathcal{G} + F_w \leq \mathbf{q} \mathbf{1}^\top \end{array} \right\}. \tag{6.37}$$

Since this set is defined entirely in terms of linear inequalities, it is possible to find a feasible policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ by solving a single linear program.

∞–**Norm Bounds:** A similar procedure leads to another polytopic set description in the case $\mathcal{W} = \mathcal{B}_\infty^{lN}$. We set $q = 1$ in (6.28) and rewrite $\Pi_N^{df}(x)$ as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (3.19)} \\ (F_x x + F_u \mathbf{v})_i + \left\| (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \right\|_1 \leq (f)_i \\ \forall i \in \mathbb{Z}_{[1,t]} \end{array} \right\}. \tag{6.38}$$

The constraints in this problem can be transformed to set of purely linear constraints through the introduction of a set of vector slack variables λ_i bounding the absolute values of the elements of the vectors $(F\mathbf{M}+G)_i^\top$, so that

$$|(F_u\mathbf{M}\mathcal{G}+F_w)_i^\top| \leq \lambda_i, \qquad \forall i \in \mathbb{Z}_{[1,t]} \tag{6.39}$$

$$\left\|(F_u\mathbf{M}\mathcal{G}+F_w)_i^\top\right\|_1 \leq \lambda_i^\top \mathbf{1}, \quad \forall i \in \mathbb{Z}_{[1,t]}. \tag{6.40}$$

Combining these vectors into a matrix $\mathbf{\Lambda} := \begin{bmatrix}\lambda_1 & \dots & \lambda_t\end{bmatrix}^\top$, the set $\Pi_N^{df}(x)$ can then be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{c} (\mathbf{M},\mathbf{v}) \text{ satisfies (3.19)}, \exists \mathbf{\Lambda} \\ F_x x + F_u \mathbf{v} + \mathbf{\Lambda}\mathbf{1} \leq f \\ -\mathbf{\Lambda} \leq F_u\mathbf{M}\mathcal{G} + F_w \leq \mathbf{\Lambda}. \end{array} \right\}. \tag{6.41}$$

As in the 1–norm bounded disturbance case, this set is defined entirely in terms of linear inequalities, so it is possible to find a feasible policy $(\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x)$ by solving a single linear program.

Remark 6.8. *Since each of the sets $\mathcal{B}_1^{lN}$ and $\mathcal{B}_\infty^{lN}$ is polytopic, it is also possible to derive the results of this section via direct application of the methods described in Section 6.2.2. For example, in the case $W = \mathcal{B}_\infty^{lN}$ the set description* (6.25) *defined for general polytopes is easily converted to the form* (6.41) *by setting*

$$\mathbf{Z} = \begin{bmatrix}\mathbf{\Lambda}_1 & \mathbf{\Lambda}_2\end{bmatrix}, \; S = \begin{bmatrix}+I \\ -I\end{bmatrix}, \;\; \text{and } h = \begin{bmatrix}\mathbf{1} \\ \mathbf{1}\end{bmatrix},$$

and defining $\mathbf{\Lambda} = \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2$.

6.2.4 L-Nonzero Disturbance Sets

As a final example, we consider disturbances generated from a less conventional set – the set of all disturbances with elements taking values in $\{-1,0,1\}$, and with at most L nonzero terms. We define such a set in $\mathbb{R}^n$ as

$$\mathcal{V}_L^n = \left\{ x \in \mathbb{R}^n \;\middle|\; x_i \in \{-1,0,1\}, \sum_{i=1}^n |x_i| \leq L \right\}, \tag{6.42}$$

and would like to model the disturbance set as $\mathcal{W} = \mathcal{V}_L^{lN}$ for some $L \in \mathbb{Z}_{[1,lN]}$.

Recalling the discussion of Section 3.4.1, it is sufficient for our purposes to work with convex hulls, i.e. to set $\mathcal{W} = \operatorname{conv}(\mathcal{V}_L^{lN})$. Since $\mathcal{V}_L^{lN}$ is the union of a finite number of points, its convex hull is a polytope; it is therefore conceptually possible to use the methods described in Section 6.2.2 directly for this disturbance model. However, this approach is ill-advised, since, as noted in [Gra04], the set $\mathcal{V}_L^{lN}$ has $2^L(lN)!/(L!(lN-L)!)$ vertices. In this section we take an alternative approach based on a direct characterization of the polar set $(\mathcal{V}_L^n)^\circ$.

Lemma 6.9. *If $\mathcal{V}_L^n$ is defined as in* (6.42) *then, for any $c \geq 0$, the set $c(\mathcal{V}_L^n)^\circ$ is*

$$c(\mathcal{V}_L^n)^\circ = \left\{ w \;\middle|\; \exists(\lambda, q) \geq 0, \begin{array}{c} |w| \leq (\lambda + \mathbf{1}q) \\ Lq + \mathbf{1}^\top\lambda \leq c \end{array} \right\}. \tag{6.43}$$

Proof. Taking the convex hull of the set (6.42) and exploiting the properties of polar sets (Prop. 2.12), we can write $\operatorname{conv}(\mathcal{V}_L^n)$ as

$$\operatorname{conv}(\mathcal{V}_L^n) = (L\mathcal{B}_1^n)\bigcap(\mathcal{B}_\infty^n) = \left((\frac{1}{L}\mathcal{B}_\infty^n)\bigcup(\mathcal{B}_1^n)\right)^\circ, \tag{6.44}$$

so that the polar of $\mathcal{V}_L^n$ can be written as the convex hull of a union of sets

$$(\mathcal{V}_L^n)^\circ = (\operatorname{conv}(\mathcal{V}_L^n))^\circ = \left((\frac{1}{L}\mathcal{B}_\infty^n)\bigcup(\mathcal{B}_1^n)\right)^{\circ\circ} = \operatorname{conv}\left((\frac{1}{L}\mathcal{B}_\infty^n)\bigcup(\mathcal{B}_1^n)\right). \tag{6.45}$$

We then write $c(\mathcal{V}_L^n)^\circ$ directly as the set of all convex combinations of vectors in $\frac{c}{L}\mathcal{B}_\infty^n$ and $c\mathcal{B}_1^n$:

$$\begin{aligned}
c(\mathcal{V}_L^n)^\circ &= \left\{ w \;\middle|\; \begin{array}{ll} \exists(x,y,\alpha), & w = \alpha x + (1-\alpha)y \\ 0 \leq \alpha \leq 1, & x \in \frac{c}{L}\mathcal{B}_\infty^n,\; y \in c\mathcal{B}_1^n \end{array} \right\} \\
&= \left\{ w \;\middle|\; \begin{array}{ll} \exists(\tilde{x},\tilde{y},\alpha), & w = \tilde{x} + \tilde{y} \\ 0 \leq \alpha \leq 1, & \|\tilde{x}\|_\infty \leq \frac{\alpha c}{L}, \quad \|\tilde{y}\|_1 \leq (1-\alpha)c \end{array} \right\} \\
&= \left\{ w \;\middle|\; \begin{array}{lll} \exists(\tilde{x},\tilde{y},\alpha,q,\lambda), & w = \tilde{x} + \tilde{y} & \\ 0 \leq \alpha \leq 1, & -\mathbf{1}q \leq \tilde{x} \leq \mathbf{1}q, & Lq \leq \alpha c \\ (q,\lambda) \geq 0, & -\lambda \leq \tilde{y} \leq \lambda, & \lambda^\top\mathbf{1} \leq (1-\alpha)c \end{array} \right\} \\
&= \left\{ w \;\middle|\; \begin{array}{ll} \exists(q,\lambda), & -(\mathbf{1}q + \lambda) \leq w \leq (\mathbf{1}q + \lambda) \\ (q,\lambda) \geq 0, & Lq + \mathbf{1}^\top\lambda \leq c \end{array} \right\},
\end{aligned}$$

where the last equality comes from addition of constraints to eliminate $(\tilde{x}, \tilde{y}, \alpha)$. ∎

Remark 6.10. *It is easy to show from the definition* (6.42) *that the set* $(\mathcal{V}_L^n)^\circ$ *is the unit ball in the* largest-L *norm, which we denote* $\|\cdot\|_{[L]}$*, and thus that* $\gamma_{(\mathcal{V}_L^n)^\circ}(\cdot) = \|\cdot\|_{[L]}$*. The largest-L norm is defined as*

$$\|w\|_{[L]} := \sum_{i=1}^{L} |w_{[i]}|,$$

where $w_{[i]}$ *is the* i^{th} *largest element of the vector* w *in absolute value terms, so that*

$$|w_{[1]}| \geq |w_{[2]}| \geq \cdots \geq |w_{[n-1]}| \geq |w_{[n]}|.$$

Given the polar set representation (6.43), it is now possible to define the set of constraint admissible control policies $\Pi_N^{df}(x)$ given the disturbance set $\mathcal{W} = \mathcal{V}_L^{lN}$. By direct substitution into (6.16), the set $\Pi_N^{df}(x)$ can be written as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) \text{ satisfies } (3.19) \\ \exists(\lambda_1, q_1, \cdots, \lambda_t, q_t) \geq 0,\ \forall i \in \mathbb{Z}_{[1,t]} \\ (F_x x + F_u \mathbf{v})_i + L q_i + \lambda_i^\top \mathbf{1} \leq (f)_i \\ -(\lambda_i + \mathbf{1} q_i) \leq (F_u \mathbf{M} \mathcal{G} + F_w)_i^\top \leq (\lambda_i + \mathbf{1} q_i) \end{aligned} \right\}, \tag{6.46}$$

or, in matrix form, as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) \text{ satisfies } (3.19),\ \exists(\mathbf{\Lambda}, \mathbf{q}) \geq 0 \\ F_x x + F_u \mathbf{v} + \mathbf{q} L + \mathbf{\Lambda} \mathbf{1} \leq f \\ -(\mathbf{\Lambda} + \mathbf{q} \mathbf{1}^\top) \leq F_u \mathbf{M} \mathcal{G} + F_w \leq (\mathbf{\Lambda} + \mathbf{q} \mathbf{1}^\top) \end{aligned} \right\}. \tag{6.47}$$

In this case, it is once again possible to find a feasible policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ via the solution of a single linear program in a polynomial number of decision variables and constraints. Note that the set description (6.47) for the case $W = \mathcal{V}_L^{lN}$ has more variables than either of the cases $\mathcal{W} = \mathcal{B}_1^{lN}$ and $\mathcal{W} = \mathcal{B}_\infty^{lN}$ (cf. (6.37) and (6.41) respectively).

Remark 6.11. *As in the case of conic set descriptions for* $\mathcal{W}$*, it is possible to derive the set description* (6.47) *directly from convex programming duality (cf. Remark 6.4). Recalling that the support function of a set matches the support function of its convex hull, we can write*

$$\sigma_{\mathcal{V}_L^n}(x) = \max \left\{ x^\top w \;\mid\; \|w\|_\infty \leq 1,\ \|w\|_1 \leq L \right\}$$

which, by LP duality, is equivalent to

$$\sigma_{\mathcal{V}_L^n}(x) = \max\left\{ x^\top w \;\middle|\; \exists z, \begin{array}{c} -z \leq w \leq z \leq \mathbf{1} \\ \mathbf{1}^\top z \leq L \end{array} \right\} = \min\left\{ \mathbf{1}^\top \lambda + Lq \;\middle|\; \begin{array}{c} -(\lambda + \mathbf{1}q) \leq x \leq (\lambda + \mathbf{1}q) \\ \lambda \geq 0, \quad q \geq 0 \end{array} \right\}.$$

This is the preferred approach, for example, in [Gra04, Sec2.9].

Remark 6.12. *Although the disturbance set $W = \mathcal{V}_L^{lN}$ is integer valued, the methods in this section are applicable to cases where L is any positive (potentially non-integer) value satisfying $L \in (1, lN)$ and the disturbance is modelled as $\mathcal{W} = L\mathcal{B}_1^{lN} \bigcap \mathcal{B}_\infty^{lN}$. Note that if $L \leq 1$ then $L\mathcal{B}_1^{lN} \subset \mathcal{B}_\infty^{lN}$; likewise if $L \geq lN$, then $L\mathcal{B}_1^{lN} \supset \mathcal{B}_\infty^{lN}$. In these cases the problem reduces to the simpler* 1- *and* ∞*-norm bounded cases discussed in Section 6.2.3.*

6.2.5 Computational Complexity

The common feature of all of the particular disturbance sets discussed in this section is that, given an initial state x, a feasible policy pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ can be calculated by solving a convex optimization problem in a tractable number of variables. The total number of variables and constraints for each of the problem types considered thus far is outlined in Table 6.1. Note that the number of variables and constraints grows quadratically in the horizon length N (recall that the number of constraints in $\mathcal{Z}$ is $t := sN + s_f$). However, the table is useful only as a rough guide to computational complexity, since no consideration is paid to the number of variables appearing in each constraint (i.e. the problem sparsity), nor to any underlying problem structure that may be exploited. This issue is dealt with in more detail for the ∞–norm bounded disturbance case in Chapter 7.

6.3 Expected Value Problems

In Chapter 4 we defined an optimal control policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ to be one that minimized the expected value of a certain quadratic cost function. It was shown that such a minimizer could be found as a solution to the optimization problem

$$\begin{aligned} \min_{(\mathbf{M},\mathbf{v})} \quad & J_N(x, \mathbf{M}, \mathbf{v}) \\ \text{subject to:} \quad & (\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x), \end{aligned} \tag{6.48}$$

Table 6.1: Computational Complexity for Different Disturbance Classes

Disturbance Class	Variables	Equalities	Linear Inequalities	Nonlinear Inequalities	Solution Method
Polytopic‡	$\frac{1}{2}mnN(N-1)$ $+mN+ta$	tlN	$t+ta$	–	LP
2–Norm Bounded	$\frac{1}{2}mnN(N-1)$ $+mN$	–	–	t	SOCP
1–Norm Bounded	$\frac{1}{2}mnN(N-1)$ $+mN+t$	–	$t+2tlN$	–	LP
∞–Norm Bounded	$\frac{1}{2}mnN(N-1)$ $+mN+tlN$	–	$t+2tlN$	–	LP
L-Largest	$\frac{1}{2}mnN(N-1)$ $+mN+t+tlN$	–	$2t+3tlN$	–	LP

where the cost function

$$J_N(x,\mathbf{M},\mathbf{v}) := \|H_x x + H_u \mathbf{v}\|_2^2 + \operatorname{tr}\left(\mathbf{C}_w^{\frac{1}{2}}(H_u\mathbf{M}\mathcal{G}+H_w)^\top(H_u\mathbf{M}\mathcal{G}+H_w)\mathbf{C}_w^{\frac{1}{2}}\right), \tag{6.49}$$

is convex and quadratic in the policy variables $(\mathbf{M},\mathbf{v})$.

Remark 6.13. *Recalling the Kronecker product identities* $\operatorname{vec}(AXB) = (B^\top \otimes A)\operatorname{vec}(X)$ *and* $\operatorname{tr}(A^\top B) = \operatorname{vec}(A)^\top \operatorname{vec}(B)$, (6.49) *can be written in a more familiar vectorized form as*

$$J_N(x,\mathbf{M},\mathbf{v}) = \left\|H_x x + H_u \mathbf{v}\right\|_2^2 + \left\|[(\mathbf{C}_w^{\frac{1}{2}}\mathcal{G}^\top)\otimes H_u]\operatorname{vec}(\mathbf{M}) + \operatorname{vec}(H_w\mathbf{C}_w^{\frac{1}{2}})\right\|_2^2. \tag{6.50}$$

This is the more suitable representation of $J_N(x,\mathbf{M},\mathbf{v})$ for use with most convex optimization software packages.

In this section, we show that the problem (6.48) can be solved using standard convex optimization methods for several of the most common disturbance types encountered in Section 6.2.

Polytopic Sets and Quadratic Programming

When the set $\mathcal{W}$ is a polytope, the optimization problem (6.48) can be posed as a convex quadratic program. Substitution of the set description for $\Pi_N^{df}(x)$ in (6.25) into the problem (6.48) gives the following QP:

$$\begin{aligned}
\min_{\mathbf{Z},\mathbf{M},\mathbf{v}} \quad & \|H_x x + H_u \mathbf{v}\|_2^2 + \operatorname{tr}\left(\mathbf{C}_w^{\frac{1}{2}}(H_u\mathbf{M}\mathcal{G} + H_w)^\top (H_u\mathbf{M}\mathcal{G} + H_w)\mathbf{C}_w^{\frac{1}{2}}\right) \\
\text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \leq j \\
& F_x x + F_u \mathbf{v} + \mathbf{Z}h \leq f \\
& \mathbf{Z}S = (F_u\mathbf{M}\mathcal{G} + F_w),\ \mathbf{Z} \geq 0.
\end{aligned} \tag{6.51}$$

2–Norm Bounded Sets and Second-Order Cone Programming

If the disturbance set $\mathcal{W}$ is 2–norm bounded, the optimization problem (6.48) can be written as a second-order cone program (SOCP). However, some care must be taken to convert the problem to the standard SOCP form described in Section 2.3.2, which uses a linear objective function expected by most SOCP solvers, since the objective function in (6.49) is quadratic in the decision variables $\mathbf{M}$ and $\mathbf{v}$. The following lemma is useful for making this conversion:

Lemma 6.14 (Hyperbolic Constraints ([LVBL98])). *Given a vector $x \in \mathbb{R}^n$ and real scalars y and z,*

$$\|w\|_2^2 \leq xy,\ x \geq 0 \ \text{and}\ y \geq 0 \iff \left\| \begin{pmatrix} 2w \\ x - y \end{pmatrix} \right\|_2 \leq x + y. \tag{6.52}$$

Using the policy set description (6.31) for the 2–norm bounded disturbance case, the optimization problem (6.48) can be written as

$$\begin{aligned}
\min_{\mathbf{M},\mathbf{v}} \quad & \|H_x x + H_u \mathbf{v}\|_2^2 + \operatorname{tr}\left(\mathbf{C}_w^{\frac{1}{2}}(H_u\mathbf{M}\mathcal{G} + H_w)^\top (H_u\mathbf{M}\mathcal{G} + H_w)\mathbf{C}_w^{\frac{1}{2}}\right) \\
\text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \leq j \\
& (F_x x + F_u \mathbf{v})_i + \left\|(F_u\mathbf{M}\mathcal{G} + F_w)_i^\top\right\|_2 \leq (f)_i, \quad \forall i \in \mathbb{Z}_{[1,t]}.
\end{aligned} \tag{6.53}$$

Writing the cost function in the vectorized form (6.50), adding additional variables t_1 and t_2 to serve as upper bounds for the quadratic components of the cost, and applying Lem. 6.14 yields the following problem:

$$
\begin{aligned}
\min_{\mathbf{M},\mathbf{v},t_1,t_2} \quad & t_1 + t_2 \\
\text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \le j \\
& \left\| \begin{pmatrix} 2\left[H_x x + H_u \mathbf{v}\right] \\ t_1 - 1 \end{pmatrix} \right\|_2 \le t_1 + 1 \\
& \left\| \begin{pmatrix} 2\left[[(\mathbf{C}_w^{\frac{1}{2}} \mathcal{G}^\top) \otimes H_u]\mathrm{vec}(\mathbf{M}) + \mathrm{vec}(H_w \mathbf{C}_w^{\frac{1}{2}}) \right] \\ t_2 - 1 \end{pmatrix} \right\|_2 \le t_2 + 1 \\
& (F_x x + F_u \mathbf{v})_i + \left\| (F_u \mathbf{M}\mathcal{G} + F_w)_i^\top \right\|_2 \le (f)_i, \ \ \forall i \in \mathbb{Z}_{[1,t]},
\end{aligned}
\tag{6.54}
$$

which is an SOCP in standard form.

∞–Norm Bounded Sets and Quadratic Programming

As in the polytopic disturbance case, the policy optimization problem (6.48) can be posed as a convex quadratic program if the disturbance set $\mathcal{W}$ is 1– or ∞–norm bounded, or if $\mathcal{W}$ is characterized using the L-nonzero set description of Section 6.2.4; recall that all of these sets are actually special cases of the general polytopic disturbance class. For example, in the ∞–norm case substitution of the set description (6.41) into (6.48) yields the following quadratic program:

$$
\begin{aligned}
\min_{\mathbf{\Lambda},\mathbf{M},\mathbf{v}} \quad & \|H_x x + H_u \mathbf{v}\|_2^2 + \mathrm{tr}\left(\mathbf{C}_w^{\frac{1}{2}} (H_u \mathbf{M}\mathcal{G} + H_w)^\top (H_u \mathbf{M}\mathcal{G} + H_w) \mathbf{C}_w^{\frac{1}{2}} \right) \\
\text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \le j \\
& F_x x + F_u \mathbf{v} + \mathbf{\Lambda}\mathbf{1} \le f \\
& -\mathbf{\Lambda} \le F_u \mathbf{M}\mathcal{G} + F_w \le \mathbf{\Lambda}.
\end{aligned}
\tag{6.55}
$$

Similar problem formulations are easily constructed for the 1–norm bounded and L–nonzero disturbance set cases using the set descriptions (6.37) and (6.47) respectively. A simplified version of the quadratic program (6.55) will be the topic of particular interest in Chapter 7.

6.3.1 Soft Constraints and Guaranteed Feasibility

An important practical consideration for control applications is the handling of potential infeasibility of the optimization problem (6.48). If the RHC law proposed in Chapter 4 is to be implemented on-line for a real system, it is important to guarantee reasonable controller behavior if the plant enters a state x such that $\Pi_N^{df}(x)$ is empty (equivalently, if $x \notin X_N^{df}$). A common approach in the literature in receding horizon control is to treat some or all of the constraints in Z or X_f as so-called *soft constraints*, i.e. constraints that may be violated if necessary to guarantee that the optimization problem (6.48) remains feasible for all x. Techniques for soft constraint handling are well established in the literature on linear predictive control for undisturbed systems [RWR98, SR99, Mac02], and we show briefly how these ideas can be extended to cover the robust control problems considered here.

We use the problem (6.55) as an example, since it will be of further interest in Chapter 7. We consider the simplest case where every constraint is a soft constraint, and replace the hard state and input constraints with constraints of the form

$$Cx_i + Du_i \leq b + \xi_i, \quad \xi_i \geq 0, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{6.56a}$$

$$Yx_N \leq z + \xi_N, \ \xi_N \geq 0. \tag{6.56b}$$

We also augment the objective function with convex linear-quadratic terms $(\gamma_i^\top \xi_i + \xi_i^\top \Gamma_i \xi_i)$ penalizing the soft constraint violations ξ_i, where $\Gamma_i \succeq 0$ and $\gamma_i \geq 0$ for each i. The optimization problem (6.55) becomes

$$\begin{aligned} \min_{\mathbf{\Lambda},\mathbf{M},\mathbf{v}} \quad & J_N(x,\mathbf{M},\mathbf{v}) + \sum_{i=0}^{N}(\gamma_i^\top \xi_i + \xi_i^\top \Gamma_i \xi_i) \\ \text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \leq j \\ & F_x x + F_u \mathbf{v} + \mathbf{\Lambda}\mathbf{1} \leq f + \boldsymbol{\xi} \\ & -\mathbf{\Lambda} \leq F_u \mathbf{M}\mathcal{G} + F_w \leq \mathbf{\Lambda}, \ \boldsymbol{\xi} \geq 0. \end{aligned} \tag{6.57}$$

where $\boldsymbol{\xi} := \text{vec}(\xi_0, \ldots, \xi_N)$. Note that the quadratic program (6.57) is feasible for all x, so that a receding horizon controller synthesized via repeated solution of this QP is defined everywhere on $\mathbb{R}^n$. A well-known feature of such penalty function formulations is that if, in the spirit of [Fle87, Sec. 12.3] [Mac02, Sec. 3.4], the penalty terms γ_i are chosen large enough, then solutions to (6.57) correspond exactly to solution of (6.55) for all $x \in X_N^{df}$.

6.4 Min-Max Problems

In Chapter 5 we defined an optimal control policy $(\mathbf{M},\mathbf{v}) \in \Pi_N^{\gamma}(x,\gamma)$ to be one that minimized the maximum value of a quadratic cost, while satisfying an additional LMI constraint in the matrix $\mathbf{M}$ and the ℓ_2 gain parameter γ. This led to the following optimization problem

$$\begin{aligned} \min_{\mathbf{M},\mathbf{v}} \max_{\mathbf{w}\in\mathcal{W}} \quad & \|(H_x x + H_u\mathbf{v}) + (H_u\mathbf{M}\mathcal{G} + H_w)\mathbf{w}\|_2^2 - \gamma^2\|\mathbf{w}\|_2^2 \\ \text{subject to:} \quad & (\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x) \\ & \begin{pmatrix} -\gamma I & (H_u\mathbf{M}\mathcal{G}+H_w) \\ (H_u\mathbf{M}\mathcal{G}+H_w)^\top & -\gamma I \end{pmatrix} \preceq 0, \end{aligned} \tag{6.58}$$

where the gain parameter γ was treated as a pre-specified constant.

In this section, we show that, like the expected value problem described in the previous section, the optimization problem (6.58) can be solved using standard convex optimization methods for several of the most common disturbance types encountered in Section 6.2.

Remark 6.15. *Note that if one instead wishes to minimize the achievable gain γ given an initial state x, i.e. to determine*

$$\gamma_N^*(x) := \inf\left\{\gamma \;\middle|\; \exists(\mathbf{M},\mathbf{v}) \in \Pi_N^{df}(x),\ \begin{pmatrix} -\gamma I & (H_u\mathbf{M}\mathcal{G}+H_w) \\ (H_u\mathbf{M}\mathcal{G}+H_w)^\top & -\gamma I \end{pmatrix} \preceq 0\right\}$$

as defined in (5.15)*, then it is obvious from the discussion thus far that this constitutes a standard form convex optimization problem when the disturbance set $\mathcal{W}$ takes any of the forms considered in Section 6.2.*

The principal difficulty with the optimization problem (6.58) lies in the elimination of the maximization operation from the cost function. Recall from Prop. 5.3 that the LMI constraint in (6.58) is equivalent to the QMI condition

$$(H_{\mathbf{u}}\mathbf{M}\mathcal{G}+H_{\mathbf{w}})^\top(H_{\mathbf{u}}\mathbf{M}\mathcal{G}+H_{\mathbf{w}}) - \gamma^2 I \preceq 0, \tag{6.59}$$

and is imposed in order to ensure that the maximization problem

$$\max_{\mathbf{w}\in\mathcal{W}} \left(\|(H_x x + H_u\mathbf{v}) + (H_u\mathbf{M}\mathcal{G} + H_w)\mathbf{w}\|_2^2 - \gamma^2\|\mathbf{w}\|_2^2\right) \tag{6.60}$$

is concave (i.e. equivalent to the minimization of a convex function). This enforced concavity enables exploitation of the dual of (6.60), so that (6.58) can be written as a single convex optimization problem.

6.4.1 Conic Disturbance Sets

Suppose that the disturbance set $\mathcal{W}$ is defined by an affine conic inequality as in Section 6.2.1, so that

$$\mathcal{W} := \{\mathbf{w} \mid S\mathbf{w} \preceq_K h\}, \tag{6.61}$$

where K is a convex cone[4] with dual cone K^*. When writing the dual of the maximization problem (6.60), we must bear in mind that the matrix $\mathbf{M}$ is itself a decision variable in the outer minimization problem of (6.58), so care must be taken to avoid a formulation that involves products or inverses of matrices involving this term. We therefore formulate the dual of (6.60) in a slightly unconventional fashion using an LMI:

Proposition 6.16. *If the QMI* (6.59) *holds, then the following duality result holds:*

$$\max_{\mathbf{w}} \left\{ \left(\|(H_x x + H_u \mathbf{v}) + (H_u \mathbf{M}\mathcal{G} + H_w)\mathbf{w}\|_2^2 - \gamma^2 \|\mathbf{w}\|_2^2 \right) \mid S\mathbf{w} \preceq_K h \right\} = \tag{6.62}$$

$$\min_{\delta,\, \mathbf{y} \succeq_{K^*} 0} \left\{ 2h^\top \mathbf{y} + \delta \;\middle|\; \begin{pmatrix} \delta & \mathbf{y}^\top S & (H_x x + H_u \mathbf{v})^\top \\ S^\top \mathbf{y} & \gamma^2 I & (H_u \mathbf{M}\mathcal{G} + H_w)^\top \\ (H_x x + H_u \mathbf{v}) & (H_u \mathbf{M}\mathcal{G} + H_w) & I \end{pmatrix} \succeq 0 \right\}. \tag{6.63}$$

Proof. Define V to be the maximum attainable value in the problem (6.62), and define the vector $\beta := -(H_x x + H_u \mathbf{v})$ and matrices $\Gamma := (H_u \mathbf{M}\mathcal{G} + H_w)$ and $\Theta := (\gamma^2 I - \Gamma^\top \Gamma)$, so that

$$V := \max_{\mathbf{w}} \left\{ \beta^\top \beta - 2\beta^\top \Gamma \mathbf{w} - \mathbf{w}^\top \Theta \mathbf{w} \mid S\mathbf{w} \preceq_K h \right\}. \tag{6.64}$$

We first write (6.64) as a minimization and take the dual, so that

$$V = -\min_{\mathbf{w}} \left\{ -\beta^\top \beta + 2\beta^\top \Gamma \mathbf{w} + \mathbf{w}^\top \Theta \mathbf{w} \mid S\mathbf{w} \preceq_K h \right\} \tag{6.65}$$

$$= -\max_{\lambda \succeq_{K^*} 0} \left\{ -h^\top \lambda - \beta^\top \beta + \min_{\mathbf{w}} \left\{ (2\Gamma^\top \beta + S^\top \lambda)^\top \mathbf{w} + \mathbf{w}^\top \Theta \mathbf{w} \right\} \right\}, \tag{6.66}$$

where equality is achieved in (6.66) since $0 \in \mathcal{W}$ by assumption, so that strong duality

[4] Recalling Rem. 6.3, we again make the assumption that K is a subset of a finite dimensional inner product space equipped with the inner product $\langle x, y \rangle := x^\top y$ and adjoint operator $S^* := S^\top$.

holds [BV04, Sec. 5.9]. Considering the inner unconstrained minimization of (6.66) in isolation, a well known result [BV04, Sec. A.5.5] when $\Theta \succeq 0$ (not necessarily positive definite) is

$$\min_{\mathbf{w}\in\mathcal{W}}\Big((2\Gamma^\top\beta+S^\top\lambda)^\top\mathbf{w}+\mathbf{w}^\top\Theta\mathbf{w}\Big)=\begin{cases}-\frac{1}{4}(2\Gamma^\top\beta+S^\top\lambda)^\top\Theta^\dagger(2\Gamma^\top\beta+S^\top\lambda) & \text{if } (2\Gamma^\top\beta+S^\top\lambda)\in\mathcal{R}(\Theta),\\ -\infty & \text{otherwise,}\end{cases}$$

where

$$(2\Gamma^\top\beta + S^\top\lambda) \in \mathcal{R}(\Theta) \;\Leftrightarrow\; (I-\Theta\Theta^\dagger)(2\Gamma^\top\beta + S^\top\lambda) = 0. \tag{6.67}$$

We can then make the constraint (6.67) explicit and substitute into (6.66) to get

$$\begin{aligned} V &= -\max_{\lambda\succeq_{K^*}0}\left\{-h^\top\lambda-\beta^\top\beta-\frac{1}{4}(2\Gamma^\top\beta+S^\top\lambda)^\top\Theta^\dagger(2\Gamma^\top\beta+S^\top\lambda)\;\middle|\;(I-\Theta\Theta^\dagger)(2\Gamma^\top\beta+S^\top\lambda)=0\right\}\\ &= \min_{\lambda\succeq_{K^*}0}\left\{+h^\top\lambda+\beta^\top\beta+\frac{1}{4}(2\Gamma^\top\beta+S^\top\lambda)^\top\Theta^\dagger(2\Gamma^\top\beta+S^\top\lambda)\;\middle|\;(I-\Theta\Theta^\dagger)(2\Gamma^\top\beta+S^\top\lambda)=0\right\}. \end{aligned} \tag{6.68}$$

Adding a variable δ to serve as an upper bound to the quadratic terms in (6.68) and applying the Schur complement Lemma 2.20 then gives

$$V = \min_{\delta,\,\lambda\succeq_{K^*}0}\left\{h^\top\lambda+\delta\;\middle|\;\begin{aligned}\beta^\top\beta+\frac{1}{4}(2\Gamma^\top\beta+S^\top\lambda)^\top\Theta^\dagger(2\Gamma^\top\beta+S^\top\lambda)\le\delta\\ (I-\Theta\Theta^\dagger)(2\Gamma^\top\beta+S^\top\lambda)=0\end{aligned}\right\} \tag{6.69}$$

$$= \min_{\delta,\,\lambda\succeq_{K^*}0}\left\{h^\top\lambda+\delta\;\middle|\;\begin{pmatrix}(\delta-\beta^\top\beta) & (\Gamma^\top\beta+\frac{1}{2}S^\top\lambda)^\top\\ (\Gamma^\top\beta+\frac{1}{2}S^\top\lambda) & \Theta\end{pmatrix}\succeq 0\right\}. \tag{6.70}$$

Finally, setting $\mathbf{y}=\frac{1}{2}\lambda$ and once again applying the Schur complement Lemma to (6.70),

$$V = \min_{\delta,\,\mathbf{y}\succeq_{K^*}0}\left\{2h^\top\mathbf{y}+\delta\;\middle|\;\begin{pmatrix}\delta & \mathbf{y}^\top S\\ S^\top\mathbf{y} & \gamma^2 I\end{pmatrix}-\begin{pmatrix}-\beta^\top\\ \Gamma^\top\end{pmatrix}\begin{pmatrix}-\beta & \Gamma\end{pmatrix}\succeq 0\right\} \tag{6.71}$$

$$= \min_{\delta,\,\mathbf{y}\succeq_{K^*}0}\left\{2h^\top\mathbf{y}+\delta\;\middle|\;\begin{pmatrix}\delta & \mathbf{y}^\top S & -\beta^\top\\ S^\top\mathbf{y} & \gamma^2 I & \Gamma^\top\\ -\beta & \Gamma & I\end{pmatrix}\succeq 0\right\}, \tag{6.72}$$

gives the desired result. ∎

Remark 6.17. *Note that if the LMI in the minimization problem* (6.63) *holds, then the QMI* (6.59) *(equivalently, the LMI in the optimization problem* (6.58)*) holds automatically.*

Using this result, we can write the min-max optimization problem (6.58) as a single minimization problem when the disturbance set is defined as in (6.61). We consider the most important of these situations in the remainder of this section.

Polytopic Disturbance Sets

We first consider the case where the disturbance set is polytopic, so that

$$\mathcal{W} = \{\mathbf{w} \mid S\mathbf{w} \leq h\},$$

which is equivalent to (6.61) with the cone K defined as the positive orthant in $\mathbb{R}^{lN}$. Recalling the characterization of the set $\Pi_N^{df}(x)$ in (6.25), the optimization problem (6.58) can be written as

$$\begin{aligned}
\min_{\delta,\mathbf{y},\mathbf{Z},\mathbf{M},\mathbf{v}} \quad & 2h^\top \mathbf{y} + \delta \\
\text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \leq j \\
& F_x x + F_u \mathbf{v} + \mathbf{Z}h \leq f \\
& \mathbf{Z}S = (F_u \mathbf{M}\mathcal{G} + F_w) \\
& \mathbf{Z} \geq 0,\ \mathbf{y} \geq 0 \\
& \begin{pmatrix} \delta & \mathbf{y}^\top S & (H_x x + H_u \mathbf{v})^\top \\ S^\top \mathbf{y} & \gamma^2 I & (H_u \mathbf{M}\mathcal{G} + H_w)^\top \\ (H_x x + H_u \mathbf{v}) & (H_u \mathbf{M}\mathcal{G} + H_w) & I \end{pmatrix} \succeq 0,
\end{aligned} \tag{6.73}$$

by direct application of the results of Prop. 6.16. The problem (6.73) is recognizable as a semidefinite program in a tractable number of variables (cf. Table 6.1).

Norm Bounded Disturbance Sets

We next consider the case where the disturbance set is p–norm bounded. Recalling Rem. 6.6, such a set can be written as in (6.61) by setting

$$S = \begin{pmatrix} -I \\ 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix},$$

with $K = K_p$ defined as in (6.29), and with dual cone $K_p^* = K_q$. In this case the optimization problem (6.58) can be written as

$$\begin{aligned}
\min_{\delta,\theta,\bar{\mathbf{y}},\mathbf{M},\mathbf{v}} \quad & 2\theta + \delta \\
\text{subject to:} \quad & M_{i,j} = 0,\ \forall i \le j \\
& \|\bar{\mathbf{y}}\|_q \le \theta \\
& (F_x x + F_u \mathbf{v})_i + \left\|(F_u \mathbf{M}\mathcal{G} + F_w)_i^\top\right\|_q \le (f)_i,\ \forall i \in \mathbb{Z}_{[1,t]} \\
& \begin{pmatrix} \delta & -\bar{\mathbf{y}}^\top & (H_x x + H_u \mathbf{v})^\top \\ -\bar{\mathbf{y}} & \gamma^2 I & (H_u \mathbf{M}\mathcal{G} + H_w)^\top \\ (H_x x + H_u \mathbf{v}) & (H_u \mathbf{M}\mathcal{G} + H_w) & I \end{pmatrix} \succeq 0,
\end{aligned} \tag{6.74}$$

where $\mathbf{y} =: \left[\begin{smallmatrix} \bar{\mathbf{y}} \\ \theta \end{smallmatrix}\right]$. As in the polytopic disturbance case, this optimization problem is a semidefinite program when $p \in \{1, 2, \infty\}$.

2–**Norm Bounded Disturbances:** We examine only the 2–norm bounded disturbance case in detail, since the 1– and ∞ norm bounded cases are straightforward. If $p = 2$, the optimization problem (6.74) can be written as

$$\begin{aligned}
\min_{\delta,\theta,\bar{\mathbf{y}},\mathbf{M},\mathbf{v}} \quad & 2\theta + \delta \\
\text{subject to:} \quad & M_{i,j} = 0,\ \forall i \le j \\
& \|\bar{\mathbf{y}}\|_2 \le \theta \\
& (F_x x + F_u \mathbf{v})_i + \left\|(F_u \mathbf{M}\mathcal{G} + F_w)_i^\top\right\|_2 \le (f)_i,\ \forall i \in \mathbb{Z}_{[1,t]} \\
& \begin{pmatrix} \delta & -\bar{\mathbf{y}}^\top & (H_x x + H_u \mathbf{v})^\top \\ -\bar{\mathbf{y}} & \gamma^2 I & (H_u \mathbf{M}\mathcal{G} + H_w)^\top \\ (H_x x + H_u \mathbf{v}) & (H_u \mathbf{M}\mathcal{G} + H_w) & I \end{pmatrix} \succeq 0.
\end{aligned} \tag{6.75}$$

The only consideration is the conversion of the SOCP constraints in (6.75) to LMI constraints so that the problem is in standard SDP form. Using straightforward procedures from [LVBL98], the convex optimization problem (6.75) can be written as the following standard form semidefinite program:

$$
\begin{aligned}
\min_{\delta,\theta,\bar{\mathbf{y}},\mathbf{M},\mathbf{v}} \quad & 2\theta + \delta \\
\text{subject to:} \quad & M_{i,j} = 0,\ \forall i \leq j \\
& \begin{pmatrix} \theta I & \bar{\mathbf{y}} \\ \bar{\mathbf{y}}^\top & \theta \end{pmatrix} \succeq 0 \\
& \begin{pmatrix} ((f)_i - (F_x x + F_u \mathbf{v})_i)\, I & (F_u \mathbf{M}\mathcal{G} + F_w)_i^\top \\ (F_u \mathbf{M}\mathcal{G} + F_w)_i & ((f)_i - (F_x x + F_u \mathbf{v})_i) \end{pmatrix} \succeq 0,\ \forall i \in \mathbb{Z}_{[1,t]} \\
& \begin{pmatrix} \delta & -\bar{\mathbf{y}}^\top & (H_x x + H_u \mathbf{v})^\top \\ -\bar{\mathbf{y}} & \gamma^2 I & (H_u \mathbf{M}\mathcal{G} + H_w)^\top \\ (H_x x + H_u \mathbf{v}) & (H_u \mathbf{M}\mathcal{G} + H_w) & I \end{pmatrix} \succeq 0.
\end{aligned}
\tag{6.76}
$$

Remark 6.18. *As in the case of the expected value minimization problems discussed in the previous section, it is straightforward to treat some or all of the constraints in the optimization problems posed in this section as soft constraints, so that feasibility can be guaranteed for all $x \in \mathbb{R}^n$.*

6.5 Conclusions

In this chapter we have shown that when the state and input constraint sets are polytopic, the policy optimization problems introduced in Chapters 3, 4 and 5 are solvable as standard convex optimization problems for a wide variety of disturbance classes. Additionally, for all of the problems considered in this chapter, the number of decision variables and constraints increases polynomially with the data defining the problem. The importance of this result should not be understated, since efficient computation is a critical consideration if the receding horizon controls proposed in Chapters 4 and 5 are to be implemented in real systems.

In the next chapter, we consider the efficient numerical solution of one of these optimization problems in greater detail, and will show that considerable efficiency gains are achievable by exploiting the special structure of the problem.

Chapter 7. Efficient Computation for ∞–norm Bounded Disturbances

7.1 Introduction

All of the finite horizon control problems defined in Chapter 6 for various classes of disturbance sets are convex problems – in principle, such problems are solvable in an amount of time which is a polynomial function of the number of variables and constraints by which they are defined [NN94, Wri97, BTN01]. However, these problems may still require considerable computational effort to reach a solution, since the dimensions of the problem can be very large.

This is particularly problematic if the problems in Chapter 6 are to be solved *on-line* in the implementation of a receding horizon control law, since the maximum update rate of the controller is dictated by the speed with which the underlying finite horizon problem can be solved. Additionally, the applications for which robust constrained control methods are of greatest potential benefit (including, for example, those in the aerospace or automotive industries) may have much faster dynamics than those typically associated with receding horizon control techniques. Computational speed and efficiency are therefore essential if the ideas presented thus far are to be of practical use.

We consider in greater detail the problem of finding an optimal control policy in the case where the disturbance set W is modelled as an ∞-norm bounded set

$$W = \{w \in \mathbb{R}^l \mid \|w\|_\infty \leq 1\}, \tag{7.1}$$

with $\mathcal{W} := W^N$, and the cost to be minimized is quadratic in the *nominal* or *disturbance-free* state and input sequences (cf. Section 4.4.1) . Note that this problem is a version of the problem introduced in Section 6.3 on page 104 with $\mathbf{C}_w = 0$. From (6.55), the optimization problem of interest is a quadratic program in the variables $\mathbf{M}$, $\mathbf{\Lambda}$, and $\mathbf{v}$:

$$\begin{aligned}
\min_{\mathbf{M},\mathbf{\Lambda},\mathbf{v}} \quad & \|H_x x + H_u \mathbf{v}\|_2^2 \\
\text{subject to:} \quad & M_{i,j} = 0, \quad \forall i \leq j \\
& F_x x + F_u \mathbf{v} + \mathbf{\Lambda} \mathbf{1} \leq f \\
& -\mathbf{\Lambda} \leq F_u \mathbf{M} \mathcal{G} + F_w \leq \mathbf{\Lambda}.
\end{aligned} \tag{7.2}$$

In this chapter, we will develop an efficient computational solution method for this problem.

We continue to use much of the notation introduced in previous chapters, and we make the following assumptions throughout:

A7.1 (Standing Assumptions) *The following conditions hold:*

*i. The assumptions **A3.1** hold.*

ii. The matrix D_z is full column rank, with $C_z^\top D_z = 0$.

iii. The matrix D_c is full column rank.

iv. The matrix P is positive semidefinite.

Note that these assumptions are a subset of those made in developing the theoretical results of Chapter 4, and that the full rank assumption on D_c is implied assumption in **A3.1** that the system is bounded in the inputs. For simplicity of presentation, we define the nominal state trajectory as

$$\hat{\mathbf{x}} := \mathbf{A}x + \mathbf{B}\mathbf{v} \tag{7.3a}$$

$$=: \operatorname{vec}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N), \tag{7.3b}$$

and define matrices $Q := C_z^\top C_z$ and $R := D_z^\top D_z$, so that the cost function in (7.2) can be written as

$$\|H_x x + H_u \mathbf{v}\|_2^2 = \|\hat{x}_N\|_P^2 + \sum_{i=0}^{N-1} \left(\|\hat{x}_i\|_Q^2 + \|v_i\|_R^2 \right).$$

Remark 7.1. *Recall from Section 6.2.5 that the total number of decisions variables in (7.2) is mN in $\mathbf{v}$, $mnN(N-1)/2$ in $\mathbf{M}$, and $(slN^2 + s_f lN)$ in $\mathbf{\Lambda}$, with the number of constraints equal to $(sN + s_f) + 2(slN^2 + s_f lN))$, or $\mathcal{O}(N^2)$ overall. For a naive interior-point computational approach using a dense factorization method, the resulting quadratic program would therefore require computation time of $\mathcal{O}(N^6)$ at each iteration [Wri97].*

7.1.1 A QP in Separable Form

We first define the variable transformation $\mathbf{U} := \mathbf{M}\mathcal{G}$, so that the matrix $\mathbf{U} \in \mathbb{R}^{mN\times lN}$ has a block lower triangular structure similar to that defined in (3.19) for $\mathbf{M}$, i.e.

$$\mathbf{U} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ U_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ U_{N-1,0} & \cdots & U_{N-1,N-2} & 0 \end{bmatrix}. \tag{7.4}$$

Note that use of this variable transformation is tantamount to parameterizing the control policy directly in terms of the generating disturbances w_i, so that $u_i = v_i + \sum_{j=0}^{i-1} U_{i,j} w_j$, or $\mathbf{u} = \mathbf{U}\mathbf{w} + \mathbf{v}$.

When the matrix G (and thus $\mathcal{G}$) is full column rank, the QP (7.2) can be solved using this variable transformation by solving an equivalent QP in the variables $\mathbf{U}$, $\boldsymbol{\Lambda}$ and $\mathbf{v}$:

$$\min_{\mathbf{U},\boldsymbol{\Lambda},\mathbf{v}} \quad \|H_x x + H_u \mathbf{v}\|_2^2 \tag{7.5a}$$

$$\text{subject to:} \quad M_{i,j} = 0, \quad \forall i \le j \tag{7.5b}$$

$$F_x x + F_u \mathbf{v} + \boldsymbol{\Lambda}\mathbf{1} \le f \tag{7.5c}$$

$$-\boldsymbol{\Lambda} \le F_u \mathbf{U} + F_w \le \boldsymbol{\Lambda}. \tag{7.5d}$$

When $\mathcal{G}$ is full column rank as assumed in **A3.1**, the equivalence between the QPs (7.2) and (7.5) is easily demonstrated by employing a left inverse $\mathcal{G}^\dagger$ such that $\mathcal{G}^\dagger\mathcal{G} = I$, since any feasible solution $(\mathbf{M}, \boldsymbol{\Lambda}, \mathbf{v})$ satisfying the constraints in (7.5) also satisfies the constraints in (7.2) with $\mathbf{M} = \mathbf{U}\mathcal{G}^\dagger$. As in the case of the problem (7.2), calculating a solution to (7.5) using a naive interior point method would typically require $\mathcal{O}(N^6)$ computations at each iteration (cf. Remark 7.1).

The critical feature of the quadratic program (7.5) is that the columns of the variables $\mathbf{U}$ and $\boldsymbol{\Lambda}$ are decoupled in the constraint (7.5d). This allows column-wise separation of the constraint into a number of subproblems, subject to the coupling constraint (7.5c). In the sequel, we will exploit this structure to develop an efficient computational technique that reduces the solution complexity of (7.5) (and thus of (7.2)) from $\mathcal{O}(N^6)$ to $\mathcal{O}(N^3)$. The method we employ is similar to that proposed in [RWR98] in the case of receding horizon control *without* disturbances, where it was shown that a reduction in computational effort of $\mathcal{O}(N^3)$ to $\mathcal{O}(N)$ was possible in the undisturbed case.

7.2 Recovering Structure

The quadratic program (QP) defined in (7.5) can be rewritten in a more computationally attractive form by re-introducing the eliminated state variables to achieve greater structure. The remodelling process separates the original problem into subproblems; a nominal problem, consisting of that part of the state resulting from the nominal control vector $\mathbf{v}$, and a set of perturbation problems, each representing the components of the state resulting from each of the columns of (7.5d) in turn.

Nominal States and Inputs

We first define a constraint contraction vector $\mathbf{c} \in \mathbb{R}^{sN+s_f}$ such that

$$\mathbf{c} := \mathrm{vec}(c_0, \ldots, c_N) = \mathbf{\Lambda 1}, \tag{7.6}$$

so that the constraint (7.5c) becomes

$$F_x x + F_u \mathbf{v} + \mathbf{c} \leq f. \tag{7.7}$$

Recalling that the nominal states $\hat{x}_i$ are defined in (7.3) as the expected states given no disturbances, the constraint (7.7) can be written explicitly in terms of the nominal controls v_i and states $\hat{x}_i$ as

$$\hat{x}_{i+1} - A\hat{x}_i - Bv_i = 0, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{7.8a}$$

$$C_c\hat{x}_i + D_c v_i + c_i \leq b, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{7.8b}$$

$$Y_c\hat{x}_N + c_N \leq z, \tag{7.8c}$$

where $\hat{x}_0 = x$, which is in a form that is *exactly the same* as that in conventional receding horizon control problem with no disturbances, but with the right-hand-sides of the state and input constraints at each stage i modified by the constraint contractions terms c_i; compare (7.8a)–(7.8c) with (3.1), (6.2) and (6.3) respectively.

Perturbed States and Inputs

We next consider the contribution of each of the columns of $(F_u\mathbf{U} + F_w)$ in turn, and construct a set of problems similar to that in (7.8). We treat each column as the output of a system subject to a unit impulse in a single element of the disturbance sequence $\mathbf{w}$, and construct a subproblem that calculates the effect of that disturbance on the nominal problem constraints (7.8b)–(7.8c) by determining its contribution to the total constraint contraction vector $\mathbf{c}$.

From the original QP constraint (7.5d), the constraint contraction vector $\mathbf{c}$ can be written as

$$|F_u\mathbf{U} + F_w|\mathbf{1} \le \mathbf{\Lambda}\mathbf{1} =: \mathbf{c}. \tag{7.9}$$

The left-hand side of (7.9) is just a summation over the columns of the matrix $|F_u\mathbf{U} + F_w|$, so that

$$|F_u\mathbf{U} + F_w|\mathbf{1} = \sum_{p=1}^{lN} |(F_u\mathbf{U} + F_w)_{(p)}|. \tag{7.10}$$

where $(F_u\mathbf{U}+F_w)_{(p)}$ denotes the p^{th} column of the matrix $(F_u\mathbf{U}+F_w)$. Define $\mathbf{y}^p \in \mathbb{R}^{sN+s_f}$ and $\mathbf{c}^p \in \mathbb{R}^{sN+s_f}$ as

$$\mathbf{y}^p := (F_u\mathbf{U} + F_w)_{(p)} \text{ and } \mathbf{c}^p := |\mathbf{y}^p|. \tag{7.11}$$

Note that the p^{th} column of the matrix $\mathbf{U}$ represents the system control input resulting from a disturbance in some element j of the generating disturbance w_k at some time step k, with no disturbances at any other step[1]. If we denote the j^{th} column of G as $G_{(j)}$, then it is easy to recognize $\mathbf{y}^p$ as the stacked output vector of the system

$$(u_i^p, x_i^p, y_i^p) = 0, \quad \forall i \in \mathbb{Z}_{[0,k]} \tag{7.12a}$$
$$x_{k+1}^p = G_{(j)}, \tag{7.12b}$$
$$x_{i+1}^p - Ax_i^p - Bu_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} \tag{7.12c}$$
$$y_i^p - C_c x_i^p - D_c u_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} \tag{7.12d}$$
$$y_N^p - Y_c x_N^p = 0, \tag{7.12e}$$

where $\mathbf{y}^p =: \mathrm{vec}(y_0^p, \dots, y_N^p)$. The inputs u_i^p of this system come directly from the p^{th} column of the matrix $\mathbf{U}$, i.e. they are columns of the sub-matrices $U_{i,k}$ in (7.4). If the constraint terms $\mathbf{c}^p$ for each subproblem are similarly written as $\mathbf{c}^p := \mathrm{vec}(c_0^p, \dots, c_N^p)$, then

[1] Note that this implies $p = lk + j$, $k = \lfloor \frac{p-1}{l} \rfloor$ and $j = 1 + ((p-1) \bmod l)$.

each component must satisfy the linear inequality constraint

$$-c_i^p \leq y_i^p \leq c_i^p. \tag{7.13}$$

The vectors $\mathbf{c}^p$ therefore correspond exactly to the columns of the matrix $\mathbf{\Lambda}$. Note that for the p^{th} subproblem, representing a disturbance at stage $k = \lfloor \frac{p-1}{l} \rfloor$, the constraint contraction terms are zero prior to stage $(k+1)$.

By further defining

$$\bar{C} := \begin{bmatrix} +C_c \\ -C_c \end{bmatrix} \; \bar{D} := \begin{bmatrix} +D_c \\ -D_c \end{bmatrix} \; \bar{Y} := \begin{bmatrix} +Y_c \\ -Y_c \end{bmatrix} \; H := \begin{bmatrix} -I_s \\ -I_s \end{bmatrix} \; H_f := \begin{bmatrix} -I_r \\ -I_r \end{bmatrix}, \tag{7.14}$$

equations (7.12d) and (7.12e) can be combined with (7.13) to give

$$\bar{C}x_i^p + \bar{D}u_i^p + Hc_i^p \leq 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} \tag{7.15a}$$

$$\bar{Y}x_N^p + H_f c_N^p \leq 0. \tag{7.15b}$$

Complete Robust Control Problem

We can now restate the complete robust optimization problem (7.5) as:

$$\min_{\substack{\hat{x}_1,\ldots,\hat{x}_N,v_0,\ldots v_{N-1},c_0,\ldots,c_N, \\ x_0^1,\ldots,x_N^1,u_0^1,\ldots u_{N-1}^1,c_0^1,\ldots,c_N^1, \\ \ldots, \\ x_0^{lN},\ldots,x_N^{lN},u_0^{lN},\ldots u_{N-1}^{lN},c_0^{lN},\ldots,c_N^{lN}}} \quad \frac{1}{2}\left\|\hat{x}_N\right\|_P^2 + \sum_{i=0}^{N-1}\left(\frac{1}{2}\left\|\hat{x}_i\right\|_Q^2 + \frac{1}{2}\left\|v_i\right\|_R^2\right) \tag{7.16}$$

subject to (7.8), (7.12a)–(7.12c) and (7.15), which we restate here for convenience:

$$\hat{x}_{i+1} - A\hat{x}_i - Bv_i = 0, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{7.17a}$$

$$C_c\hat{x}_i + D_c v_i + c_i \leq b, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{7.17b}$$

$$Y_c\hat{x}_N + c_N \leq z, \tag{7.17c}$$

where $\hat{x}_0 = x$, and

$$c_i = \sum_{p=1}^{lN} c_i^p, \quad \forall i \in \mathbb{Z}_{[0,N]}, \tag{7.18}$$

and, for each $p \in \mathbb{Z}_{[1,lN]}$:

$$
\begin{aligned}
(u_i^p, x_i^p, c_i^p) &= 0, \quad \forall i \in \mathbb{Z}_{[0,k]} && (7.19a)\\
x_{k+1}^p &= G_{(j)}, && (7.19b)\\
x_{i+1}^p - Ax_i^p - Bu_i^p &= 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} && (7.19c)\\
\bar{C}x_i^p + \bar{D}u_i^p + Hc_i^p &\leq 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} && (7.19d)\\
\bar{Y}x_N^p + H_f c_N^p &\leq 0 && (7.19e)
\end{aligned}
$$

where $k = \lfloor \frac{p-1}{l} \rfloor$ and $j = 1 + ((p-1) \bmod l)$.

The decision variables in this problem are the nominal states and controls $\hat{x}_i$ and v_i at each stage (the initial state $\hat{x}_0$ is known, hence *not* a decision variable), plus the perturbed states, controls, and constraint contractions terms x_i^p, u_i^p, and c_i^p for each subproblem at each stage.

Remark 7.2. *Recalling the discussion of Section 6.3.1, soft constraints are easily incorporated into the optimization problem* (7.16)–(7.19) *via modification of the cost function* (7.16) *and of the constraints* (7.17b)–(7.17c)*. The important point regarding this soft constraint inclusion is that it* does not *result in a modification of any of the perturbation constraints* (7.19)*, so that the qualitative results to be presented in Section 7.3 relating to efficient solution of the QP* (7.16)–(7.19) *are not fundamentally altered by the incorporation of soft constraints.*

We can now state the following key result, proof of which follows directly from the discussion of Section 7.1.1 and of this section.

Theorem 7.3. *The convex, tractable QP* (7.16)–(7.19) *is equivalent to the robust optimal control problems* (7.2) *and* (7.5)*.*

The importance of the re-introduction of states in (7.17) and (7.19) is that significant structure and sparsity can be revealed in the problem through an interleaving of decision variables by time index. For the nominal problem, define the stacked vector of variables:

$$
\mathsf{x}_0 := \mathrm{vec}(v_0, \hat{x}_1, v_1, \ldots, \hat{x}_{N-1}, v_{N-1}, \hat{x}_N). \tag{7.20}
$$

For the p^{th} perturbation problem in (7.19), which models a unit disturbance at stage

$k = \lfloor \frac{p-1}{l} \rfloor$, define:

$$\mathsf{x}_p := \mathrm{vec}(u^p_{k+1}, c^p_{k+1}, x^p_{k+2}, u^p_{k+2}, c^p_{k+2}, \dots, x^p_{N-1}, u^p_{N-1}, c^p_{N-1}, x^p_N, c^p_N). \tag{7.21}$$

Using this reordering, the constraints (7.17)–(7.19) can be written as a single set of linear constraints in singly-bordered block-diagonal form with considerable structure and sparsity:

$$\begin{bmatrix} \mathsf{A}_0 & & & \\ & \mathsf{A}_1 & & \\ & & \ddots & \\ & & & \mathsf{A}_{lN} \end{bmatrix} \begin{bmatrix} \mathsf{x}_0 \\ \mathsf{x}_1 \\ \vdots \\ \mathsf{x}_{lN} \end{bmatrix} = \begin{bmatrix} \mathsf{b}_0 \\ \mathsf{b}_1 \\ \vdots \\ \mathsf{b}_{lN} \end{bmatrix}, \quad \begin{bmatrix} \mathsf{C}_0 & \mathsf{J}_1 & \cdots & \mathsf{J}_{lN} \\ & \mathsf{C}_1 & & \\ & & \ddots & \\ & & & \mathsf{C}_{lN} \end{bmatrix} \begin{bmatrix} \mathsf{x}_0 \\ \mathsf{x}_1 \\ \vdots \\ \mathsf{x}_{lN} \end{bmatrix} \leq \begin{bmatrix} \mathsf{d}_0 \\ \mathsf{d}_1 \\ \vdots \\ \mathsf{d}_{lN} \end{bmatrix}. \tag{7.22}$$

The coefficient matrices A_0 and C_0 in (7.22) originate from the nominal problem constraints (7.17), and are defined as

$$\mathsf{A}_0 := \begin{bmatrix} B & -I & & & & \\ & A & B & -I & & \\ & & & \ddots & & \\ & & & A & B & -I \end{bmatrix}, \quad \mathsf{C}_0 := \begin{bmatrix} D_c & & & & \\ & C_c & D_c & & \\ & & \ddots & & \\ & & & C_c & D_c \\ & & & & Y_c \end{bmatrix}, \tag{7.23}$$

with corresponding right hand sides

$$\mathsf{b}_0 := \mathrm{vec}(-Ax, 0, 0, \dots, 0), \quad \mathsf{d}_0 := \mathrm{vec}(b - C_c x, b, \dots, b, z). \tag{7.24}$$

The coefficient matrices A_p and C_p in (7.22) originate from the constraints for the p^{th} perturbation subproblem in (7.19), and are defined as

$$\mathsf{A}_p := \begin{bmatrix} B & 0 & -I & & & & \\ & A & B & 0 & -I & & \\ & & & \ddots & & & \\ & & A & B & 0 & -I & 0 \end{bmatrix}, \quad \mathsf{C}_p := \begin{bmatrix} \bar{D} & H & & & & \\ & \bar{C} & \bar{D} & H & & \\ & & \ddots & & & \\ & & & \bar{C} & \bar{D} & H \\ & & & & \bar{Y} & H_f \end{bmatrix}, \tag{7.25}$$

with corresponding right hand sides

$$\mathsf{b}_p := \mathrm{vec}(-AG_{(j)}, 0, \dots, 0), \quad \mathsf{d}_p := \mathrm{vec}(0, 0, \dots, 0, 0). \tag{7.26}$$

The coupling matrices J_p in (7.22) are then easily constructed from the coupling equation (7.18).

Remark 7.4. *It is possible to define, in a fairly obvious way, a problem structure akin to that in* (7.16)–(7.19) *for the general norm-bounded discussed in Section 6.2.3 via introduction of states in a similar manner. A similar statement can be made about the general polytopic disturbance case discussed in Section* (6.2.2)*; however, in the latter case the perturbation subproblems* (7.19) *require an additional coupling constraint for the subproblems associated with each stage.*

7.3 Interior-Point Method for Robust Control

In this section we demonstrate that, using a primal-dual interior-point solution technique, the quadratic program defined in (7.16)–(7.19) is solvable in an amount of time that is cubic in the horizon length N at each iteration, when $n+m$ is dominated by N; this situation is common, for example, in the rapidly growing number of aerospace and automotive applications of predictive control [Mac02, Sec. 3.3][QB03]. This is a major improvement on the $O(N^6)$ work per iteration associated with the compact (dense) formulation (7.2), or the equivalent problem (7.5); cf. Remark 7.1. This computational improvement comes about due to the improved structure and sparsity of the problem. Indeed, akin to the situation in [RWR98], we will show that each subproblem in the QP (7.16)–(7.19) has the same structure as that of an unconstrained optimal control problem without disturbances.

7.3.1 General Interior-Point Methods

We first outline some of the general properties of interior-point solution methods by considering the general constrained quadratic optimization problem

$$\min_{\theta} \frac{1}{2}\theta^\top \mathsf{Q}\theta \quad \text{subject to } \mathsf{A}\theta = \mathsf{b},\ \mathsf{C}\theta \le \mathsf{d}, \tag{7.27}$$

where the matrix Q is positive semidefinite. An optimal solution θ for this system exists if and only if the Karush-Kuhn-Tucker conditions are satisfied, i.e. there exist additional vectors π, λ and z satisfying the following conditions:

$$\mathsf{Q}\theta + \mathsf{A}^\top\pi + \mathsf{C}^\top\lambda = 0 \tag{7.28a}$$
$$\mathsf{A}\theta - \mathsf{b} = 0 \tag{7.28b}$$
$$-\mathsf{C}\theta + \mathsf{d} - z = 0 \tag{7.28c}$$
$$(\lambda, z) \geq 0 \tag{7.28d}$$
$$\lambda^\top z = 0. \tag{7.28e}$$

The variables λ and z in (7.28) will be referred to as the *Lagrange multipliers* and *slack variables*, respectively. In primal-dual interior point methods [Wri97], the *central path* is defined as the set of parameters $(\theta, \pi, \lambda, z)$ satisfying (7.28a)–(7.28d), with the complementarity condition (7.28e) relaxed, for each element i, to $\lambda_i z_i = \mu$, where $\mu > 0$ parameterizes the path. This guarantees that λ and z are strictly positive vectors. The central path converges to a solution of (7.28) as $\mu \downarrow 0$ if such a solution exists.

The constraints $\lambda_i z_i = \mu$ can be rewritten in a slightly more convenient form by defining diagonal matrices Λ and Z such that

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \text{ and } Z = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \tag{7.29}$$

so that the relaxed complementarity condition becomes $\Lambda Z \mathbf{1} = \mu\mathbf{1}$. Primal-dual interior-point algorithms search for a solution to the KKT conditions (7.28) by producing a sequence of iterates $(\theta^\kappa, \pi^\kappa, \lambda^\kappa, z^\kappa)$ which approximate the central path solution at some $\mu^\kappa > 0$. These iterates are updated via repeated solution of a set of Newton-like equations of the form

$$\begin{bmatrix} \mathsf{Q} & \mathsf{A}^\top & \mathsf{C}^\top & \\ \mathsf{A} & & & \\ \mathsf{C} & & & I \\ & & Z & \Lambda \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\pi \\ \Delta\lambda \\ \Delta z \end{bmatrix} = - \begin{bmatrix} r_Q \\ r_A \\ r_C \\ r_Z \end{bmatrix}, \tag{7.30}$$

where the residuals (r_Q, r_A, r_C) take the values of the left-hand sides of (7.28a)–(7.28c) respectively, evaluated at the current values $(\theta^\kappa, \pi^\kappa, \lambda^\kappa, z^\kappa)$, and the matrices (Z, Λ) are formed from the current iterates $(z^\kappa, \lambda^\kappa)$ as in (7.29). The residual vector r_Z is typically defined as $r_Z = (\Lambda Z\mathbf{1} - \mathbf{1}\bar{\mu})$, where $\bar{\mu}$ is chosen such that $\bar{\mu} \in (0, \mu^\kappa)$. Once the linear system (7.30) has been solved, the solution is updated as

$$(\theta^{\kappa+1}, \pi^{\kappa+1}, \lambda^{\kappa+1}, z^{\kappa+1}) \leftarrow (\theta^{\kappa}, \pi^{\kappa}, \lambda^{\kappa}, z^{\kappa}) + \alpha(\Delta\theta, \Delta\pi, \Delta\lambda, \Delta z),$$

where $\alpha > 0$ is chosen to maintain strict positivity of λ^{k+1} and z^{k+1}, and the path parameter μ^{κ} is updated to some $\mu^{\kappa+1} \in (0, \mu^{\kappa})$. The particular method for selecting the parameters $\bar{\mu}$ and α at each iteration depends on the specific interior-point algorithm employed; the reader is referred to [Wri97] for a thorough review.

Since all such methods maintain the strict inequalities $(\lambda, z) > 0$ at each iteration as $\mu \downarrow 0$, the matrices Λ and Z are guaranteed to remain full rank, and the system of equations in (7.30) can be simplified through elimination of the variables Δz to form the reduced system

$$\begin{bmatrix} \mathsf{Q} & \mathsf{A}^\top & \mathsf{C}^\top \\ \mathsf{A} & & \\ \mathsf{C} & & -\Lambda^{-1}Z \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\pi \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} r_Q \\ r_A \\ \left(r_C - \Lambda^{-1} r_Z\right) \end{bmatrix}. \tag{7.31}$$

Since the number of interior-point iterations required in practice is only weakly related to the number of variables [Wri97], the principal consideration is the time required to factor the Jacobian matrix (i.e. the matrix on the left-hand-side), and solve the linear system in (7.31). In the remainder of this chapter we focus on the development of an efficient solution procedure for this linear system when the problem data for the QP (7.27) is defined by the problem (7.16)–(7.19).

7.3.2 Robust Control Formulation

For the robust optimal control problem described in (7.16)–(7.19), the system of equations in (7.31) can be arranged to yield a highly structured set of linear equations through appropriate ordering of the primal and dual variables and their Lagrange multipliers at each stage. As will be shown, this ordering enables the development of an efficient solution procedure for the linear system in (7.31).

We use λ_i and λ_N to denote the Lagrange multipliers for the constraints (7.17b) and (7.17c) in the nominal system, and z_i and z_N for the corresponding slack variables. We similarly use λ_i^p and λ_N^p to denote the multipliers in (7.19d) and (7.19e) for the p^{th} perturbation subproblem, with slack variables z_i^p and z_N^p. We use π_i and π_i^p to denote the dual variables for (7.17) and (7.19).

The linear system (7.31), defined for the particular robust control problem (7.16)–(7.19), can then be reordered to form a symmetric, block-bordered, banded diagonal set of equations by interleaving the primal and dual variables within the nominal and perturbed problems, while keeping the variables from each subproblem separate. If the p^{th} perturbation subproblem corresponds to a unit disturbance at some stage $k = \lfloor \frac{p-1}{l} \rfloor$, then the components of the system of equations (7.31) corresponding to the nominal variables and the variables for the p^{th} perturbation subproblem are coupled at all stages after k.

Considering for the moment only that part of (7.19) corresponding to the first perturbation problem (with $p = 1$), this reordering yields the coupled linear system

$$
\left[\begin{array}{ccccccccccccc|ccccccccccccc}
R & D_c^\top & B^\top \\
D_c & -\Sigma_0 & 0 \\
B & 0 & 0 & -I \\
& & -I & Q & 0 & C_c^\top & A^\top \\
& & & 0 & R & D_c^\top & B^\top \\
& & & C_c & D_c & -\Sigma_1 & 0 &&&&&&&&& I \\
& & & A & B & 0 & 0 & -I \\
&&&&&& -I & Q & 0 & C_c^\top & A^\top \\
&&&&&&& 0 & R & D_c^\top & B^\top \\
&&&&&&& C_c & D_c & -\Sigma_2 & 0 &&&&&&&&&& I \\
&&&&&&& A & B & 0 & \ddots & \ddots \\
&&&&&&&&&& \ddots & P & Y_c^\top \\
&&&&&&&&&&& Y_c & -\Sigma_N &&&&&&&&&&&&& I \\
\hline
&&&&&&&&&&&&& 0 & 0 & \bar{D}^\top & B^\top \\
&&&& I &&&&&&&&& 0 & 0 & H^\top & 0 \\
&&&&&&&&&&&&& \bar{D} & H & -\Sigma_1^1 & 0 \\
&&&&&&&&&&&&& B & 0 & 0 & 0 & -I \\
&&&&&&&&&&&&&&&& -I & 0 & 0 & 0 & \bar{C}^\top & A^\top \\
&&&&&&&&&&&&&&&&& 0 & 0 & 0 & \bar{D}^\top & B^\top \\
&&&&&&&& I &&&&&&&&& 0 & 0 & 0 & H^\top & 0 \\
&&&&&&&&&&&&&&&&& \bar{C} & \bar{D} & H & -\Sigma_2^1 & 0 \\
&&&&&&&&&&&&&&&&& A & B & 0 & 0 & \ddots & \ddots \\
&&&&&&&&&&&&&&&&&&&&&& \ddots & 0 & 0 & \bar{Y}^\top \\
&&&&&&&&&&&& I &&&&&&&&&&& 0 & 0 & H_f^\top \\
&&&&&&&&&&&&&&&&&&&&&&& \bar{Y} & H_f & -\Sigma_N^1
\end{array}\right]
\left[\begin{array}{c}
\Delta v_0 \\ \Delta\lambda_0 \\ \Delta\pi_0 \\ \Delta x_1 \\ \Delta v_1 \\ \Delta\lambda_1 \\ \Delta\pi_1 \\ \Delta x_2 \\ \Delta v_2 \\ \Delta\lambda_2 \\ \vdots \\ \Delta x_N \\ \Delta\lambda_N \\ \hline \Delta u_1^1 \\ \Delta c_1^1 \\ \Delta\lambda_1^1 \\ \Delta\pi_1^1 \\ \Delta x_2^1 \\ \Delta u_2^1 \\ \Delta c_2^1 \\ \Delta\lambda_2^1 \\ \vdots \\ \Delta x_N^1 \\ \Delta c_N^1 \\ \Delta\lambda_N^1
\end{array}\right]
=
\left[\begin{array}{c}
r^{v_0} \\ r^{\lambda_0} \\ r^{\pi_0} \\ r^{x_1} \\ r^{v_1} \\ r^{\lambda_1} \\ r^{\pi_1} \\ r^{x_2} \\ r^{v_2} \\ r^{\lambda_2} \\ \vdots \\ r^{x_N} \\ r^{\lambda_N} \\ \hline r^{u_1^1} \\ r^{c_1^1} \\ r^{\lambda_1^1} \\ r^{\pi_1^1} \\ r^{x_2^1} \\ r^{u_2^1} \\ r^{c_2^1} \\ r^{\lambda_2^1} \\ \vdots \\ r^{x_N^1} \\ r^{c_N^1} \\ r^{\lambda_N^1}
\end{array}\right].
\tag{7.32}
$$

The diagonal matrices Σ_i and Σ_i^p in (7.32) correspond to the matrix products $\Lambda^{-1}Z$ in (7.31), and are defined as

$$\Sigma_i := (\Lambda_i)^{-1}Z_i, \quad \forall i \in \mathbb{Z}_{[0,N]} \tag{7.33}$$

$$\Sigma_i^p := (\Lambda_i^p)^{-1}Z_i^p, \quad \forall i \in \mathbb{Z}_{[k+1,N]}, \tag{7.34}$$

where the matrices Λ_i, Λ_i^p, Z_i, and Z_i^p are diagonal matrices formed from the Lagrange multipliers and slack variables λ_i, λ_i^p, z_i and z_i^p from the nominal and perturbation subproblems.

If all of the perturbation problems (7.19) are incorporated into a linear system of the form (7.32), the result is a system of equations whose coefficient matrix can be partitioned into block-bordered form as

$$\begin{bmatrix} \mathcal{A} & \mathcal{J}_1 & \mathcal{J}_2 & \cdots & \mathcal{J}_{lN} \\ \mathcal{J}_1^\top & \mathcal{B}_1 & & & \\ \mathcal{J}_2^\top & & \mathcal{B}_2 & & \\ \vdots & & & \ddots & \\ \mathcal{J}_{lN}^\top & & & & \mathcal{B}_{lN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{lN} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_A \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{lN} \end{bmatrix}, \tag{7.35}$$

where the banded matrix $\mathcal{A}$ is derived from the coefficients in the nominal problem (7.17), the banded matrices $\mathcal{B}_p$ are derived from the lN perturbation subproblems (7.19), and the matrices $\mathcal{J}_p$ represent the coupling between the systems. The vectors $\mathbf{b}_A$, $\mathbf{b}_p$, $\mathbf{x}_A$, and $\mathbf{x}_p$ (which should not be confused with the stacked sequence of state vectors $\mathbf{x}$) are constructed from the primal and dual variables and residuals using the ordering in (7.32). The matrices $\mathcal{J}_p$ are constructed from identity matrices coupling the rows of $\mathcal{A}$ that contain the Σ_i terms with the columns of $\mathcal{B}_p$ that contain the H terms. It should of course be noted that for the matrix $\mathcal{B}_p$, corresponding to a unit disturbance at stage $k = \lfloor \frac{p-1}{l} \rfloor$, terms from stages prior to stage $k+1$ are not required.

7.3.3 Solving for an Interior-Point Step

We can now estimate the solution time for the robust optimization problem (7.16)–(7.19) by demonstrating that the linear system in (7.35) can be solved in $\mathcal{O}((m+n)^3N^3)$ operations. We recall that, in practice, the number of interior-point iterations is only weakly dependent on the size of the problem [Wri97]. Throughout this section, we make the simplifying assumption that the number of constraints s and s_f defining Z and X_f are $\mathcal{O}(m+n)$ and $\mathcal{O}(n)$, respectively.

We first require the following preliminary results, proofs for which can be found in the appendix to this chapter.

Lemma 7.5. *For the robust control problem* (7.16)–(7.19)*, the Jacobian matrix in* (7.32) *is full rank.*

Lemma 7.6. *The sub-matrices $\mathcal{B}_p$ arising from the perturbation subproblems in* (7.35) *are full rank. Additionally, recalling that $k = \lfloor \frac{p-1}{l} \rfloor$,*

i. A solution to the linear system $\mathcal{B}_p \mathbf{x}_p = \mathbf{b}_p$ can be found in $\mathcal{O}\left((m{+}n)^3(N-k+1)\right)$ operations.

ii. If a solution to (i) above has been found, then a solution for each additional right hand side requires $\mathcal{O}\left((m{+}n)^2(N-k+1)\right)$ operations.

Remark 7.7. *Note that each of the blocks $\mathcal{B}_p$ on the diagonal of* (7.35) *is banded and symmetric indefinite. Several methods exist for the stable construction of Cholesky-like decompositions of symmetric indefinite matrices into factors of the form $LDL^\top$ [BKP76], and efficient algorithms for performing this factorization for sparse matrices are freely available [DER86, HSL02]. However, it is generally not possible to guarantee that the banded structure of an indefinite matrix, such as $\mathcal{B}_p$, will be exploited using these methods if symmetry and stability of the factorization are to be preserved. Instead, the special structure of the matrices $\mathcal{B}_p$ allows us to employ a specialized technique for solution of the linear system $\mathcal{B}_p \mathbf{x}_p = \mathbf{b}_p$ based on a Riccati recursion [Ste95, RWR98] in the proof of Lemma 7.6 in Appendix 7.A.2.*

We can now demonstrate that it is always possible to solve the linear system (7.35) in $\mathcal{O}((m{+}n)^3N^3)$ operations.

Theorem 7.8. *For the robust optimal control problem* (7.16)–(7.19)*, each primal-dual interior point iteration requires no more than $\mathcal{O}((m{+}n)^3N^3)$ operations.*

Proof. The linear system (7.35) can be factored and solved using a Schur complement technique, so that

$$\begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{lN} \end{bmatrix} = \begin{bmatrix} I & & & & \\ -\mathcal{B}_1^{-1}\mathcal{J}_1^\top & I & & & \\ -\mathcal{B}_2^{-1}\mathcal{J}_2^\top & & I & & \\ \vdots & & & \ddots & \\ -\mathcal{B}_{lN}^{-1}\mathcal{J}_{lN}^\top & & & & I \end{bmatrix} \begin{bmatrix} \Delta^{-1} & & & & \\ & \mathcal{B}_1^{-1} & & & \\ & & \mathcal{B}_2^{-1} & & \\ & & & \ddots & \\ & & & & \mathcal{B}_{lN}^{-1} \end{bmatrix} \begin{bmatrix} I & -\mathcal{J}_1\mathcal{B}_1^{-1} & -\mathcal{J}_1\mathcal{B}_2^{-1} & \dots & -\mathcal{J}_{lN}\mathcal{B}_{lN}^{-1} \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix} \begin{bmatrix} \mathbf{b}_A \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{lN} \end{bmatrix},$$

with

$$\Delta := \mathcal{A} - \sum_{p=1}^{lN} \mathcal{J}_p \mathcal{B}_p^{-1} \mathcal{J}_p^\top.$$

where, by virtue of Lemma 7.5, the matrix Δ is always full rank [HJ85, Thm. 0.8.5]. The

$\mathcal{O}((m{+}n)^3N^3)$ complexity bound can then be attained by solving (7.35) using the following procedure:

$$\begin{array}{llll}
 & \textbf{Operation} & & \textbf{Complexity} \\
\textit{solve:} & \tilde{\mathbf{x}}_p = \mathcal{B}_p^{-1}\mathbf{b}_p & \forall p \in \mathbb{Z}_{[1,lN]} & lN \cdot \mathcal{O}((m{+}n)^3N) \quad (7.36a) \\
 & \mathcal{S}_p = \mathcal{J}_p\left(\mathcal{B}_p^{-1}\mathcal{J}_p^\top\right) & \forall p \in \mathbb{Z}_{[1,lN]} & lN \cdot \mathcal{O}((m{+}n)^3N^2) \quad (7.36b) \\
\textit{factor:} & \Delta = \mathcal{A} - \sum_{p=1}^{lN}\mathcal{S}_p & & lN \cdot \mathcal{O}((m{+}n)N) \quad (7.36c) \\
 & \quad = L_\Delta D_\Delta L_\Delta^\top & & \mathcal{O}((m{+}n)^3N^3) \quad (7.36d) \\
\textit{solve:} & \mathbf{z}_A = \mathbf{b}_A - \sum_{p=1}^{lN}(\mathcal{J}_p\tilde{\mathbf{x}}_p), & & lN \cdot \mathcal{O}((m{+}n)N) \quad (7.36e) \\
 & \mathbf{x}_A = (L_\Delta^\top)^{-1}(D_\Delta^{-1}(L_\Delta^{-1}\mathbf{z}_A)), & & \mathcal{O}((m{+}n)^2N^2) \quad (7.36f) \\
 & \mathbf{z}_p = \mathcal{J}_p^\top\mathbf{x}_A, & \forall p \in \mathbb{Z}_{[1,lN]} & lN \cdot \mathcal{O}((m{+}n)N) \quad (7.36g) \\
 & \mathbf{x}_p = \tilde{\mathbf{x}}_p - \mathcal{B}_p^{-1}\mathbf{z}_p. & \forall p \in \mathbb{Z}_{[1,lN]} & lN \cdot \mathcal{O}((m{+}n)^2N). \quad (7.36h)
\end{array}$$

The complexity of the solution to the linear system (7.36a) follows from Lemma 7.6(i). The complexity of the solution to (7.36b) and (7.36h) follows from Lemma 7.6(ii), where each of the matrices $\mathcal{J}_p^\top$ in (7.36b) have $\mathcal{O}((m+n)N)$ nonzero columns. □

Remark 7.9. *In the solution procedure* (7.36)*, it is important to note that since the coupling matrices $\mathcal{J}_i$ have no more than a single* 1 *on every row and column, matrix products involving left or right multiplication by $\mathcal{J}_i$ or $\mathcal{J}_i^\top$ do not require* any *floating point operations to calculate. The reader is referred to [BV04, App. C] for a more complete treatment of complexity analysis for matrix operations.*

Remark 7.10. *If the solution procedure* (7.36) *is employed, then the robust optimization problem is an obvious candidate for parallel implementation*[2]*. However, it is generally not necessary to hand implement the suggested variable interleaving and block factorization procedure to realize the suggested block-bordered structure in* (7.35) *and $\mathcal{O}((m+n)^3N^3)$ solution time, as any reasonably efficient sparse factorization code can be expected to perform similar steps automatically; see [DER86]. Note that the "arrowhead" structure in* (7.35) *should be reversed (i.e. pointing down and to the right) in order for direct $LDL^\top$ factorization to produce sparse factors.*

[2] Note that if a parallel scheme is employed to factor the lN matrices $\mathcal{B}_p$, the overall complexity of $\mathcal{O}(N^3)$ is not reduced – it would still require $\mathcal{O}(N^3)$ operations to factor the dense matrix Δ.

Remark 7.11. *Recalling the discussion of soft constraint handling in Section 6.3.1 and Remark 7.2, it is easy to show that the inclusion of soft constraints does not qualitatively alter the complexity results of Theorem 7.8, since the inclusion of such constraints amounts only to a modification of the matrix $\mathcal{A}$ (and thus of the dense matrix Δ) in* (7.36c)*, and does* not *effect the complexity of any of the operations involving the banded matrices $\mathcal{B}_i$.*

7.4 Numerical Results

Two sparse QP solvers were used to evaluate the proposed formulation. The first, OOQP [GW03], uses a primal-dual interior-point approach configured with the sparse factorization code MA27 from the HSL library [HSL02] and the OOQP version of the multiple-corrector interior-point method of Gondzio [Gon96].

The second sparse solver used was the QP interface to the PATH [DF95] solver. This code solves mixed complementarity problems using an active-set method, and hence can be applied to the stationary conditions of any quadratic program. Note that since we are dealing with convex QPs, each optimization problem and its associated complementarity system have equivalent solution sets.

All results reported in this section were generated on a single processor machine with a 3 GHz Pentium 4 processor and 2GB of RAM. We restrict our attention to sparse solvers as the amount of memory required in the size of the problems considered is prohibitively large for dense factorization methods.

A set of test cases was generated to compare the performance of the two sparse solvers using the $(\mathbf{M}, \mathbf{v})$ formulation in (7.2) and the decomposition-based method of Section 7.2. Each test case is defined by its number of states n and horizon length N. The remaining problem parameters were chosen using the following rules:

- There are twice as many states as inputs.
- The constraint sets W, Z and X_f represent randomly selected symmetric bounds on the states and inputs subjected to a random similarity transformation.
- The state space matrices A and B are randomly generated, with (A, B) controllable, and with A potentially unstable.
- The dimension l of the generating disturbance is chosen as half the number of states, with randomly generated G of full column rank.
- All test cases have feasible solutions. The initial state is selected such that at least some of the inequality constraints in (7.17b) are active at the optimal solution.

The average computational times required by each of the two solvers for the two problem formulations for a range of problem sizes are shown in Table 7.1. Each entry represents the average of ten test cases, unless otherwise noted. It is clear from these results that, as expected, the decomposition-based formulation can be solved much more efficiently than the original $(\mathbf{M}, \mathbf{v})$ formulation for robust optimal control problems of nontrivial size, and that the difference in solution times increases dramatically with increased problem dimension. Additionally, the decomposition formulation seems particularly well suited to the interior-point solver (OOQP), rather than the active set method (PATH). Nevertheless we expect the performance of active set methods to improve relative to interior-point methods when solving a sequence of similar QPs that would occur in a receding horizon control scheme, where a good estimate of the optimal active set is typically available at the start of computation. That is, interior-point methods are particularly effective in "cold start" situations, while the efficiency of active set methods is likely to improve given a "warm start".

Table 7.1: Average Solution Times (sec)

	$(\mathbf{M}, \mathbf{v})$		Decomposition	
Problem Size	OOQP	PATH	OOQP	PATH
2 states, 4 stages	0.004	0.004	0.005	0.005
2 states, 8 stages	0.020	0.010	0.016	0.019
2 states, 12 stages	0.061	0.027	0.037	0.052
2 states, 16 stages	0.172	0.091	0.072	0.198
2 states, 20 stages	0.432	0.123	0.132	1.431
4 states, 4 stages	0.024	0.026	0.018	0.024
4 states, 8 stages	0.220	0.316	0.099	0.357
4 states, 12 stages	0.969	1.162	0.264	2.019
4 states, 16 stages	3.755	17.50	0.576	16.63
4 states, 20 stages	11.67	41.45	1.047	22.26
8 states, 4 stages	0.667	1.282	0.136	0.261
8 states, 8 stages	7.882	81.50	0.858	14.89
8 states, 12 stages	46.97	257.9†	2.81	183.8†
8 states, 16 stages	189.75	2660†	6.781	288.9†
8 states, 20 stages	620.3	x	13.30	x
12 states, 4 stages	6.292	75.608	0.512	5.044
12 states, 8 stages	132.1	1160†	4.671	388.9†
12 states, 12 stages	907.4	x	14.08	x
12 states, 16 stages	x	x	37.99	x
12 states, 20 stages	x	x	82.06	x

x – Solver failed all test cases

† – Based on limited data set due to failures

Figure 7.1 shows that the interior-point solution time increases cubicly with horizon length for randomly generated problems with 2, 4, 8 and 12 states. The performance closely matches the predicted behavior described in Section 7.2. For the particular problems shown, the number of iterations required for the OOQP algorithm to converge varied from 12 to 20 over the range of horizon lengths and state dimensions considered. The total number of interior-point iterations required to solve each problem as a function of horizon length is shown in Figure 7.2

7.5 Conclusions

In this chapter we have derived a highly efficient computational method for calculation of affine state feedback policies for robust control of constrained systems with bounded disturbances. This is done by exploiting the structure of the underlying optimization problem and deriving an equivalent problem with considerable structure and sparsity, resulting in a problem formulation that is particularly suited to an interior-point solution method. As a result, the robustly stabilizing receding horizon control law proposed in Chapter 4 is practically realizable, even for systems of significant size or with long horizon lengths.

In Section 7.3 we proved that, when applying an interior-point solution technique to our robust optimal control problem, each iteration of the method can be solved using a number of operations proportional to the cube of the control horizon length. We appeal to the Riccati based factorization technique in [Ste95, RWR98] to support this claim. However, it should be stressed that the results in Section 7.4, which demonstrate this cubic-time behavior numerically, are based on freely available optimization and linear algebra packages and *do not* rely on any special factorization methods.

A number of open research issues remain. It may be possible to further exploit the structure of the control problem discussed in this chapter by developing specialized factorization algorithms for the factorization of each interior-point step, e.g. through the parallel block factorization procedure alluded to in Remark 7.10.

It is also possible to extend, in a fairly obvious way, the results of this chapter to certain of the other $\mathcal{H}_2$ control problems for various disturbance classes proposed in Chapter 6. Of particular interest, however, would be an extension of the results presented here allowing the efficient solution of the various semidefinite programming problems arising in Chapter 6 in relation to constrained $\mathcal{H}_\infty$ control problems.

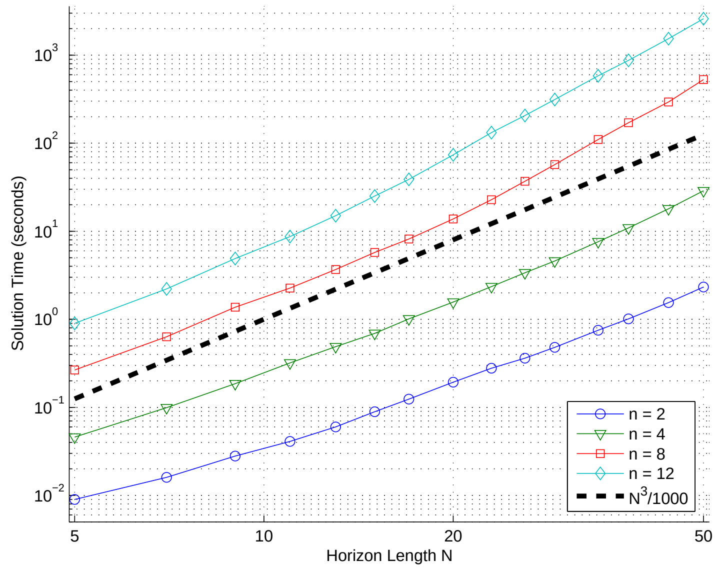

Figure 7.1: Computation time vs. horizon length for systems of increasing state dimension, using the decomposition method and OOQP solver. Also shown is the constant line $N^3/1000$ for comparison.

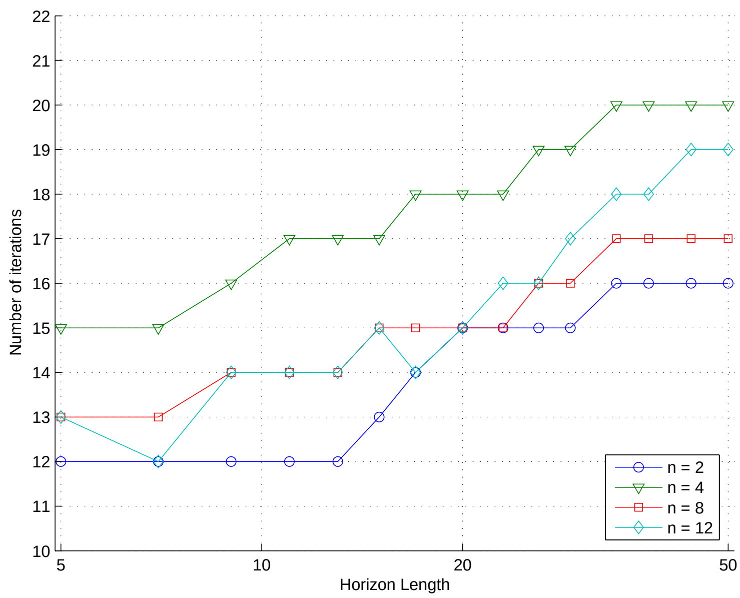

Figure 7.2: Iterations vs. horizon length for systems of increasing state dimension, using decomposition method and OOQP solver.

7.A Proofs

7.A.1 Rank of the Robust Control Problem Jacobian (Proof of Lemma 7.5)

We demonstrate that the Jacobian matrix defined in (7.32) is always full rank. Recalling the discussion in Section 7.3.1, for *any* quadratic program the Jacobian matrix is full rank if the only solution to the system

$$\begin{bmatrix} \mathsf{Q} & \mathsf{A}^\top & \mathsf{C}^\top \\ \mathsf{A} & 0 & 0 \\ \mathsf{C} & 0 & -\Sigma \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\pi \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.37}$$

satisfies $\Delta\theta = 0$, $\Delta\pi = 0$, and $\Delta\lambda = 0$, where $\Sigma := \Lambda^{-1}Z \succ 0$, $\mathsf{Q} \succeq 0$ and the coefficient matrices A and C come from the equality and inequality constraints of the QP respectively (cf. (7.27)). From the first two rows of this system,

$$\Delta\theta^\top\mathsf{Q}\Delta\theta + (\Delta\theta^\top\mathsf{A}^\top)\Delta\pi + \Delta\theta^\top\mathsf{C}^\top\Delta\lambda = \Delta\theta^\top\mathsf{Q}\Delta\theta + \Delta\theta^\top\mathsf{C}^\top\Delta\lambda = 0. \tag{7.38}$$

Incorporating the final block row, $\mathsf{C}\Delta\theta = \Sigma\Delta\lambda$, we have

$$\Delta\theta^\top\mathsf{Q}\Delta\theta + \Delta\lambda^\top\Sigma\Delta\lambda = 0. \tag{7.39}$$

Since $\mathsf{Q} \succeq 0$ for a convex QP and $\Sigma \succ 0$ for a strictly interior point, we conclude that $\Delta\lambda = 0$. We next make use of the following matrix condition, which is easily verified:

Fact 7.12. *The matrix* $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right]$ *is full column rank for any* Y *if both* X *and* Z *are full column rank.*

Since $\Delta\lambda = 0$ always holds, sufficient conditions to guarantee $\Delta\theta = 0$ and $\Delta\pi = 0$ in (7.37) are that:

(i) A is full row rank.

(ii) $\left[\begin{smallmatrix} \mathsf{A} \\ \mathsf{C} \end{smallmatrix}\right]$ is full column rank.

For the quadratic program defined by the robust control problem (7.17)–(7.19), the equality and inequality constraints are defined as in (7.22). For this convex QP, it is straightforward to show that the above rank conditions on A and C are equivalent to requiring that:

(i) Each of the matrices $\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_{lN}$ is full row rank.

(ii) Each of the matrices $\begin{bmatrix} \mathsf{A}_0 \\ \mathsf{C}_0 \end{bmatrix}, \begin{bmatrix} \mathsf{A}_1 \\ \mathsf{C}_1 \end{bmatrix}, \dots, \begin{bmatrix} \mathsf{A}_{lN} \\ \mathsf{C}_{lN} \end{bmatrix}$ is full column rank.

The condition (ii) is derived by noting that, for the particular problem (7.17)–(7.19), the general rank condition on $\begin{bmatrix} \mathsf{A} \\ \mathsf{C} \end{bmatrix}$ is equivalent to requiring that the matrix

$$\begin{bmatrix} \mathsf{C}_0 & \mathsf{J}_1 & \mathsf{J}_2 & \dots & \mathsf{J}_{lN} \\ \mathsf{A}_0 & & & & \\ & \mathsf{C}_1 & & & \\ & \mathsf{A}_1 & & & \\ & & \mathsf{C}_2 & & \\ & & \mathsf{A}_2 & & \\ & & & \ddots & \\ & & & & \mathsf{C}_{lN} \\ & & & & \mathsf{A}_{lN} \end{bmatrix}$$

is full column rank, which reduces to (ii) upon repeated application of Fact 7.12 above to eliminate the coupling terms J_p. If **A7.1**(iii) holds so that D_c is full column rank, then both of these rank conditions are easily verified by examination of the definitions in (7.23) and (7.25). The Jacobian matrix for the QP defined in (7.17)–(7.19) is thus full rank, and it remains full rank if its rows and columns are reordered as in (7.32).

7.A.2 Solution of $\mathcal{B}_p\mathbf{x}_p = \mathbf{b}_p$ via Riccati Recursion (Proof of Lemma 7.6)

We demonstrate that the system of equations $\mathcal{B}_p\mathbf{x}_p = \mathbf{b}_p$ has a unique solution for every $\mathbf{b}_p$, where $\mathcal{B}_p$, $\mathbf{x}_p$ and $\mathbf{b}_p$ are defined as

$$\mathcal{B}_p := \begin{bmatrix} 0 & 0 & \bar{D}^\top & B^\top & & & & & & & & & \\ 0 & 0 & H^\top & 0 & & & & & & & & & \\ \bar{D} & H & -\Sigma^p_{k+1} & 0 & & & & & & & & & \\ B & 0 & 0 & 0 & -I & & & & & & & & \\ & & & -I & 0 & 0 & 0 & \bar{C}^\top & A^\top & & & & \\ & & & & 0 & 0 & 0 & \bar{D}^\top & B^\top & & & & \\ & & & & 0 & 0 & 0 & H^\top & 0 & & & & \\ & & & & \bar{C} & \bar{D} & H & -\Sigma^p_{k+2} & 0 & \ddots & & & \\ & & & & A & B & 0 & 0 & & \ddots & & & \\ & & & & & & & & & \ddots & 0 & 0 & \bar{Y}^\top \\ & & & & & & & & & & 0 & 0 & H_f^\top \\ & & & & & & & & & & \bar{Y} & H_f & -\Sigma^p_N \end{bmatrix}, \tag{7.40a}$$

$$\begin{aligned} \mathbf{x}_p &:= \operatorname{vec}(\Delta u^p_{k+1}, \Delta c^p_{k+1}, \Delta\lambda^p_{k+1}, \Delta\pi^p_{k+1}, \Delta x^p_{k+2}, \Delta u^p_{k+2}, \Delta c^p_{k+2}, \Delta\lambda^p_{k+2}, \dots, \Delta x^p_N, \Delta c^p_N, \Delta\lambda^p_N), \\ \mathbf{b}_p &:= \operatorname{vec}(\ r^{u^p_{k+1}}\ ,\ r^{c^p_{k+1}}\ ,\ r^{\lambda^p_{k+1}}\ ,\ r^{\pi^p_{k+1}}\ ,\ r^{x^p_{k+2}}\ ,\ r^{u^p_{k+2}}\ ,\ r^{c^p_{k+2}}\ ,\ r^{\lambda^p_{k+2}}\ , \dots,\ r^{x^p_N}\ ,\ r^{c^p_N}\ ,\ r^{\lambda^p_N}) \end{aligned}$$

and $k = \lfloor \frac{p-1}{l} \rfloor$, and that this solution is obtainable in $\mathcal{O}((m+n)^3 N)$ operations. We first perform a single step of block elimination on the variables $\Delta\lambda_i^p$ and Δc_{k+1}^p, so that the resulting linear system is solvable via specialized methods based on Riccati recursion techniques [Ste95, RWR98] (see also related results in [DB89] for the unconstrained case).

It is straightforward to eliminate the terms $\Delta\lambda_i^p$ and Δc_i^p from each of the subproblems, yielding a linear system $\tilde{\mathcal{B}}_p \tilde{\mathbf{x}}_p = \tilde{\mathbf{b}}_p$. The coefficient matrix $\tilde{\mathcal{B}}_p$ is:

$$\tilde{\mathcal{B}}_p := \begin{bmatrix} R_{k+1}^p & B^\top & & & & & & & \\ B & 0 & -I & & & & & & \\ & -I & Q_{k+2}^p & M_{k+2}^p & A^\top & & & & \\ & & (M_{k+2}^p)^\top & R_{k+2}^p & B^\top & & & & \\ & & A & B & 0 & -I & & & \\ & & & & -I & Q_{k+3}^p & M_{k+3}^p & A^\top & \\ & & & & & (M_{k+3}^p)^\top & R_{k+3}^p & B^\top & \ddots \\ & & & & & A & B & \ddots & \ddots \\ & & & & & & & \ddots & Q_N^p \end{bmatrix} \tag{7.41}$$

where, for stages $i \in \mathbb{Z}_{[k+1,N-1]}$:

$$\Phi_i^p := H^\top (\Sigma_i^p)^{-1} H \tag{7.42a}$$
$$\Theta_i^p := (\Sigma_i^p)^{-1} - (\Sigma_i^p)^{-1} H (\Phi_i^p)^{-1} H^\top (\Sigma_i^p)^{-1} \tag{7.42b}$$
$$Q_i^p := \bar{C}^\top \Theta_i^p \bar{C} \tag{7.42c}$$
$$R_i^p := \bar{D}^\top \Theta_i^p \bar{D} \tag{7.42d}$$
$$M_i^p := \bar{C}^\top \Theta_i^p \bar{D}, \tag{7.42e}$$

and for stage N:

$$\Phi_N^p := H_f^\top (\Sigma_N^p)^{-1} H_f \tag{7.42f}$$
$$\Theta_N^p := (\Sigma_N^p)^{-1} - (\Sigma_N^p)^{-1} H_f (\Phi_N^p)^{-1} H_f^\top (\Sigma_N^p)^{-1} \tag{7.42g}$$
$$Q_N^p := \bar{Y}^\top \Theta_N^p \bar{Y}. \tag{7.42h}$$

The vectors $\tilde{\mathbf{x}}_p$ and $\tilde{\mathbf{b}}_p$ are defined as:

$$\tilde{\mathbf{x}}_p := \mathrm{vec}(\Delta u_{k+1}^p, \Delta\pi_{k+1}^p, \Delta x_{k+2}^p, \Delta u_{k+2}^p, \Delta\pi_{k+2}^p, \ldots, \Delta x_N^p) \tag{7.43}$$
$$\tilde{\mathbf{b}}_p := \mathrm{vec}(\ \tilde{r}^{u_{k+1}^p}\ ,\ r^{\pi_{k+1}^p}\ ,\ \tilde{r}^{x_{k+2}^p}\ ,\ \tilde{r}^{u_{k+2}^p}\ ,\ r^{\pi_{k+2}^p}\ , \ldots,\ \tilde{r}^{x_N^p})\ , \tag{7.44}$$

where, for stages $i \in \mathbb{Z}_{[k+1,N-1]}$:

$$\tilde{r}^{x_i^p} := r^{x_i^p} + \bar{C}\left(\Theta_i^p r^{\lambda_i^p} - (\Sigma_i^p)^{-1} H (\Phi_i^p)^{-1} r^{c_i^p}\right) \tag{7.45a}$$

$$\tilde{r}^{u_i^p} := r^{u_i^p} + \bar{D}\left(\Theta_i^p r^{\lambda_i^p} - (\Sigma_i^p)^{-1} H (\Phi_i^p)^{-1} r^{c_i^p}\right), \tag{7.45b}$$

and, for stage N:

$$\tilde{r}^{x_N^p} := r^{x_N^p} + \bar{Y}\left(\Theta_N^p r^{\lambda_N^p} - (\Sigma_N^p)^{-1} H_f (\Phi_N^p)^{-1} r^{c_N^p}\right). \tag{7.45c}$$

Remark 7.13. *The matrix $\tilde{\mathcal{B}}_p$ is equivalent to the KKT matrix for the unconstrained control problem:*

$$\min_{\substack{u_{k+1},\dots,u_{N-1},\\ x_{k+1},\dots,x_N}} \left(\frac{1}{2} x_N^\top Q_N^p x_N + \sum_{i=(k+1)}^{N-1} \frac{1}{2}(x_i^\top Q_i^p x_i + u_i^\top R_i^p u_i + 2 x_i M_i^p u_i)\right) \tag{7.46}$$

subject to:

$$x_k = G_{(j)}, \tag{7.47a}$$

$$x_{i+1} = A x_i + B u_i, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]}. \tag{7.47b}$$

Lemma 7.14. *Each of the matrices R_i^p, Q_i^p and Q_N^p are positive semidefinite. If **A7.1** holds, then R_i^p is positive definite.*

Proof. Recall that the matrix Θ_i^p is defined as

$$\Theta_i^p = (\Sigma_i^p)^{-1} - (\Sigma_i^p)^{-1} H \left(H'(\Sigma_i^p)^{-1} H\right)^{-1} H^\top (\Sigma_i^p)^{-1} \tag{7.48}$$

Partition the diagonal and positive definite matrix Σ_i^p into $\Sigma_i^p = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$. Recalling that $H := -\begin{bmatrix} I \\ I \end{bmatrix}$, Θ_i^p can be written as

$$\Theta_i^p = \begin{bmatrix} \Sigma_1^{-1} - \Sigma_1^{-1}(\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}\Sigma_1^{-1} & -\Sigma_1^{-1}(\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}\Sigma_2^{-1} \\ -\Sigma_2^{-1}(\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}\Sigma_1^{-1} & \Sigma_1^{-1} - \Sigma_2^{-1}(\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}\Sigma_2^{-1} \end{bmatrix} \tag{7.49}$$

$$= \begin{bmatrix} I \\ -I \end{bmatrix} (\Sigma_1 + \Sigma_2)^{-1} \begin{bmatrix} I & -I \end{bmatrix} \tag{7.50}$$

which is easily verified using standard matrix identities and the fact that the matrices Σ_1 and Σ_2 are diagonal.

Recalling that $\bar{D} := \left[\begin{smallmatrix} D_c \\ -D_c \end{smallmatrix}\right]$, it follows that R_i^p is positive semidefinite since it can be written as

$$R_i^p = \bar{D}^\top \begin{bmatrix} I \\ -I \end{bmatrix} (\Sigma_1 + \Sigma_2)^{-1} \begin{bmatrix} I & -I \end{bmatrix} \bar{D} \tag{7.51}$$

$$= 4D^\top (\Sigma_1 + \Sigma_2)^{-1} D \succeq 0. \tag{7.52}$$

If **A7.1**(iii) holds so that D_c is full column rank, then R_i^p is positive definite. A similar argument establishes the result for Q_i^p and Q_N^p. $\square$

We are now in a position to prove Lemma 7.6. Since R_i^p is positive definite and Q_i^p and Q_N^p are positive semidefinite, the linear system $\tilde{\mathcal{B}}_p \tilde{\mathbf{x}}_p = \tilde{\mathbf{b}}_p$ (and consequently the original system $\mathcal{B}_p \mathbf{x}_p = \mathbf{b}_p$) has a unique solution that can found in $\mathcal{O}((m+n)^3(N-k+1))$ operations using the Riccati recursion procedure described in [Ste95, RWR98]. Once such a solution has been obtained, a solution for each additional right hand side requires $\mathcal{O}\left((m+n)^2(N-k+1)\right)$ operations [RWR98, Sec. 3.4]. We note that in [RWR98] the Riccati factorization procedure is shown to be numerically stable, and that similar arguments can be used to show that factorization of (7.41) is also stable.

Chapter 8. Constrained Output Feedback

In this chapter we extend the results for the state feedback problem considered in the majority of this dissertation to the output feedback case. The results presented here broadly parallel the results of Chapter 3; a class of feedback policies will be introduced that guarantees constraint satisfaction for all time for a linear system subject to convex constraints and bounded disturbances and output measurement errors, and that can be computed via the solution of a convex optimization problem.

8.1 Problem Definition

Throughout, we consider the following discrete-time linear time-invariant system:

$$x^+ = Ax + Bu + w \tag{8.1}$$

$$y = Cx + \eta \tag{8.2}$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the system input, $w \in \mathbb{R}^n$ is a disturbance, $y \in \mathbb{R}^r$ is the system output and $\eta \in \mathbb{R}^r$ is a measurement error. We will assume that, at each time step, a measurement of the output y is available, but a measurement of the state x is not.

We assume that the disturbances w are unknown but contained within a compact set W containing the origin, and that the measurement errors η are unknown but contained in a compact set $H \subset \mathbb{R}^r$, also containing the origin. We will also assume that an initial state estimate $s \in \mathbb{R}^n$ is provided along with a compact estimation error set $\mathcal{E}$, such that the state x is initially known to satisfy $x \in \{s\} \oplus \mathcal{E}$.

As in Chapter 3, the system is subject to mixed constraints on the states and inputs, so that a design goal is to guarantee that $(x, u) \in Z \in \mathbb{R}^n \times \mathbb{R}^m$ for all time; note that such a constraint may include constraints on the output y. For finite horizon problems, one may additionally specify some closed and convex target set X_T which the state x must reach after a fixed number of time steps.

We will only consider the problem of finding a *feasible* (i.e. constraint admissible) output feedback control policy for the system (8.1)–(8.2), without regard to optimality. We make use of the following assumptions about the system (8.1)–(8.2) throughout this chapter:

A8.1 (Standing Assumptions)

i. The pairs (A, B) and (C, A) are stabilizable and detectable respectively.

ii. The state and input constraint set $Z \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is closed, convex, contains the origin in its interior and is bounded in the inputs, i.e. there exists a bounded set B such that $Z \subseteq \mathbb{R}^n \times B$.

8.2 Control Policy and Observer Structure

As in the full state information case considered in Chapter 3, finding an *arbitrary* finite horizon control policy that satisfies the constraints of the system (8.1)–(8.2) for all possible uncertainty realizations is extremely difficult in general. As a result, we will restrict the class of control policies considered to those in which each control input u_i is affine in the measurements $\{y_0, \ldots, y_{i-1}\}$, i.e.

$$u_i = g_i + \sum_{j=0}^{i-1} K_{i,j} y_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{8.3}$$

and consider the problem of finding a policy in this form that is guaranteed to satisfy the system constraints for all possible uncertainty realizations. Note that such a control policy is the logical counterpart, in the output feedback case, to the class of state feedback policies introduced in Section 3.2. For infinite horizon problems, we will consider the construction of constraint admissible control laws based on receding horizon implementations of policies in the form (8.3).

8.2.1 Observers and Terminal Sets

For infinite horizon problems, we will find it helpful to introduce a linear control law and associated robust positively invariant set to serve as a target/terminal constraint in a manner similar to that used in Chapter 3 (cf. the assumptions of **A3.2**). The obvious extension of **A3.2** when implementing receding horizon controllers based on output feedback policies in the form (8.3) is to define a feedback matrix K_f such that $(A + BK_f C)$ is Hurwitz, and to find an associated robust positively invariant for the system (8.1)–(8.2) in closed loop with the control law $u = K_f y$. However, the problem of finding such a K_f is generally believed to be NP hard [BT97].

As an alternative, we specify a Luenberger type observer gain L such that

$$s^+ = As + Bu + L(y - Cs) \tag{8.4}$$

and define the state estimation error $e \in \mathbb{R}^n$ as $e := x - s$, such that

$$e^+ = (A - LC)e - L\eta + w, \tag{8.5}$$

where s^+ and e^+ represent a state estimate and estimation error at the next time instant. In conjunction with this observer, we define a closed and convex target/terminal constraint set X_f for the joint state estimate and estimation error, so that a design goal (for finite horizon problems) will be to drive the system (8.4)–(8.5) to satisfy $(s, e) \in X_f$ at the end of the planning horizon[1]. For receding horizon implementations, we will define X_f such that it is robust positively invariant for the system (8.4)–(8.5) in closed loop with a stabilizing control law of the form $u = K_f s$.

Note that if one only wishes to ensure that $x \in X_T$ for some target set X_T at the end of a finite planning horizon, then one can define

$$X_f := \{(s, e) \mid s + e \in X_T\}. \tag{8.6}$$

In such a case no observer is required, and nothing is gained or lost by specifying one. However, in order to maintain consistency of notation throughout, we will use a target set X_f as defined above with the understanding that it can be defined as in (8.6) if required.

Note that if it is known initially that $x \in \{s\} \oplus \mathcal{E}$, then the state at the next time instant will be known to satisfy

$$x^+ \in \{s^+\} \oplus (A - LC)\mathcal{E} \oplus W \oplus (-L)H \tag{8.7}$$

[1]This definition of the terminal constraint differs from that used in previous chapters, where X_f was defined as a constraint on the true state x, rather than on the joint state estimate and estimation error pair (s, e).

if an observer of the form (8.4)–(8.5) is used. The main technical issue to be addressed in this chapter is the problem of synthesizing a receding horizon control law from policies in the form (8.3) that is both time-invariant *and* that can be calculated by solving a finite-dimensional optimization problem. The difficulty of computing such a control law arises mainly from the fact that given an initial estimation error set $\mathcal{E}$ of fixed complexity (e.g. one characterized by a finite number of linear inequalities), the complexity of future state estimation error sets generally increases without bound if one employs a linear state observer. As will be shown, this problem can be circumvented via the use of an outer approximation to the initial error set which is robust positively invariant in closed loop with the state estimator.

Remark 8.1. *Note that unlike in previous chapters, the disturbance input to the system* (8.1) *is* not *multiplied by a matrix G, and the disturbances are drawn from a set W containing the origin in its interior (note that the interior of the set GW might be empty if G is not full row rank, even when W has nonempty interior). The reason for this distinction is twofold; first, the results to follow will require the construction of an invariant outer approximation to the minimal robust positively invariant set for the error dynamics* (8.5) *using the results of [RKKM04, RKKM05]. The methods for constructing such a set require that $W \oplus (-L)H$ has nonempty interior. Second, the method to be proposed for defining an invariant control law for the system* (8.1) *is based on a modification of W such that the disturbance inputs to the system* (8.1) *are drawn from a slightly larger set with nonempty interior; this method would be incompatible with any G that is not invertible.*

8.2.2 Alternative Observer Schemes

In this chapter we propose a method for calculating robust finite horizon control policies for the system (8.1)–(8.2), where the control input u_k at each time k is determined as an affine function of the measurements $\{y_0, \ldots, y_{k-1}\}$. The decision not to allow the control input u_k to also depend on the measurement y_k will be consistent with the choice of the Luenberger form observer in (8.4) when developing the invariance results of Section 8.5.

As an alternative, one could employ a predictor-corrector type observer of the form

$$s_{k+1} = (I - LC)(As_k + Bu_k) + Ly_{k+1} \tag{8.8a}$$

$$e_{k+1} = (I - LC)(Ae_k + w_k) - L\eta_{k+1}, \tag{8.8b}$$

when constructing an invariant terminal set. Such an observer would be consistent with a control policy where each input u_k is modelled as an affine function of the measurements $\{y_0, \ldots, y_k\}$. Such an alternative formulation is not qualitatively different from the method used here, and observers in both forms are in widespread use. Note that if one were to employ an observer in the form (8.8) with $H = \{0\}$, $C = I$ and $L = I$, and were then to define an affine control policy in $\{y_0, \ldots, y_k\}$, then one would recover exactly the affine state feedback control problem of Section 3.2.

We elect to use an observer of the form (8.4) primarily for consistency and convenience of comparison with related work within the predictive control literature on output feedback [LK01, RH05, MRFA06]. Additionally, the proposed parameterization is slightly more relaxed, since it does not require the measurement y_k, which may not always be available, to calculate u_k. It should be stressed that use of the alternative state estimate and error dynamics (8.8) would not by itself alleviate the main technical difficulties to be addressed in Section 8.5.

Finally, this chapter will not directly address the issue of computational delay since this issue can be adequately managed using available methods in the literature. For example, [Mac02, Sec. 2.5] gives a procedure for transforming a system with computational delay into the standard form considered in (8.1)–(8.4).

8.2.3 Notation

We will use of much of the notation introduced in Section 3.1.1, and further define stacked versions of the state estimate, estimation error, output, and measurement error vectors $\mathbf{s} \in \mathbb{R}^{n(N+1)}$, $\mathbf{e} \in \mathbb{R}^{n(N+1)}$, $\mathbf{y} \in \mathbb{R}^{rN}$ and $\boldsymbol{\eta} \in \mathbb{R}^{rN}$ respectively, as

$$\mathbf{s} := \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_N \end{bmatrix}, \ \mathbf{e} := \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{N-1} \end{bmatrix}, \ \mathbf{y} := \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}, \ \text{and } \boldsymbol{\eta} := \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}, \tag{8.9}$$

where $s_0 := s$ and $e_0 := e$ denote the current values of the state estimate and estimation error respectively, and $s_{i+1} := (A - LC)s_i + Bu_i + Ly_i$ and $e_{i+1} := (A - LC)e_i - L\eta_i + w_i$ for all $i \in \mathbb{Z}_{[0,N-1]}$. The predicted measurements after i time instants are $y_i = C(s_i + e_i) + \eta_i$ for all $i \in \mathbb{Z}_{[0,N-1]}$. The actual values of the state, state estimate, estimation error, input and output at time instant k are denoted $x(k)$, $s(k)$, $e(k)$, $u(k)$ and $y(k)$, respectively.

We define a closed and convex set $\mathcal{Z}$, appropriately constructed from Z and X_f, such that the constraints to be satisfied are equivalent to $(\mathbf{s}, \mathbf{e}, \mathbf{u}) \in \mathcal{Z}$, i.e.

$$\mathcal{Z} := \left\{ (\mathbf{s}, \mathbf{e}, \mathbf{u}) \;\middle|\; \begin{array}{c} (s_i + e_i, u_i) \in Z,\ \forall i \in \mathbb{Z}_{[0,N-1]} \\ (s_N, e_N) \in X_f \end{array} \right\}. \tag{8.10}$$

Note that, as with the set X_f, this definition of $\mathcal{Z}$ differs from that used in previous chapters since it is defined in terms of the joint state estimate and estimation error sequences $\mathbf{s}$ and $\mathbf{e}$, rather than the sequence of true states $\mathbf{x}$.

Define $A_L := (A - LC)$ and matrices $\Phi \in \mathbb{R}^{n(N+1)\times n}$, $\Gamma \in \mathbb{R}^{n(N+1)\times nN}$, $\mathcal{L} \in \mathbb{R}^{nN\times rN}$ and $\mathbf{C} \in \mathbb{R}^{nN\times rN}$ as

$$\Phi := \begin{bmatrix} I_n \\ A_L \\ A_L^2 \\ \vdots \\ A_L^N \end{bmatrix}, \quad \Gamma := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ A_L & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_L^{N-1} & A_L^{N-2} & \cdots & I_n \end{bmatrix},$$

$\mathcal{L} := I_N \otimes L$, and $\mathbf{C} := [(I_N \otimes C)\ \ 0]$ respectively. Finally, define affine functions f_e and f_s such that the vectors $\mathbf{s}$ and $\mathbf{e}$ can be written as

$$\mathbf{s} = f_s(s, e, \mathbf{u}, \mathbf{w}, \boldsymbol{\eta}) := \mathbf{A}s + \mathbf{B}\mathbf{u} + \mathbf{E}\mathcal{L}(\mathbf{C}\mathbf{e} + \boldsymbol{\eta}) \tag{8.11}$$

$$\mathbf{e} = f_e(e, \mathbf{w}, \boldsymbol{\eta}) := \Phi e - \Gamma\mathcal{L}\boldsymbol{\eta} + \Gamma\mathbf{w}. \tag{8.12}$$

Note that using these definitions, the state estimates $\mathbf{s}$ can alternatively be expressed directly as an affine function of $\mathbf{y}$, i.e.

$$\mathbf{s} = \Phi s + \Gamma\mathcal{B}\mathbf{u} + \Gamma\mathcal{L}\mathbf{y}. \tag{8.13}$$

We define E to be the set of all compact subsets of $\mathbb{R}^n$, and W to be the set of all compact subsets of $\mathbb{R}^n$ containing the origin in their interior.

The reason for defining the sets E and W in this way is to allow for some flexibility in subsequent control policy definitions, where the set of feasible policies will be defined in terms of the disturbances and initial errors to which they are robust. We will typically specify that the estimation error e and true initial state x are such that $e \in \mathcal{E}$ and $x \in \{s\} \oplus \mathcal{E}$, for some $\mathcal{E} \in \mathsf{E}$, and that the disturbances w are unknown but drawn from some known set $W \in \mathsf{W}$.

8.3 Affine Feedback Parameterizations

8.3.1 Output Feedback

Recalling (8.3), we restrict our attention to the class of control policies that model the control inputs u_i as affine functions of the measurements $\{y_0, \dots, y_{i-1}\}$, i.e.

$$u_i = g_i + \sum_{j=0}^{i-1} K_{i,j} y_j, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{8.14}$$

where each $K_{i,j} \in \mathbb{R}^{m\times r}$ and $g_i \in \mathbb{R}^m$. For notational convenience we define the vector $\mathbf{g} \in \mathbb{R}^{mN}$ and matrix $\mathbf{K} \in \mathbb{R}^{mN\times rN}$ as

$$\mathbf{K} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ K_{1,0} & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ K_{N-1,0} & \cdots & K_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{g} := \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}, \tag{8.15}$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{g}$.

For a given initial state estimate s, estimation error set $\mathcal{E} \in \mathsf{E}$ and disturbance set $W \in \mathsf{W}$, the set of feasible output feedback policies that are guaranteed to satisfy the state and input constraints $\mathcal{Z}$ for all possible uncertainty realizations (assuming that the true initial state $x \in \{s\} \oplus \mathcal{E}$) is

$$\Pi_N^{of}(s,\mathcal{E},W) = \bigcap_{\substack{\mathbf{w}\in W^N \\ \boldsymbol{\eta}\in H^N,\, e\in\mathcal{E}}} \left\{ (\mathbf{K},\mathbf{g}) \;\middle|\; \begin{aligned} (\mathbf{K},\mathbf{g}) \text{ satisfies } (8.15) \\ \mathbf{s} = f_s(s,e,\mathbf{u},\mathbf{w},\boldsymbol{\eta}) \\ \mathbf{e} = f_e(e,\mathbf{w},\boldsymbol{\eta}) \\ \mathbf{y} = \mathbf{C}(\mathbf{s}+\mathbf{e}) + \boldsymbol{\eta} \\ \mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{g},\ (\mathbf{s},\mathbf{e},\mathbf{u}) \in \mathcal{Z} \end{aligned} \right\}. \tag{8.16}$$

For a given estimation error set $\mathcal{E} \in \mathsf{E}$ and disturbance set $W \in \mathsf{W}$, define the set of all initial state estimates for which a constraint admissible policy exists as

$$\mathcal{S}_N^{of}(\mathcal{E},W) := \left\{ s \;\middle|\; \Pi_N^{of}(s,\mathcal{E},W) \neq \emptyset \right\}.$$

Remark 8.2. *It is important to recognize that for a finite horizon problem with a terminal state constraint in the form $x_N \in X_T$, it is* not *necessary to specify an observer, and nothing is gained or lost by doing so. For problems of this type, one can rewrite* (8.16) *as*

$$\Pi_N^{of}(s,\mathcal{E},W) = \bigcap_{\substack{\mathbf{w}\in W^N \\ \boldsymbol{\eta}\in H^N,\, e\in\mathcal{E}}} \left\{ (\mathbf{K},\mathbf{g}) \;\middle|\; \begin{aligned} (\mathbf{K},\mathbf{g}) \text{ satisfies } (8.15) \\ \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \boldsymbol{\eta},\ \mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{g} \\ (x_i,u_i)\in Z,\ \forall i\in\mathbb{Z}_{[0,N-1]} \\ x = s+e,\ x_N \in X_T \end{aligned} \right\}, \tag{8.17}$$

which is not related to any choice of observer, nor is the set $\mathcal{S}_N^{of}(\mathcal{E},W)$. The advantage of accounting explicitly for the action of a linear observer in defining the policy set (8.16) *is that it will allow us to construct an invariant receding horizon control law when the constraint set X_f is defined as a robust positively invariant set for the system* (8.4)–(8.5) *in closed loop with a linear feedback control law.*

Remark 8.3. *The feedback policy* (8.14) *includes the class of "pre-stabilizing" control policies in which the control is based on perturbations $\{c_i\}_{i=0}^{N-1}$ to a* fixed *linear state feedback gain K, so that $u_i = c_i + Ks_i$, since the estimated state s_i can be expressed as an affine function of the measurements $\{y_0,\ldots,y_{i-1}\}$ (cf.* (8.13)*). Such a scheme is commonly employed in conjunction with a stabilizing linear observer gain L for output feedback [LK01, RH05, YB05]. The method proposed can also be shown to subsume tube-based schemes such as [MSR05, MRFA06] when the invariant sets defining the tube are based on linear state feedback, though these methods also confer additional stability properties which we do not address here. Finally, note that unlike the certainty-equivalence based method proposed in [BR71], we* do not *combine the state estimation error set $\mathcal{E}$ with the disturbance set W into a single lumped disturbance at each time step, but rather consider the effect of the estimation error $e \in \mathcal{E}$ at the initial time propagated over the planning horizon.*

Remark 8.4. *As in the state feedback case considered in Chapter 3, the set $\Pi_N^{of}(s,\mathcal{E},W)$ is nonconvex, in general, due to the nonlinear relationship between the estimated states* $\mathbf{s}$ *and feedback gains* $\mathbf{K}$ *in* (8.16).

8.3.2 Output Error Feedback

As an alternative to the parameterization (8.14), we consider a control policy parameterized as

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j}(y_j - Cs_j), \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{8.18}$$

$$= v_i + \sum_{j=0}^{i-1} M_{i,j}(Ce_j + \eta_j), \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{8.19}$$

where each $M_{i,j} \in \mathbb{R}^{m\times r}$ and $v_i \in \mathbb{R}^m$. We further define the matrix $\mathbf{M} \in \mathbb{R}^{mN\times rN}$ and vector $\mathbf{v} \in \mathbb{R}^{mN}$ as

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \tag{8.20}$$

so that the control input sequence can be written as

$$\mathbf{u} = \mathbf{M}(\mathbf{y} - \mathbf{C}\mathbf{s}) + \mathbf{v} \tag{8.21}$$

$$= \mathbf{M}(\mathbf{C}\mathbf{e} + \boldsymbol{\eta}) + \mathbf{v}. \tag{8.22}$$

By virtue of the relation (8.12), this control parameterization is *affine* in the unknown parameters e, $\mathbf{w}$ and $\boldsymbol{\eta}$. The control parameterization (8.19) is therefore the logical counterpart, in the output feedback case, to the class of disturbance feedback policies introduced in Section 3.3. For a given initial state estimate s, estimation error set $\mathcal{E} \in \mathsf{E}$ and disturbance set $W \in \mathsf{W}$, the set of feasible feedback policies that are guaranteed to satisfy the system constraints for all possible uncertainty realizations (assuming that the true initial state $x \in \{s\} \oplus \mathcal{E}$) is

$$\Pi_N^{ef}(s,\mathcal{E},W) = \bigcap_{\substack{\mathbf{w}\in W^N \\ \boldsymbol{\eta}\in H^N,\ e\in\mathcal{E}}} \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M},\mathbf{v}) &\text{ satisfies (8.20)} \\ \mathbf{s} &= f_s(s,e,\mathbf{u},\mathbf{w},\boldsymbol{\eta}) \\ \mathbf{e} &= f_e(e,\mathbf{w},\boldsymbol{\eta}) \\ \mathbf{u} &= \mathbf{M}(\mathbf{C}\mathbf{e}+\boldsymbol{\eta}) + \mathbf{v} \\ (\mathbf{s},\mathbf{e},\mathbf{u}) &\in \mathcal{Z} \end{aligned} \right\}. \tag{8.23}$$

For a given error set $\mathcal{E} \in \mathsf{E}$ and disturbance set $W \in \mathsf{W}$, define the set of all constraint admissible initial state estimates to be

$$\mathcal{S}_N^{ef}(\mathcal{E}, W) := \left\{ s \;\middle|\; \Pi_N^{ef}(s, \mathcal{E}, W) \neq \emptyset \right\}. \tag{8.24}$$

Remark 8.5. *As in the affine output feedback case, the set $\mathcal{S}_N^{ef}(\mathcal{E}, W)$ is independent of any choice of observer for finite horizon problems with a terminal constraint in the form $x_N \in X_T$ (cf. Remark 8.2).*

We next characterize two critical properties of the parameterization (8.19) which make it attractive in application to control of the system (8.1), and which parallel the results in Section 3.3 for the disturbance feedback case.

8.4 Convexity and Equivalence

8.4.1 Convexity and Closedness

We first establish convexity and closedness of the sets $\Pi_N^{ef}(s, \mathcal{E}, W)$ and $\mathcal{S}_N^{ef}(\mathcal{E}, W)$. Proof of the following results closely parallels the proof of the corresponding results for the disturbance feedback case in Section 3.4, and so will not be repeated here.

Theorem 8.6 (Convexity). *For any $\mathcal{E} \in \mathsf{E}$, $W \in \mathsf{W}$ and $s \in \mathcal{S}_N^{ef}(\mathcal{E}, W)$, the set of constraint admissible feedback policies $\Pi_N^{ef}(s, \mathcal{E}, W)$ is closed and convex. Furthermore, the set of state estimates $\mathcal{S}_N^{ef}(\mathcal{E}, W)$, for which at least one admissible affine output error feedback policy exists, is also closed and convex.*

Proposition 8.7 (Convexification of Uncertainty Sets). *Given sets $W \in \mathsf{W}$ and $\mathcal{E} \in \mathsf{E}$, the sets $\Pi_N^{ef}(s, W, \mathcal{E})$ and $S_N^{ef}(\mathcal{E}, W)$, defined in* (8.23) *and* (8.24) *respectively, are unchanged if $\mathcal{E}$, W and H are replaced with their convex hulls.*

Corollary 8.8 (Polyhedral Sets). *If the constraint sets Z and X_f are polyhedral and the sets $\mathcal{E}$, W and H are polytopes, then the set $\mathcal{S}_N^{ef}(\mathcal{E}, W)$ is polyhedral and $\Pi_N^{ef}(s, \mathcal{E}, W)$ is polyhedral for each $s \in \mathcal{S}_N^{ef}(\mathcal{E}, W)$.*

8.4.2 Equivalence of Affine Policy Parameterization

We next show that, as with the state and disturbance feedback parameterizations introduced in Chapter 3, the two affine feedback policies introduced in this chapter are equivalent.

Theorem 8.9. *Given any initial state estimation error set $\mathcal{E} \in \mathsf{E}$ and any disturbance set $W \in \mathsf{W}$, the sets $\mathcal{S}_N^{ef}(\mathcal{E}, W)$ and $\mathcal{S}_N^{of}(\mathcal{E}, W)$ are equal. Additionally, given any $s \in \mathcal{S}_N^{of}(\mathcal{E}, W)$, for any admissible $(\mathbf{K}, \mathbf{g})$ an admissible $(\mathbf{M}, \mathbf{v})$ can be found which yields the same state and input sequence for all allowable disturbance sequences, and vice-versa.*

Proof. $\mathcal{S}_N^{of}(\mathcal{E}, W) \subseteq \mathcal{S}_N^{ef}(\mathcal{E}, W)$: By definition, for any $s \in \mathcal{S}_N^{of}(\mathcal{E}, W)$, there exists a pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \mathcal{E}, W)$. A bit of algebra shows that, given any uncertainty realization e, $\boldsymbol{\eta}$ and $\mathbf{w}$, the input sequence $\mathbf{u}$ can be written as

$$\mathbf{u} = \Delta^{-1}\mathbf{K}\left[\mathbf{C}(\mathbf{A}e + \mathbf{E}\mathbf{w}) + \boldsymbol{\eta}\right] + \Delta^{-1}(\mathbf{K}\mathbf{C}\mathbf{A}s + \mathbf{g}), \tag{8.25}$$

where $\Delta := (I - \mathbf{KCB})$, and the matrix Δ is always invertible since $\mathbf{KCB}$ is strictly lower triangular. Noting the identity $\mathbf{C}(\mathbf{A}e + \mathbf{E}\mathbf{w}) + \boldsymbol{\eta} = (I + \mathbf{CE}\mathcal{L})(\mathbf{y} - \mathbf{C}\mathbf{s})$, the input sequence $\mathbf{u}$ can be written as

$$\mathbf{u} = \Delta^{-1}\mathbf{K}(I + \mathbf{CE}\mathcal{L})(\mathbf{y} - \mathbf{C}\mathbf{s}) + \Delta^{-1}(\mathbf{KCA}s + \mathbf{g}).$$

A constraint admissible policy $(\mathbf{M}, \mathbf{v}) \in \mathcal{S}_N^{ef}(\mathcal{E}, W)$ can then be found by selecting

$$\mathbf{M} = \Delta^{-1}\mathbf{K}(I + \mathbf{CE}\mathcal{L}), \ \ \mathbf{v} = \Delta^{-1}(\mathbf{KCA}s + \mathbf{g}). \tag{8.26}$$

Thus, $s \in \mathcal{S}_N^{ef}(\mathcal{E}, W)$ for all $s \in \mathcal{S}_N^{of}(\mathcal{E}, W)$, so $\mathcal{S}_N^{of}(\mathcal{E}, W) \subseteq \mathcal{S}_N^{ef}(\mathcal{E}, W)$.

$\mathcal{S}_N^{ef}(\mathcal{E}, W) \subseteq \mathcal{S}_N^{of}(\mathcal{E}, W)$: By definition, for any $s \in \mathcal{S}_N^{ef}(\mathcal{E}, W)$, there exists a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{ef}(s, \mathcal{E}, W)$. Using the relation (8.13), the output error terms can be written as $\mathbf{y} - \mathbf{C}\mathbf{s} = (I - \mathbf{C}\Gamma\mathcal{L})\mathbf{y} - \mathbf{C}\Phi s - \mathbf{C}\Gamma\mathcal{B}\mathbf{u}$, and the control input sequence $\mathbf{u} = \mathbf{M}(\mathbf{y} - \mathbf{C}\mathbf{s}) + \mathbf{v}$ as

$$\mathbf{u} = \hat{\Delta}^{-1}\mathbf{M}(I - \mathbf{C}\Gamma\mathcal{L})\mathbf{y} + \hat{\Delta}^{-1}(\mathbf{v} - \mathbf{MC}\Phi s),$$

where $\hat{\Delta} := (I + \mathbf{MC}\Gamma\mathcal{B})$, and the matrix $\hat{\Delta}$ is always invertible since $\mathbf{MC}\Gamma\mathcal{B}$ is strictly lower triangular. A constraint admissible policy $(\mathbf{K}, \mathbf{g}) \in \mathcal{S}_N^{of}(\mathcal{E}, W)$ can then be found by selecting

$$\mathbf{K} = \hat{\Delta}^{-1}\mathbf{M}(I - \mathbf{C}\Gamma\mathcal{L}), \ \ \mathbf{g} = \hat{\Delta}^{-1}(\mathbf{v} - \mathbf{M}\mathbf{C}\Phi s). \tag{8.27}$$

Thus, $s \in \mathcal{S}_N^{of}(\mathcal{E}, W)$ for all $s \in \mathcal{S}_N^{ef}(\mathcal{E}, W)$, so $\mathcal{S}_N^{ef}(\mathcal{E}, W) \subseteq \mathcal{S}_N^{of}(\mathcal{E}, W)$. □

Remark 8.10. *A control policy based on the measurement prediction error terms* $(\mathbf{y} - \mathbf{C}\mathbf{s})$ *was proposed in [vHB05], and independently in the context of robust optimization in [BBN06]. The latter gives an equivalence proof similar to that presented here, but without the inclusion of a nonzero initial state estimate or observer dynamics. In the sequel, we make explicit use of these error dynamics to derive conditions under which receding horizon control laws based on the parameterization* (8.19) *can be guaranteed to satisfy constraints for the resulting closed-loop system for all time.*

8.5 Geometric and Invariance Properties

In this section, we characterize some of the geometric and invariance properties associated with control laws synthesized from the feedback parameterization (8.19). We first require the following assumption about the terminal constraint set X_f:

A8.2 (Terminal Constraint) *For a given disturbance set* $W \in \mathsf{W}$, *a state feedback gain matrix* K_f, *observer gain* L *and terminal constraint set* X_f *have been chosen such that:*

i. *The matrices* $A + BK_f$ *and* $A - LC$ *are Hurwitz.*

ii. X_f *is contained inside the set for which the constraints* $\big((s+e), u\big) \in Z$ *are satisfied under the control* $u = K_f s$, *i.e.* $X_f \subseteq \big\{(s, e) \ \big| \ \big((s+e), K_f s\big) \in Z\big\}$.

iii. X_f *is robust positively invariant for the closed-loop system* $s^+ = (A+BK_f)s + L(Ce+\eta)$ *and* $e^+ = A_L e - L\eta + w$, *i.e.* $(s^+, e^+) \in X_f$ *for all* $(s, e) \in X_f$, *all* $w \in W$ *and all* $\eta \in H$.

Remark 8.11. *If the set* $W \times H$ *is a polytope or affine map of a p-norm ball and the constraints* Z *are polyhedral, then one can calculate an invariant set which satisfies the conditions* ***A8.2*** *by applying the techniques in [Bla99, KG98] to the augmented system*

$$\begin{bmatrix} s^+ \\ e^+ \end{bmatrix} = \begin{bmatrix} (A + BK_f) & LC \\ 0 & (A - LC) \end{bmatrix} \begin{bmatrix} s \\ e \end{bmatrix} + \begin{bmatrix} 0 & L \\ I & -L \end{bmatrix} \begin{bmatrix} w \\ \eta \end{bmatrix}. \tag{8.28}$$

In general, one wishes to select the terminal set X_f such that it is a maximal invariant set, so that the set $\Pi_N(\mathcal{E}, W)$ is as large as possible. Alternatively, one can use the techniques in [LK99] for calculating a target set of a given complexity.

Remark 8.12. *If one wishes to employ the alternative observer scheme* (8.8) *in conjunction with a control policy of the form $u_i = g_i + \sum_{j=0}^{i} K_{i,j} y_j$, then construction of an invariant terminal set X_f would need to be modified accordingly. One could, for example, replace the augmented system* (8.28) *with the system*

$$\begin{bmatrix} s^+ \\ e^+ \end{bmatrix} = \begin{bmatrix} (A+BK) & LC \\ 0 & (I-LC)A \end{bmatrix} \begin{bmatrix} s \\ e \end{bmatrix} + \begin{bmatrix} LC & L \\ (I-LC) & -L \end{bmatrix} \begin{bmatrix} w \\ \eta \end{bmatrix}, \tag{8.29}$$

and then employ all of the same methods suggested in Rem. 8.11 to calculate the set X_f.

8.5.1 Monotonicity of $\mathcal{S}_N^{of}(\mathcal{E}, W)$ and $\mathcal{S}_N^{ef}(\mathcal{E}, W)$

Proposition 8.13 (Monotonicity). *If **A8.2** holds, then the following set inclusions hold for any $\mathcal{E} \in \mathsf{E}$ and any $W \in \mathsf{W}$:*

$$\mathcal{S}_1^{of}(\mathcal{E}, W) \subseteq \cdots \subseteq \mathcal{S}_{N-1}^{of}(\mathcal{E}, W) \subseteq \mathcal{S}_N^{of}(\mathcal{E}, W) \subseteq \mathcal{S}_{N+1}^{of}(\mathcal{E}, W) \ldots \tag{8.30}$$

$$\mathcal{S}_1^{ef}(\mathcal{E}, W) \subseteq \cdots \subseteq \mathcal{S}_{N-1}^{ef}(\mathcal{E}, W) \subseteq \mathcal{S}_N^{ef}(\mathcal{E}, W) \subseteq \mathcal{S}_{N+1}^{ef}(\mathcal{E}, W) \ldots \tag{8.31}$$

Proof. The proof of the first relation is by induction. Suppose that $s \in \mathcal{S}_N^{of}(\mathcal{E}, W)$ and $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \mathcal{E}, W)$. Recalling the relation (8.13), the state estimates $\mathbf{s}$ can be found as an affine function of the measurements $\mathbf{y}$ using

$$\mathbf{s} = \Phi s + \Gamma \mathcal{B} \mathbf{g} + \Gamma(\mathcal{B}\mathbf{K} + \mathcal{L})\mathbf{y} \tag{8.32}$$

One can therefore find a pair $(\bar{\mathbf{K}}, \bar{\mathbf{g}}) \in \Pi_{N+1}^{of}(s, \mathcal{E}, W)$, where $\bar{\mathbf{K}} := \begin{bmatrix} \mathbf{K} & 0 \\ \bar{K}_1 & \bar{K}_2 \end{bmatrix}$ and $\bar{\mathbf{g}} := \begin{bmatrix} \mathbf{g} \\ \bar{g} \end{bmatrix}$, by defining

$$\bar{g} := K_f \left(\tilde{A}_L (\mathcal{B}\mathbf{K} + \mathcal{L})\mathbf{g} + A_L^N s \right) \tag{8.33}$$

$$\bar{K}_1 := K_f \left(\tilde{A}_L (\mathcal{B}\mathbf{K} + \mathcal{L}) \right), \quad \bar{K}_2 := 0 \tag{8.34}$$

where

$$\tilde{A}_L := \begin{pmatrix} A_L^{N-1} & \cdots & A_L & I \end{pmatrix},$$

so that the final stage input is $u_N = K_f s_N$. Since $s \in \mathcal{S}_N^{of}(\mathcal{E}, W)$ implies $(s_N, e_N) \in X_f$ by definition, then it follows that $(s_N + e_N, u_N) \in Z$ and $(s_{N+1}, e_{N+1}) \in X_f$ for all $w \in W$ and all $\eta \in H$ if **A8.2** holds. Thus $(\bar{\mathbf{K}}, \bar{\mathbf{g}}) \in \Pi_{N+1}^{of}(s, \mathcal{E}, W)$ and $s \in \mathcal{S}_{N+1}^{of}(\mathcal{E}, W)$. The second relation then follows from Theorem 8.9. □

8.5.2 Time-Varying and mRPI-based RHC Laws

We next consider some properties of receding horizon control laws synthesized from the parameterization (8.14) (equivalently, (8.19)). In particular, we develop conditions under which such an RHC law can be guaranteed to be robust positively invariant for the resulting closed-loop system.

We define the set-valued map $\kappa_N : \mathbb{R}^n \times \mathsf{E} \times \mathsf{W} \to 2^{\mathbb{R}^m}$ as

$$\kappa_N(s, \mathcal{E}, W) := \left\{ u \;\middle|\; \exists (\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \mathcal{E}, W) \text{ s.t. } u = g_0 \right\} \tag{8.35}$$

$$= \left\{ u \;\middle|\; \exists (\mathbf{M}, \mathbf{v}) \in \Pi_N^{ef}(s, \mathcal{E}, W) \text{ s.t. } u = v_0 \right\}, \tag{8.36}$$

where $2^{\mathbb{R}^m}$ is the set of all subsets of $\mathbb{R}^m$, and (8.36) follows directly from Theorem 8.9. We define a function $\mu_N : \mathbb{R}^n \times \mathsf{E} \times \mathsf{W} \to \mathbb{R}^m$ as any selection from the set κ_N, i.e. given $\mathcal{E} \in \mathsf{E}$ and $W \in \mathsf{W}$, $\mu_N(\cdot, \mathcal{E}, W)$ must satisfy

$$\mu_N(s, \mathcal{E}, W) \in \kappa_N(s, \mathcal{E}, W),\ \forall s \in \mathcal{S}_N^{of}(\mathcal{E}, W). \tag{8.37}$$

We wish to develop conditions under which time-varying or time-invariant control schemes based on (8.37) can be guaranteed to satisfy the system constraints Z for all time. We first introduce the following standard definition from the theory of invariant sets [KG98, RKKM05]:

Definition 8.14. *The set $\mathcal{E}_i$ is defined as*

$$\mathcal{E}_i := \bigoplus_{j=0}^{i} A_L^j (W \oplus L(-H)),\ \ \forall i \in \{0, 1, \dots\}. \tag{8.38}$$

The minimal robust positively invariant (mRPI) set $\mathcal{E}_\infty$ is defined as the limit set of the sequence $\{\mathcal{E}_i\}$, i.e. $\mathcal{E}_\infty := \lim_{i\to\infty} \mathcal{E}_i$.

Remark 8.15. *As noted in [KG98], unless the observer gain L is selected such that there exists an integer $k \geq 0$ and $0 \leq \alpha < 1$ such that $A_L^k = \alpha A_L$ (e.g. when L is a deadbeat* *observer, so that A_L is nilpotent), then the set $\mathcal{E}_\infty$ may not be characterized by a finite number of inequalities, since it is a Minkowski sum with an infinite number of terms*[2]*.*

We consider the implementation of a time-varying RHC law based on the function $\mu_N(\cdot)$. Taking the initial time to be 0 (which is always possible since the system (8.4)–(8.5) is time-invariant), and given an initial state estimate $s(0)$, initial state estimation error set $\mathcal{E} \in \mathsf{E}$ and disturbance set $W \in \mathsf{W}$, we define the *time-varying* RHC control law $\nu : \mathbb{R}^n \times \mathbb{N} \times \mathsf{E} \times \mathsf{W} \to \mathbb{R}^m$ as

$$\nu(s(k), k, \mathcal{E}, W) := \begin{cases} \mu_N(s(k), \mathcal{E}, W), & \text{if } k = 0 \\ \mu_N(s(k), A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1}, W), & \text{if } k > 0. \end{cases} \tag{8.39}$$

Note that the error sets required in the calculation of $\mu_N(s(k), A_L^k\mathcal{E} \oplus \mathcal{E}_{k-1}, W)$ can be defined recursively, i.e. $A_L^{k+1}\mathcal{E} \oplus \mathcal{E}_k = A_L[A_L^k\mathcal{E} \oplus \mathcal{E}_{k-1}] \oplus \mathcal{E}_0$, though an explicit calculation of these sets via Minkowski summation is *not* required (cf. Section 8.6). The resulting closed-loop system can be written as:

$$x(k+1) = Ax(k) + B\nu(s(k), k, \mathcal{E}, W) + w(k) \tag{8.40}$$

$$s(k+1) = As(k) + B\nu(s(k), k, \mathcal{E}, W) + L(y(k) - Cs(k)) \tag{8.41}$$

$$e(k+1) = A_L e(k) - L\eta(k) + w(k) \tag{8.42}$$

$$y(k) = Cx(k) + \eta(k), \tag{8.43}$$

where $w(k) \in W$ and $\eta(k) \in H$ for all $k \in \{0, \dots\}$. Note that given the estimation error set $\mathcal{E}$ at time 0, the estimation errors $\{e(k)\}_{k=0}^{\infty}$ in (8.42) are only known by the controller to satisfy $e(k) \in A_L^k\mathcal{E} \oplus \mathcal{E}_{k-1}$. Our first invariance result follows immediately:

Proposition 8.16. *If **A8.2** holds and $s(0) \in \mathcal{S}_N^{of}(\mathcal{E}, W)$, then the closed-loop system* (8.40)–(8.43) *satisfies the constraints Z for all time and all possible uncertainty realizations if the true initial state $x(0) \in \{s(0)\} \oplus \mathcal{E}$.*

Proof. If $s \in \mathcal{S}_N^{of}(\tilde{\mathcal{E}}, W)$ for some $\tilde{\mathcal{E}} \in \mathsf{E}$, then there exists an output feedback policy pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \tilde{\mathcal{E}}, W)$ for which $\mu_N(s, \tilde{\mathcal{E}}, W) = g_0$. It is then easy to show that

$$s^+ = As + B\mu_N(s, \tilde{\mathcal{E}}, W) + L(Ce + \eta) \in \mathcal{S}_{N-1}^{of}(A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0, W), \; \forall e \in \tilde{\mathcal{E}},$$

since one can construct a feasible policy pair $(\tilde{\mathbf{K}}, \tilde{\mathbf{g}}) \in \Pi_{N-1}^{of}(s^+, A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0, W)$ from $(\mathbf{K}, \mathbf{g})$

[2] Note that this situation is not alleviated if one employs the alternative observer scheme (8.8).

by dropping the first component of $\mathbf{g}$ and the first block row and column of $\mathbf{K}$. If **A8.2** holds, then $s^+ \in \mathcal{S}^{of}_{N-1}(A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0, W)$ implies $s^+ \in \mathcal{S}^{of}_{N}(A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0, W)$ from Proposition 8.13, and the result follows. □

We note that if the state estimation error set $\mathcal{E} = \mathcal{E}_\infty$, then the control law $\nu(\cdot)$ defined in (8.39) is actually *time-invariant*, so that

$$\nu(s(k), k, \mathcal{E}_\infty, W) = \mu_N(s(k), \mathcal{E}_\infty, W), \quad k = 0, 1, \ldots. \tag{8.44}$$

The next result follows immediately:

Corollary 8.17. *If* ***A8.2*** *holds, then the set* $\mathcal{S}^{of}_N(\mathcal{E}_\infty, W)$ *is robust positively invariant for the closed-loop system* (8.40)–(8.43) *under the time-invariant control law* (8.44)*, i.e. if* $s(0) \in \mathcal{S}^{of}_N(\mathcal{E}_\infty, W)$ *and* $x(0) \in \{s(0)\} \oplus \mathcal{E}_\infty$*, then* $s(k) \in \mathcal{S}^{of}_N(\mathcal{E}_\infty, W)$ *and the constraints* Z *are satisfied for all time and for all possible uncertainty realizations.*

8.5.3 A Time-Invariant Finite-Dimensional RHC Law

The central difficulty with the control law defined in (8.44) is that the set $\mathcal{E}_\infty$ is not finitely determined, in general (cf. Remark 8.15). The calculation of the control law $\nu(\cdot, \cdot, \mathcal{E}, W)$ in (8.39) is thus of increasing complexity with increasing time, and the calculation of the control law $\nu(\cdot, \cdot, \mathcal{E}_\infty, W)$ in (8.44) requires the solution of an infinite-dimensional optimization problem. We therefore seek a control law that is of fixed and finite complexity, while preserving the time-invariant nature of (8.44). To this end, we define a robust positively invariant (RPI) error set $\mathcal{E}_I \in \mathsf{E}$ which satisfies the following:

A8.3 (Invariant Error Set) *For a given disturbance set* $W \in \mathsf{W}$*, the set* $\mathcal{E}_I \in \mathsf{E}$ *is chosen such that:*

i $\mathcal{E}_I$ *is robust positively invariant for the system* $e^+ = A_L e - L\eta + w$*, i.e.* $A_L e - L\eta + w \in \mathcal{E}_I$ *for all* $e \in \mathcal{E}_I$*,* $w \in W$ *and* $\eta \in H$*.*

ii For some p-norm, $\mathcal{E}_I$ *is an* ϵ*-outer approximation for* $\mathcal{E}_\infty$*, so that there exists some* $\epsilon > 0$ *such that* $\mathcal{E}_\infty \subseteq \mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \epsilon\mathcal{B}^n_p$*.*

Remark 8.18. *In [RKKM05] it is shown how one can calculate an arbitrarily close outer approximation* $\mathcal{E}_I$ *to the set* $\mathcal{E}_\infty$ *(which can be represented by a tractable number of inequal-*

ities if W and H are polytopes) such that $\mathcal{E}_\infty \subseteq \mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \epsilon\mathcal{B}_p^n$ and such that the set $\mathcal{E}_I$ is robust positively invariant. Further, it is shown in [RKKM05] that, if only the support function of the set $\mathcal{E}_I$ is required, then calculation of an explicit *representation of $\mathcal{E}_I$ via Minkowski summation is* not *necessary, a fact which we exploit in the computational results of Section 8.6.*

We can now guarantee an invariance condition similar to the one in Proposition 8.16 using the finitely determined set $\mathcal{E}_I$, by slightly enlarging the disturbance set W from which feedback policies of the form (8.19) are selected. We henceforward assume that the true disturbances are known to be drawn from some set $W \in \mathsf{W}$, and define

$$W_\epsilon := W \oplus \epsilon\mathcal{B}_p^n \tag{8.45}$$

where p and ϵ satisfy the conditions of **A8.3** for the set $\mathcal{E}_I$. Using this enlarged disturbance set, we consider the following modified assumption on the target/terminal constraint set $X_f \subseteq \mathbb{R}^n \times \mathbb{R}^n$:

A8.4 (Modified Terminal Constraint) *For a given disturbance set $W \in \mathsf{W}$, a state feedback gain matrix K_f, observer gain L and terminal constraint set X_f have been chosen such that:*

*i. The terminal conditions **A8.2** hold.*

ii. X_f is robust positively invariant for the closed-loop system $s^+ = (A+BK_f)s+L(Ce+\eta)$ and $e^+ = A_L e - L\eta + w$, i.e. $(s^+, e^+) \in X_f$ for all $(s,e) \in X_f$, all $w \in W \oplus \epsilon\mathcal{B}_p^n$ and all $\eta \in H$.

In the sequel, we will choose an invariant set $\mathcal{E}_I$ and scalar $\epsilon > 0$ satisfying the conditions of **A8.3** and **A8.4** such that a time-invariant control law constructed from $\Pi_N^{of}(s, \mathcal{E}_I, W_\epsilon)$ (equivalently, $\Pi_N^{ef}(s, \mathcal{E}_I, W_\epsilon)$) can be guaranteed to satisfy the system constraints for all time.

We define the *time-invariant* control law $\nu_I : \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon) \to \mathbb{R}^m$ as:

$$\nu_I(s) := \mu_N(s, \mathcal{E}_I, W_\epsilon). \tag{8.46}$$

When applied to the control of the system (8.1), the closed-loop system dynamics become

$$x^+ = Ax + B\nu_I(s) + w \tag{8.47}$$
$$s^+ = As + B\nu_I(s) + L(y - Cs) \tag{8.48}$$
$$e^+ = A_L e - L\eta + w \tag{8.49}$$
$$y = Cx + \eta, \tag{8.50}$$

where $w \in W$ and $\eta \in H$. It is critical to note that, though the control law $\nu_I(\cdot)$ defined in (8.46) is conservatively constructed using the *enlarged* disturbance set W_ϵ, the disturbances w in (8.47) are generated from the *true* disturbance set W. It is this conservativeness that will ensure that the *time-invariant* control law (8.46) can guarantee constraint satisfaction of the closed-loop system for all time. In particular, this conservativeness will allow us to show that, using the control law $\nu_I(\cdot)$ in (8.46), if $s \in \Pi_N^{of}(\mathcal{E}_I, W_\epsilon)$, then $s^+ \in \Pi_N^{of}(\mathcal{E}_I, W_\epsilon)$ for all $w \in W$ and all $\eta \in H$. Note that if W had been used in place of W_ϵ in (8.46), then one can only guarantee that $s \in \Pi_N^{of}(\mathcal{E}_I, W)$ implies $s^+ \in \Pi_N^{of}(A_L\mathcal{E}_I \oplus \mathcal{E}_0, W)$. The latter situation is less desirable, since in general the set $A_L\mathcal{E}_I \oplus \mathcal{E}_0$ will be more complex (i.e. characterized by more inequalities) than the set $\mathcal{E}_I$. One would also be forced to employ a time-varying control scheme of the form (8.39), where the control calculation becomes increasingly complex with increasing time.

We can now state our final result:

Theorem 8.19. *If* ***A8.3*** *and* ***A8.4*** *hold, then the set* $\mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon)$ *is robust positively invariant for the closed-loop system* (8.47)–(8.50)*, i.e. if* $s \in \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon)$ *and* $x(0) \in \{s(0)\} \oplus \mathcal{E}_I$*, then* $s^+ \in \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon)$ *and the constraints* Z *are satisfied for all time and for all possible uncertainty realizations.*

Proof. If **A8.4** holds then it can be shown, using arguments identical to those in the proof of Proposition 8.16, that $s \in \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon)$ implies that the successor state

$$s^+ \in \mathcal{S}_N^{of}(A_L\mathcal{E}_I \oplus W_\epsilon \oplus L(-H), W_\epsilon), \tag{8.51}$$

or, equivalently, that

$$s^+ \in \mathcal{S}_N^{of}(A_L\mathcal{E}_I \oplus \mathcal{E}_0 \oplus \epsilon\mathcal{B}_p^n, W_\epsilon). \tag{8.52}$$

If **A8.3** holds, so that $\mathcal{E}_\infty \subseteq A_L\mathcal{E}_I \oplus \mathcal{E}_0 \subseteq \mathcal{E}_I$, then

$$\mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \epsilon\mathcal{B}_p^n \subseteq A_L\mathcal{E}_I \oplus \mathcal{E}_0 \oplus \epsilon\mathcal{B}_p^n.$$

Using the set intersection notation in (8.16) it is easily verified that, for any sets $\mathcal{E}' \in \mathsf{E}$ and $\mathcal{E}'' \in \mathsf{E}$, $\mathcal{E}' \subseteq \mathcal{E}''$ implies

$$\Pi_N^{of}(s, \mathcal{E}'', W_\epsilon) \subseteq \Pi_N^{of}(s, \mathcal{E}', W_\epsilon) \text{ for all } s \in \mathbb{R}^n,$$

and consequently that $\mathcal{S}_N^{of}(\mathcal{E}'', W_\epsilon) \subseteq \mathcal{S}_N^{of}(\mathcal{E}', W_\epsilon)$. It follows that

$$\mathcal{S}_N^{of}(A_L\mathcal{E}_I \oplus \mathcal{E}_0 \oplus \epsilon\mathcal{B}_p^n, W_\epsilon) \subseteq \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon),$$

so that $s^+ \in \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon)$ for all $e \in \mathcal{E}_I$, $\eta \in H$ and $w \in W$. Finally we verify that the closed-loop system (8.47)–(8.50) satisfies the constraints Z for all time. We again use set intersection arguments to confirm that $\Pi_N^{of}(s, \mathcal{E}_I, W_\epsilon) \subseteq \Pi_N^{of}(s, \mathcal{E}_I, W)$. This implies that $\kappa_N(s, \mathcal{E}_I, W_\epsilon) \subseteq \kappa_N(s, \mathcal{E}_I, W)$, which guarantees that $(s + e, \nu_I(s)) \in Z$ for all $e \in \mathcal{E}_I$ if $s \in \mathcal{S}_N^{of}(\mathcal{E}_I, W_\epsilon)$. □

Remark 8.20. *If **A8.3** holds, then $A_L\mathcal{E}_I \oplus (W \oplus L(-H)) \subseteq \mathcal{E}_I$ and $\mathcal{E}_\infty \subseteq \mathcal{E}_I$. Such a set can be calculated in a variety of ways using standard techniques (cf. Remark 8.11), or, more usefully, as an invariant outer approximation to the mRPI set using results from [RKKM05]. In both cases, the resulting set is polytopic when all of the relevant constraints and uncertainty sets are polytopic, and the set $\mathcal{E}_I$ can be characterized by a finite number of linear inequalities, though an* explicit *representation of the set $\mathcal{E}_I$ is not required (cf. Remark 8.15 and the results of Section 8.6).*

In general, one should expect that the initial error set $\mathcal{E}$ will be provided as a part of the problem description, and will not be an RPI set. In addition, if the initial estimation error set $\mathcal{E}$ is large, it may be undesirable to employ a time-invariant policy of the form (8.46) with $\mathcal{E} \subseteq \mathcal{E}_I$.

In such cases a variety of methods may be devised for mitigating computational complexity while preserving a large region of attraction. For example, given some initial error set $\mathcal{E}$ and disturbance set W, one could elect to use a time-varying controller (see (8.39)) over some interval, and then switch to a time-invariant controller (see (8.46)) once $A_L^k\mathcal{E} \oplus \mathcal{E}_{k-1} \subseteq \mathcal{E}_I$, where $\mathcal{E}_I$ satisfies **A8.3**. Such a control scheme is easily shown to satisfy the system constraints for all time if the control inputs are

$$u(k) = \begin{cases} \mu_N(s(k), k, \mathcal{E}, W_\epsilon), & \text{if } k = 0 \\ \mu_N(s(k), k, A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1}, W_\epsilon), & \text{if } k \in \{1, \dots, q\} \\ \nu_I(s(k), \mathcal{E}_I, W_\epsilon), & \text{if } k \in \{q+1, \dots\}, \end{cases} \tag{8.53}$$

where the integer q is chosen large enough so that $A_L^q \mathcal{E} \oplus \mathcal{E}_{q-1} \subseteq \mathcal{E}_I$[3]. In order to increase the region of attraction for such a scheme, it is advantageous to minimize the size of the set $\mathcal{E}_I$, i.e. to choose ϵ as small as computational resources allow.

8.6 Computation of Feedback Control Laws

Finally, we comment briefly on the computational problem of finding feedback policies of the form (8.23) for the implementation of the control law (8.39). As in Chapter 6, we consider the particular case when the constraint sets Z and X_f and uncertainty sets W, H and $\mathcal{E}$ are polytopes, while the uncertainty sets $\mathcal{E}$, W and H can be any compact and convex sets. Suppose that one defined matrices S, T and U and a vector b of appropriate dimensions such that $\mathcal{Z}$ could be written as a set of linear inequalities

$$\mathcal{Z} = \{(\mathbf{s}, \mathbf{e}, \mathbf{u}) \mid S\mathbf{s} + T\mathbf{e} + U\mathbf{u} \leq b\}. \tag{8.54}$$

In this case the set of feasible control policies can be written as

$$\Pi_N^{ef}(s, \mathcal{E}, W) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies } (8.20) \\ S\mathbf{A}s + P\mathbf{v} + \delta_e + \delta_w + \delta_\eta \leq b \\ \delta_e = \max\limits_{e \in \mathcal{E}} \; (P\mathbf{M}Q_e + R_e)e \\ \delta_w = \max\limits_{\mathbf{w} \in W^N} (P\mathbf{M}Q_w + R_w)\mathbf{w} \\ \delta_\eta = \max\limits_{\boldsymbol{\eta} \in H^N} \; (P\mathbf{M}Q_\eta + R_\eta)\boldsymbol{\eta} \end{array} \right\}, \tag{8.55}$$

where the matrices Q_e, Q_w, Q_η, R_e, R_w, R_η and P are defined as

$$\begin{aligned} &Q_e := \mathbf{C}\Phi, && Q_w := \mathbf{C}\Gamma, && Q_\eta := (I - \mathbf{C}\Gamma\mathcal{L}), \\ &R_e := (S\mathbf{E}\mathcal{L}\mathbf{C} + T)\Phi, && R_w := (S\mathbf{E}\mathcal{L}\mathbf{C} + T)\Gamma, && R_\eta := (S\mathbf{E}(I - \mathcal{L}\mathbf{C}\Gamma) - T\Gamma)\,\mathcal{L}, \end{aligned}$$

[3]This condition holds if $A_L^q \mathcal{E} \subseteq \epsilon \mathcal{B}_p^n$. This is trivial to test, for example, in the case where the set $\mathcal{E}$ is a hypercube and $p = \infty$.

and $P := (S\mathbf{B} + U)$ respectively[4], where all of the maximizations are performed row-wise. Note that all of the maxima in (8.55) are attained since the sets $\mathcal{E}$, W and H are assumed compact. Using the methods of Chapter 6, it is then a simple matter to compute a policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{ef}(s, \mathcal{E}, W)$ when each of the polar sets $\mathcal{E}^\circ$, W° and H° is easily characterized.

It is important to note that it is *not* necessary to explicitly perform the Minkowski summation of error sets in the calculation of the time-varying control law (8.39), since only the support functions of these sets is of interest. Given an initial error set $\mathcal{E}$ at time 0, one needs to calculate, at each time k, a feasible policy pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{ef}(s(k), A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1})$. In this case the vector δ_e in (8.55) can be written as

$$\delta_e = \max_{e \in A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1}} (P\mathbf{M}Q_e + R_e)e \tag{8.56}$$

$$= \max_{e \in \mathcal{E}} (P\mathbf{M}Q_e + R_e)A_L^k e + \max_{e \in \mathcal{E}_{k-1}} (P\mathbf{M}Q_e + R_e)e, \tag{8.57}$$

where

$$\max_{e \in \mathcal{E}_{k-1}} (P\mathbf{M}Q_e + R_e)e = \sum_{i=0}^{k-1} \left[\left(\max_{w \in W} (P\mathbf{M}Q_e + R_e)A_L^i w \right) + \left(\max_{\eta \in H} (P\mathbf{M}Q_e + R_e)A_L^i(-L)\eta \right) \right]$$

and one may deal with each component of this summation separately using the methods of Chapter 6. For example, if the sets $\mathcal{E}$, W and H are each polytopic then the set (8.55) can be characterized by a number of variables, and linear inequalities increasing linearly with k.

An identical procedure can be used to find an element of the set $\Pi_N^{ef}(s, \mathcal{E}_I, W_\epsilon)$ in the implementation of the time-invariant control law (8.46), resulting in a convex optimization problem of *fixed* and *finite* complexity, where once again it is not necessary to explicitly form the Minkowski sum in (8.45), and where the support function of $\mathcal{E}_I$ can be determined using an *implicit* representation of a Minkowski sum of a finite number of polytopes as in [RKKM05]. In particular, the results in [RKKM05] demonstrate that, for any p-norm and approximation accuracy $\epsilon > 0$, there exists a finite integer q and a scalar $\beta > 1$, such that $\mathcal{E}_\infty \subseteq \mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \epsilon \mathcal{B}_p^n$, where

$$\mathcal{E}_I = \beta \bigoplus_{i=0}^{q-1} A_L^i (W \oplus L(-H)). \tag{8.58}$$

[4] A bit of algebra confirms that the matrix identities $\mathbf{E} = (I + \mathbf{E}\mathcal{L}\mathbf{C})\Gamma$ and $\mathbf{A} = (I + \mathbf{E}\mathcal{L}\mathbf{C})\Phi$ hold, so that one may also use the equivalent matrix definitions $R_e := S\mathbf{A} - (S - T)\Phi$, $R_w := S\mathbf{E} - (S - T)\Gamma$ and $R_\eta := (S - T)\Gamma\mathcal{L}$.

It is of particular importance to note that the number of terms q in this summation is primarily dependent on the spectral radius of the matrix $(A-LC)$, which is selected by the designer of the controller [RKKM04]. Note that all of $\mathcal{E}_I$, L, q, β and ϵ are fixed and can be determined off-line. One can thus write

$$\delta_e = \beta \sum_{i=0}^{q-1} \left[\left(\max_{w \in W \oplus \epsilon \mathcal{B}_p^n} (P\mathbf{M}Q_e + R_e) A_L^i w \right) + \left(\max_{\eta \in H} (P\mathbf{M}Q_e + R_e) A_L^i (-L)\eta \right) \right] \tag{8.59}$$

in (8.55), and once again apply the methods of Chapter 6 to each element of this summation. It is therefore possible to obtain a representation of the set $\Pi_N^{ef}(s, \mathcal{E}_I, W)$ in a number of variables and constraints that is polynomial in the size of the problem data, and that does *not* increase with time.

8.6.1 Numerical Example

We consider the discrete-time system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u + w \tag{8.60}$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + \eta \tag{8.61}$$

with stable feedback gain K_f and observer gain L chosen as

$$K_f = \begin{bmatrix} -0.75 & -1.85 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1.15 & 0.65 \end{bmatrix}^\top. \tag{8.62}$$

The sets $\mathcal{Z}$, $\mathcal{E}$, W and H are defined as

$$Z := \left\{ (x,u) \in \mathbb{R}^2 \times \mathbb{R} \;\middle|\; \begin{array}{r} -3 \le x_1 \le 25 \\ -3 \le x_2 \le 25 \\ |u| \le 5 \end{array} \right\} \tag{8.63}$$

$$\mathcal{E} := \left\{ e \in \mathbb{R}^2 \mid \|e\|_\infty \le 0.4 \right\} \tag{8.64}$$

$$W := \left\{ w \in \mathbb{R}^2 \mid \|w\|_\infty \le 0.1 \right\} \tag{8.65}$$

$$H := \left\{ \eta \in \mathbb{R}^2 \mid \|\eta\|_\infty \le 0.1 \right\}, \tag{8.66}$$

where x_i is the i^{th} element of x. In order to obtain the set X_f, we calculate the maximal RPI set compatible with Z for the system (8.28) using the method of [KG98, Alg. 6.2].

We consider the set of feasible initial state estimates $\mathcal{S}_i^{of}(\mathcal{E}, W)$ (equivalently $\mathcal{S}_i^{ef}(\mathcal{E}, W)$) for this system. For comparison, we also consider the sets $S_i^K(\mathcal{E}, W)$ for which a feasible control policy can be found when the policy is parameterized in terms of perturbations to a *fixed* state feedback gain, such that $u_j = c_j + Ks_j$. Recall that $S_i^K(\mathcal{E}, W) \subseteq S_i^{of}(\mathcal{E}, W)$ for all $i \in \{0, 1, \dots\}$ (cf. Remark 8.3). The resulting sets of feasible initial state estimates for this system are shown in Figure 8.1.

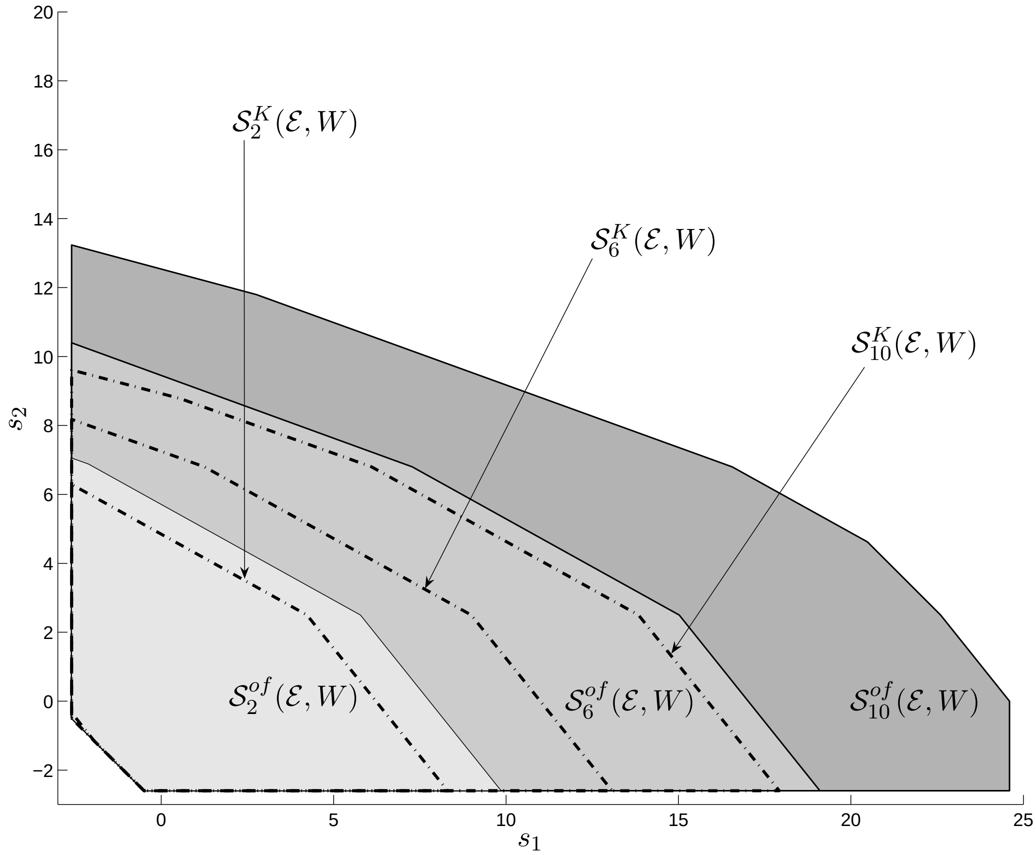

Figure 8.1: Feasible initial state estimate sets $\mathcal{S}_i^{of}(\mathcal{E}, W)$ and $\mathcal{S}_i^K(\mathcal{E}, W)$ for $i \in \{2, 6, 10\}$

8.7 Conclusions

The main contribution of this chapter is to propose a class of time-invariant receding horizon output feedback control laws for control of linear systems subject to bounded disturbances and measurement errors. The proposed method is based on a fixed linear state observer combined with optimization over the class of feedback policies which are affine in the sequence of prior outputs.

As in the state feedback case considered in Chapter 3, it is possible to compute such an output feedback policy using an appropriate convexifying reparameterization. As a consequence, receding horizon control laws in the proposed class can be computed using the methods of Chapter 3, while providing a larger region of attraction than methods based on calculating control perturbations to a static linear feedback law.

Chapter 9. Conclusions

9.1 Contributions of this Dissertation

The main focus of this dissertation has been the use of affine feedback policies in robust receding horizon control of constrained systems. Specific contributions are as follows:

Affine Feedback Policies

The main idea underpinning all of the results in this dissertation is that, for a constrained linear system subject to bounded disturbances, the class of robust finite horizon feedback policies modelling the input at each time step as an affine function of prior *states* is equivalent to the class of policies modelling each input as an affine function of prior *disturbances*. This equivalence result is reminiscent of the well-known Youla parameterization in linear system theory [YJB76].

The main advantage of affine disturbance feedback policies is that, given an initial state, the set of constraint admissible policies for a system with convex constraints is a convex set — one can therefore compute a policy in this class using standard convex optimization tools in many cases.

One the other hand, affine state feedback policies generally are *not* computable using convex optimization techniques. However, this class of policies has a close connection to existing research in robust predictive control, and many standard predictive control techniques for ensuring invariance of receding horizon control laws can be preserved for control laws synthesized from policies in this class. The equivalence result described above allows one to exploit the advantageous properties of both affine disturbance and state feedback policies simultaneously.

Stability Results

Using the aforementioned properties of affine feedback policies, a family of receding horizon control laws was defined that guarantee infinite horizon constraint satisfaction. If one wishes to guarantee that such a control law is also stabilizing, then it is necessary to define a cost or objective function to help discriminate from amongst feasible policies at each time step. Properties of control laws developed using two such cost functions were presented in Chapters 4 and 5:

Expected Value Costs and Input-to-State Stability
If one chooses to minimize the expected value of a quadratic cost function of states and control inputs, then a receding horizon control law can be constructed that guarantees invariance and input-to-state stability for a system with arbitrary convex constraints. The behavior of this control law matches that of a linear-quadratic or $\mathcal{H}_2$ control law when the system operates far from its constraints.

Central to this result is a proof of the existence of minimizers and convexity of the value function in the underlying finite horizon optimal control problem. In order to derive this result, we also provided conditions under which input-to-state stability can be established using convex Lyapunov functions.

Min-Max Costs and ℓ_2 Gain Minimization
If one instead chooses to minimize the maximum value of quadratic cost, where the disturbances are negatively weighted as in $\mathcal{H}_\infty$ control, then an invariant receding horizon control law can be constructed with guaranteed bounds on the ℓ_2 gain of the closed-loop system.

The proposed control law requires the imposition of additional LMI constraints of the set of feedback policies over which one must optimize at each time step – this constraint ensures that the min-max problem to be solved is convex-concave, making the method a suitable candidate for on-line implementation.

Computational Results

It was shown in Chapter 6 that when all the system constraints are linear, then a finite horizon control policy can be calculated using standard techniques in convex optimization for a wide variety of disturbance classes — the same statement applies to the computation of policies with respect to either the expected value or min-max costs employed in deriving the aforementioned stability results. In all of these cases, the size of the optimization problem to be solved grows polynomially with the problem data, and in particular with the horizon length. This result is a major improvement on the complexity one would expect using methods based on dynamic programming, where exponential growth with horizon length is generally to be expected.

Robust Output Feedback

In Chapter 8, the main results of the dissertation were extended to the output feedback case. A family of receding horizon control laws were proposed based on optimization over the class of feedback policies that are affine in the sequence of prior outputs — as in the state feedback case, it was shown that this class of policies can be rendered convex via an appropriate reparameterization. Appropriate conditions were established to guarantee that such a receding horizon control law would satisfy the system constraints for all time. The main technical issue addressed was the problem of synthesizing a control law that is time-invariant *and* that can be calculated by solving a finite-dimensional optimization problem using policies in this class.

9.2 Directions for Future Research

Possible directions for future research are outlined as follows:

Degree of Approximation

A central motivation for this work is that the problem of finding a finite horizon robust control policy composed of arbitrary functions (e.g. using dynamic programming) is far too difficult in almost all cases of practical interest. By restricting the class of functions considered to affine functions the problem becomes tractable, but it is unclear how closely this class approximates the ideal case or in what cases the two approaches achieve the same result. Development of a method to quantify the degree of approximation, preferably *without* explicitly calculating the relevant control laws, is of considerable interest.

It would also be interesting to compare the performance of the proposed method with that of methods based on *linear* control design, and in particular with ℓ_l control design methods.

Computational Methods

The results of Chapter 7 on efficient computational methods are particular to the combination of a quadratic cost function of nominal state and input sequences and a disturbance set modelled as an ∞–norm bounded ball. However, the basic idea of re-introduction of states appears to be easily generalizable to the other disturbance models considered, and it is likely that significant improvements in computational efficiency can be achieved in these cases as well, particularly for those problems using the expected value cost function of Chapter 4.

A more difficult problem is the extension of these ideas to the $\mathcal{H}_\infty$ problem developed in Chapter 5, since it is not immediately clear whether the method of re-introduction of states can be applied to the additional LMI constraint that appears in these problems.

An entirely different approach to improve computational efficiency would be to define a class of control policies where the input at each time step is an affine function of a *fixed* number of prior disturbances only. Use of such an approximation would invalidate nearly all of the invariance results presented in this dissertation, since the key equivalence result of Chapter 3 could no longer be exploited. However, it might be possible to find alternative terminal conditions than those used in Chapter 3 to guarantee invariance, or one might find a suitably modified class of state feedback policies, e.g. policies based on deadbeat control, with which an equivalence result might hold.

Stability and Output Feedback

In the output feedback results of Chapter 8, we have only considered the problem of finding a *feasible* control policy at each time, without regard to optimality. As in the state feedback case, it is possible to define a variety of cost functions to motivate the selection from amongst this feasible set of policies, and we have not addressed any stability results which may be derived based on this selection. In order to be practically useful, the results could also be extended to handle the cases of setpoint tracking and offset-free control.

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