A Whitehead Programme Rehabilitated

Abstract

Whitehead's programme for defining points by his method of "Abstractive Extension" was deemed a failure, and he has been criticized for not establishing mereology on a proper rigorous basis. But if mereological presuppositions are avoided, Extensive Abstraction will yield an adequate definition of points within a logicist programme, and provide a basis from which topology and the whole of geometry can be developed.

I History

The fourth volume of *Principia Mathemtica* was never written. The first World War supervened, with Whitehead and Russell taking different stances. But some of Whitehead's thinking was published in a different format. In his *The Principles of Natural Knowledge*, Cambridge, 1919, (§30, pp.101-106) he sketched a programme of defiing points and providing a firm basis for topology, which he touches on again in his *Process and Reality*, Cambridge, 1929, Part 4, Chapter 2. Russell gave a pre-view of it in his *Our Kowledge of the External World*, London, 1914, pp.7-8, and 114-115..

Whitehead tried a "Russian doll" approach, to define points in terms of 'events', $a, b, c, \ldots$ *etc.* by the method of "Extensive Abstraction" (so called, because at that stage he was talking of 'extensions' rather than 'regions').[1] "Abstractive Classes" were convergent sequences that would define limits of regions. They were a generalisation of Cauchy sequences and a predecessor of modern filters and directed sets; Whitehead used them in much the same way as Cantor had used nested intervals to define point-like real numbers. Each extension in an Abstractive Classe was enclosed in, or nested in, its predecessors, but with none that was enclosed in all

[1] The term 'events', (3+1)-dimensional bounded objects, though appropriate in view of Whitehead's empiricist programme, is needlessly specific; he used 'extensions' as a general term in his early work, and 'regions' later; here 'regions' will be used as the most general term, but 'portions' will used in strictly mereological contexts, and 'wholes' where regions are connected, and 'parts' where the parallel between *proper part of* and *inner part of* (which will be used as an alternative to Whithead's *enclosed in*) needs to be drawn.

its predecessors.[2] Such sequences converge to a limit, with limits of volumes being surfaces, limits of surfaces being lines, and limits of lines being points. Geometers normally postulate points, but in a volume of *Principia Mathemtica* they needed to be constructed by honest toil. Moreover, having defined points, Whitehead could then hope to define dimensionality and introduce lines, surfaces and higher-dimensional entities. Abstractive Classes should also serve to characterize the topological concept of neighbourhood, and thus to define a Hausdorff space, which has a concept of nearness, and thus lay the foundations of topology.

II Modern Mereology

Whitehead has been hailed as one of the fathers of "Mereology", and castigated for being an unrigorous parent.[3] Mereology, from the Greek *μέρος meros*), a part, can be based on the *proper part of* relation, (here symbolized by $<$), which is transitive and irreflexive, and thus an ordering relation. Mereology can be seen as a sibling of simple set theory, based on the *proper subset* relation, (here symbolized by $\subset$). But whereas the *proper subset* relation is discrete, the *proper part of* relation, is dense (and possibly continuous). And whereas set theory forms a lattice, mereology has the shape of the roots of a tree, each root tapering downwards and downwards and multiplying without limit. More formally, whereas of any two sets each has the null set as a subset of itself, two portions that do

[2] Whitehead spoke of one extension *extending over* another, but for the sake of uniformity with the standard practice of having the smaller item preceding the larger, the converse relation 'enclosed in' will be used.

[3] Mereology has been much studied since Whitehead's time. Much of the work was done in Poland, notably by Leśniewski, Lejewski and Tarski. See also N.Goodman and H.S.Leonard, "The Calculus of Individuals and its Uses", *Journal of Symbolic Logic*, **5**, 1940, pp.45-55. More recent work has been done by J.E.Tiles, *Things that Happen*, Aberdeen, 1981, §8. B.L.Clarke, "A Calculus of Individuals based on 'Connection' ", *Notre Dame Journal of Formal Logiv*, **22**, 1981, pp.204-218. B.L.Clarke, "Individuals and Points", *Notre Dame Journal of Formal Logiv*, **26**, 1985, pp.204-218. P.Roeper, "Region-based Topology", *Journal of Philosophical Logic*, **26**, 1997, pp.251-309. An excellent survey of the different mereological systems is to be found in P. Simons, *Parts*, Oxford, 1987, ch.2, pp.46-100.

Set Theory and Mereology Compared
Both *proper subset of*, $\subset$, and *proper part of*, $<$ are transitive and irreflexive, and so ordering, relations.
But
proper subset of is discrete, while *proper part of* is dense
proper subset of is directed, and forms a lattice, while *proper part of* is not downward directed
Set theory has a minimum element, the null set while for every portion there is another one which is a proper part of it

not overlap do not have any portion that is a *proper part of* them both, and the *proper part of* relation is downwards serial, that is, for every portion there is another one which is a proper part of it.

We can define *overlaps* and *is disjoint from* in terms of *is a proper part of*, $<$: x *overlaps* y iff[4] there is a z which *is a proper part of* them both, and x *is disjoint from* y iff there is no z which *is a proper part of* them both
(in symbols

$x \circ y \;\text{ iff }\; (\exists z)(z < x \wedge z < y)$

$x \mid y \;\text{ iff }\; \neg(\exists z)(z < x \wedge z < y)$).

Equality can be defined by a two-way implication, x *equals* y iff every z that *is a proper part of* x *is a proper part of* y and *vice versa*;
(in symbols $x = y \;\text{ iff }\; (\forall z)(z <\leftrightarrow z < y)$).

Suitable axioms can be framed, but they are cumbersome, and Bostock offers an extremely elegant axiomatization based on the symmetric irreflexive relation of *being disjoint from*, $|$.[5] Overlap is simply the negation of disjointedness: x *overlaps* y iff x *is not disjoint from* y

$x \circ y \;\text{ iff }\; \neg(x \mid y)$

4 short for 'if and only if'.

5 David Bostock, "Whitehead and Russell on Points" in *Philosophia Mathematica* (III), **18** (2010), pp. 1-57, esp. pp.4-6; I am deeply indebted to him for his patience in finding flaws in my tentative arguments as I wrote this paper.

Equality can be defined by a two-way implication, x *equals* y iff every z that *is disjoint from* x

is disjoint from y and *vice versa*;

$x = y$ iff $(\forall z)(z \mid x \leftrightarrow z \mid y)$

The one-way implicatiion provides a definition of *being er or improper part of*, $\leq$

It is requisite in this formulation to define *being a proper part of*, $<$, in terms of *being disjoint from*, $\mid$. The one-way implication above allows the possibility of equality, and so defines *a proper or improper part of* (that is, either x *is a proper part of* y, or x *equals* y)

x is *a proper or improper part of* y iff every z that is *disjoint from* y is *disjoint from* x (in symbols,

$x \leq y$ iff $(\forall z)(z \mid y \rightarrow z \mid x))$.

from this we can define

x *is a proper part of* y as holding when *a proper or improper part of* holds, but not equality.

$x < y$ iff $(x \leq y) \land \neg(x = y)$.

Further axioms and definitions to secure the existence of a sum, that is that for any two (Whitehead), or set of (Bostock), portions there exists a portion, of which they both, or all, are parts. and to establish least upper, and greatest lower, bounds, which in turn secure the existence of unique sums and products. Although different axiomatizations sometimes axiomatize different systems, they have a common core, which succeeds in capturing our informal concept of *proper part of.*

III Whitehead's Abstractive Extension

Whitehead's Russian dolls were constituted by the method of "Extensive Abstraction". Whitehead defined an Abstractive Class as a set of regions where:

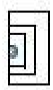
1. of any two members of the Abstractive Class one was a proper part of the other, and
2. there was no region which was a proper part of every region of the set.

Abstractive Classes will be symbolized capital Greek letters, Φ, Ψ, X, K, Θ, *etc.*

Whitehead then introduced the concept of *covering*: one Abstractive Class *covered* another Abstractive Class if every member of the former enclosed some member of the latter. *covering* was an antisymmetric relation between Abstractive Classes, and could be used to define equivalence classes of Abstractive Classes, each of which both *covered* and was *covered by* the others. By factoring out these equivalence classes, he obtained *covering* as an ordering relation over equivalence classes of Abstractive Classes. Points were then to be defined by a minimality condition: a point was an equivalence class of Abstractive Classes that were *covered by* other equivalence classes of Abstractive Classes, but did not themselves *cover* any. Granted that points could be defined as minimal entities with zero dimensionality, Whitehead could then go on to characterize lines, surfaces and higher-dimensional entities, as limit elements whose equivalence class of Abstractive Classes *covered*, but were not *covered by*, the limit elements of lower dimensionality.

IV Failure

But Extensive Abstraction does not work in mereology. Besides the genuinely "punctiform" Abstractive Classes envisaged by Whitehead, there are other "pathological" ones, which also fit the definitions but do not satisfy the minimality condition. The set of open intervals $(0, \delta_n)$, where δ_n tends to 0, satisfies Whitehead's definition, but fails to characterize the point 0 uniquely, since the set of open intervals $(-\delta_n, 0)$, also converges to 0, although no member of either set encloses any member of the other. Nor is this peculiar to one-dimensional cases. In the example below there are successive members of an Abstractive Class sharing some part of their boundaries, so that they converge to a point that is on the boundary of the regions that are members of the Abstractive Class.



There is thus no unique Abstractive Class which can be defined as converging to a point. hus, although points can be defined, they cannot be identified uniquely.

Whitehead failed because his Abstractive Classes did not have reliable minimums. Although a genuinely punctiform Abstractive Class might converge to a point, Whitehead could not pick it out by its being a minimum. For, like Abstractive Classes that converge to a line or to several points, it *covered* other Abstractive

Classes which it was not *covered by.* Although genuinely punctiform Abstractive Classes differed from all these by not *covering*, though *being covered by*, them, the criterion could not be used, since genuinely punctiform Abstractive Classes, likewise *covered* pathological punctiform Abstractive Classes but were not *covered by* them.

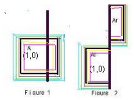
The underlying reason for the failure on minimality is that in mereology the principle of extensionality holds not only for equality, =, but for the relation *proper part of*, <, as well. The principle of extensionality in mereology is like the thesis of extensionality in set theory. Just as the latter lays down that only the *constituents* (here represented by *subsets* rather than *members*) of a set matter, and not the intension—the way they are described—so too in mereology only portions matter. And in both disciplines the principle of extensionality is itself extended to the fundamental relations, *subset of* and *proper part of*, as well as *equals*, =. The extended principle of extensionality[6] lays down that only parts matter, and not how they are related in any other respect—a canon of mereological meritocracy. In that case, besides the two-way implication noted above, which defines equality, we have the one-way implication, characterizing *proper or improper part of*, <,

If every portion of x is a portion of y, then x is either a potrion of *y or equal to it*. It is the former alternative that allows shared boundaries, and precludes a merelogical definition of Russian dolls, as illutrated in the figure below, where every portion that *is a proper part of x is a proper part of y*, thus ensuring that *x is a proper part of y*, but allowing x and y to share a common boundary.

Bostock gives a formal proof that no adequate grounding of topology can be given in mereological terms alone. He considers a 2-dimensionsal Abstractive Class in which each member is an *inner part of* its predecessors, and which converges to the point $(1, 0)$. It can be subjected to a transformation that displaces all points to

[6] Called the Proper Parts Principle' by P. Simons, *Parts*, Oxford, 1987, ch.1, §1.4, p.28; in J.R.Lucas, *The Conceptual Roots of Mathematics*, London, 2000, §9.9, pp.253-255, I use the term 'converse transitivity from below'; but but 'extended extensionality' seems more appropriate in a Whiteheadian context.

the right of the vertical line $x = 1$ one unit upwards, while leaving the rest of the plane unaltered as in the diagram below:



Mereologically figure 2 is the same as figure 1; the relation *proper part of*, $<$, is preserved under the transforamtion. But topologically the figures are different. Hence topological relations cannot be defined in terms of $<$.

This is so. But was Whitehead really a mereologist in the making? Did he need to characterize his Russian dolls in terms of *proper part of*, $<$?

V Non-mereology

Rather than the mereological *proper part of* relation, Whitehead may have been thinking all along of the stronger *enclosed in*, or *inner part of* relation, which would guarantee that each shell of the Russian doll had some thickness throughout, and thus would exclude pathological Abstractive Classes. His term for the converse relation, *extends over*, which has been taken to mean the mereological *has as a proper part*, could be construed as extending over like a table-cloth, which needs to come down over the table on every side, and, as it were, enclose it. Such an interpretation would ensure that each shell of the Russian doll was nowhere of zero thickness, with the consequence that the minimality condition would not be subverted by pathological cases even more minimal than the genuinely punctiform Abstractive Classes needed to define and identify points.

Proper part of and *inner part of* compared

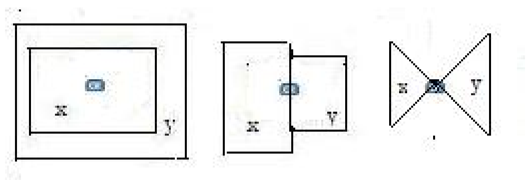
$\prec$ stands for a *part of* relation which could be either *proper part of*, $<$, or *inner part of*, $\ll$.
The definitions of overlap and equality are the same: regions x and y overlap iff there is some region that is *part of* both; $x \circ y \leftrightarrow (\exists z)(z \prec x \wedge z \prec y)$;
regions x and y are equal to each other iff every region z that is *part of* one is *part of* the other; $x = y \leftrightarrow (\forall z)(z \prec x \leftrightarrow z \prec y)$.
$\prec$ is irreflexive, $(\forall x)(\neg x \prec x)$;
transitive, $(\forall x)(\forall y)(\forall z)((x \prec y \wedge y \prec z) \rightarrow x \prec z))$;
dense, $(\forall x)(\forall y)(\exists z)(x \prec y \rightarrow (x \prec z \wedge z \prec y))$
serial downwards, $(\forall x)(\exists y)(y \prec x)$ not downward directed, but upward directed; $(\forall x)(\forall y)(\exists z)(x \prec z \wedge y \prec z)$

Inner part of, which can be symbolized by $\ll$, resembles *proper part of*, $<$ in being irreflexive, transitive, dense, serial downwards, not downward directed, but upward directed. *Inner part of*, $\ll$, is a stronger relation. For if x is an *inner part of* y, every z that is an *inner part of* x is an *inner part of* y, and so the complement of z, (symbolized by $\underline{z}$),[7] that is *disjoint from* y is *disjoint from* x; from which it follows that x is a *proper part of* y. If $x \ll y$, then $x < y$, but not necessarily *vice versa*; for crucially, conditions which imply $x \leq y$ need not imply $x \ll y$.

It is not enough, however, simply to omit extended extensionality as a condition that the relation being an *inner part of* must satisfy, since often the condition is satisfied, sometimes vacuously and sometimes in a straightforward way; and though an informal discussion of the critical cases is illuminating, it does not yield an adequate characterization. Intuitively the condition which would ensure that boundaries never merge is that there are always some granules, as it were ball-bearings, between them, to keep them apart. Such granules are postulated by R.L. Moore.[8] But Whitehead would want to establish them by honest toil. This can be done, but it is simpler to make a negative approach , in terms of lack of connectivity, so that we can stipulate that erf x is an *inner part of* y, then x is not *pointwise connected with* $\underline{y}$, the complement of y. For that we need a discussion of pointwise connection.

VI Connectedness

Whitehead's concept of *covering* provides a definition of *pointwise* connectedness. If x and y are pointwise connected, then there is a region w which is a member of an Abstractive Class Φ, with each subsequent member overlapping both x and y.



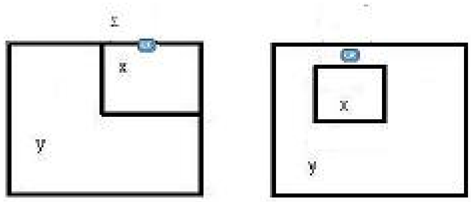
Three examples of x and y being pointwise connected.

[7] Whitehead does not specify what is the universe of discourse in which *extends over* operates, and hence what absolute complements are. But it is enough to consider complements relative to a suitably large region.

[8] R.L.Moore, *Fundamenta Mathematicae*,"A Set of Axioms for Plane Analysis Situs" **25** 1935, pp.13-28.

But w may be disconnected. If w is disconnected, there can be other nested sequences in some disconnected bit of w. It is here that we make use of Whitehead's concept of *covering* to rule out such pathological cases. If the Abstractive Class Φ converges to two or more points, it *covers* more than one genuinely punctiform Abstractive Class Ψ that converges to just one point. Ψ does not *cover* Φ, although it is *covered by* it. So Ψ can be identified as a Abstractive Class every one of whose members overlaps both x and y, and which *covers* every Abstractive Class it is *covered by.* This is a minimality condition. It works for Abstractive Classes of regions each one of which is an *inner part of*, $\ll$, its predecessors, because then there can be no sharing of boundaries, as can occur with a Abstractive Classes of portions each one of which is a *proper part of*, $<$, its predecessors, The latter possibility subverts minimality, since for every Abstractive Class of portions there is another Abstractive Class of portions it *covers*, but is not *covered by.* Once that possibility is excluded, as is the case with Whitehead's Russian dolls, we can distinguish genuinely punctiform Abstractive Classes from pathological ones, and use them to define *pointwise* connectedness.

The characteristic condition which distinguished *inner part of* from *proper part of* is that if x is an *inner part of* y, then x is not *pointwise connected with* $\underline{y}$, the complement of y. We can reformulate this more positively as the characteristic feature of two regions, one being an *inner part of* the other, is that nowhere do their boundaries come together.



On the left, x being a proper part of y, but not an inner part, has a genuinely punctiform Abstractive Class straddling its boundary with $\underline{y}$, the complement of y: on the right x being an inner part of y, always has genuinely punctiform Abstractive Classes between its boundary with y and y's boundary with $\underline{y}$. If it did not, x would be it pointwise connected with $\underline{y}$, the complement of y.

VII Formal Presentation

There are three primitive terms: regions, denoted by lower case letters, x, y, *etc.*, a two-place relation between regions, $\ll$, and Abstractive Classes, sets denoted by capital Greek letters, Φ, Ψ *etc.*

Definitions

D1. $x \circ y$ iff $(\exists z)(z \ll x \wedge z \ll y)$;
D2. $x = y$ iff $(\forall z)(z \ll x \leftrightarrow z \ll y)$.
D3. $x \mid y$ iff $\neg(\exists z)(z \ll x \wedge z \ll y))$.
D4. Φ *covers* Ψ iff $(\forall x)(x \epsilon \Phi) \rightarrow (\exists y)(y \epsilon \Psi \wedge y \ll x))$
D5. xPy iff $(\exists \Phi)((\forall \Psi)(\Phi \textit{ covers } \Psi) \rightarrow (\Psi \textit{ covers } \Phi) \wedge ((\forall w)((w \epsilon \Phi) \rightarrow (w \circ x \wedge w \circ y)$
D6. $\underline{x}$ iff $(\forall y)((y \mid \underline{x}) \rightarrow (y \circ x) \wedge ((y \mid x) \rightarrow (y \circ \underline{x}))$
or
D6' $\underline{x}$ iff $(\forall y)((y \ll \underline{x}) \leftrightarrow \neg(y \ll x)$

Axioms

A1. $(\forall x)(\neg x \ll x)$
A2. $(\forall x)(\forall y)(\forall z)((x \ll y \wedge y \ll z) \rightarrow x \ll z))$;
A3. $(\forall x)(\forall y)(y \ll x \rightarrow (\exists z)(y \ll z \wedge z \ll x))$
A4. $(\forall x)(\exists y)(y \ll x)$
A5. $(\forall x)(\forall y)(\exists z)(x \ll y \rightarrow (x \ll z \wedge z \ll y))$
A6. $(\forall x)((x \epsilon \Phi) \rightarrow (\exists y)((y \epsilon \Phi) \wedge (y \ll x)) \wedge \neg(\exists z)(\forall u)(u \epsilon \Phi \rightarrow z \ll u)$
A7 $(\forall x)(\forall y)((y \ll x) \rightarrow \neg y P \underline{x}$

An alternaive approach is by means of the axiom below A* $(\forall x)(\forall y)(x \circ y) \wedge (x \circ \underline{y}) \rightarrow (\forall \Phi)(\forall \Psi)((z \epsilon \Phi \rightarrow z \circ x \wedge z \circ y \wedge z \circ \underline{y}) \wedge (\Phi \textit{ covers } \Psi) \rightarrow (\Psi \textit{ covers } \Phi))$,
which rules out the pssibility of disconnected regions, and resticts the range of $\ll$ to *wholes*. It would then be possible to follow the more intuitive line, and specify (rather than simply postulate, the condition that if $y \ll x$, any whole that overlapped both y and $\underline{x}$ would have within it a granule that did not overlap either.

VIII Boolean Perspectives

Though various terms have been used to denote the entities over which the relations *inner part of* and *proper part of* range—extensions, events, regions, portions, parts and wholes—these terms are needlessly specific. Whitehead's programme requires only a Boolean basis, enriched by one ordering relation, $\ll$. *Principia Mathematica* was already committed to entities obeying a Boolean Algebra—-propositions. The propositional calculus, as formulated in the first edition, was based on two Boolean operators, $\neg$ and $\vee$, and it can, as was noted in the second (1927) edition,[9] be based on Sheffer's stroke connective and just one axiom, formulated by Nicod. So a generalised treatment of Boolean entities, based on a relation specified in terms of $|$ would be unexceptionable, and clearly within the compass of a logicist programme. Instead of using $\supset$ and *modus ponens*, one could parallel the definition of $<$ in Bostock's presentation of mereology, and introduce inference, $\models$, by the rule:

If α, β and γ are well-formed formulae,
and whenever $\beta \mid \gamma$ for every γ, $\alpha \mid \gamma$ then $\alpha \models \beta$.

(Admittedly there are important differences—although $\models$ in propositional calculus is, like $<$ in mereology, upward directed, it is not downwards serial, nor dense.)

In any case, a volume IV of *Principia Mathematica* written now, a century later, might well be different. Instead of postulating a separate set of axioms for point-set topology, it would be more in accord with the ambitions of logicism to base it on Kuratowski's S4-like axioms for closure.[10] This would be natural. If propositional calculus is to be enriched in any way, the minimal addition is a monadic operator with formation rules like those for $\neg$, and some axioms governing its interaction with existing Boolean operators. If certain constraints of economy and inferential transparency are observed, and collapse into modal vacuity avoided, the resulting system must conform to the minimal modal logic known as K (for Kripke) or G (for Gödel). There may, in addition, be rules for iterated modal operators, the simplest being the characteristic axiom for S4, which lays down that $\Box$ should be idempotent, that is, that $\Box\Box p \leftrightarrow \Box p$. S5 is a stronger system, dealing also with iterated modalities separated by negation, $\neg$, reducing any such sequence to the last one. The relation between these two systems is revealed

[9] Which cited H.M.Sheffer, *Transactions of the American Mathematical Society*, Vol XIV, pp. 481-488; J.Nicod, *Proc. Cam.Phil.Soc.* vol XIX.

[10] For comparable approaches see K.Menger, "Topology without Points", *Rice Institute Pamphlets*, **37**, pp.80-107; and P.T.Johnstone, "The Point of Pointless Topology", *Bulletin of the American Mathematical Society*, **8**, 1983, pp.41-53.

by Kripke's semantic models. S4 has an accessibility relation that is transitive, S5 by one that is transitive and symmetric, and thus an equivalence relation, portraying necessity as being true in all possible worlds. We should also note that S5 operators may be iterated in a different way, each being indexed to operate on some, but not necessarily all, of the propositional variables. This yields the First-order Predicate Calculus, with the propositions becoming propositional functions, and the indexed modal operators becoming quantifiers.

Principia Mathematica was committed to S5-like quantifiers, So Kuratowski's S4-like development of topology would be within its logicist programme,[11] and would have the further merit of explaining why the standard axioms for point-set topology have to distinguish between finite and infinite intersections of open sets, and between finite and infinite unions of closed sets. The Barcan formula is valid in S5 quantifed modal logic, allowing the order of universal quantifiers and necessity operators to be interchanged, because quantifiers are S5-like; but in S4 quantified modal logic the two-way implication does not hold, because the modal operators are not S5-like, as they need to be if they are are to discriminate between regions and their complements. If every intersection, infinite as well as finite, of open sets were open, and every union, infinite as well as finite, of closed sets were closed, every open set would also be closed, and every closed set would also be open.

Whitehead's Russian dolls would still have a place in a Twenty-first Century volume IV of *Principia Mathematica*, since they could define points, instead of merely postulating them. But instead of the somewhat cumbersome specifiction of *an inner part of*, $\ll$, given in Section VI, they might avail themselves of the intuitively simpler specification that the closure of each member of the Abstractive Class was a *proper part of* the interior of its predecessor.

IX Programmatic Notes

If Whitehead had been able to make Abstractive Extension work, how would he have proceeded? Two feature of lines—linearity and

[11] Some care is needed, because the topological analogue of implication, $\rightarrow$, is being-a-subset-of, $\subseteq$, not being-a-superset-of, $\supseteq$. Since the interior of a set is always included in it, and a set is always included in its closure, $\Box$ has to be compared with the interior operator, and not, as might have been supposed, the closure operator.

continuity—would have distinguished them from limit elements consisting merely of a multiplicity of points. First, the limit elements *covered by* a line would be linearly ordered, whereas those *covering* a mere multiplicity of points would not; that is, given any one "anchor" point, one of the lines *covering* both it and one of two other different points, would *cover* the other line that *covers* the anchor point and the other different point. And secondly, given any minimal linear limit element *covering* two distinct points, the set of all those linear limit elements *covered by* it, and *covering* a particular one of those points would have a least upper bound that was itself *covered by* the line. Lines could thus be characterized as *covering*, but not being *covered by*, points, and being linear and continuous.

Straightness would be more difficult. The Greek definition, that a line lies evenly on itself, might be used: a straight line is invariant under translation along itself or rotation around itself. But in view of his leanings towards empiricism, Whitehead might have made use of the practical procedure for fashioning an optical plane, by grinding three planes each against the others, until all are smooth. The rationale is that although any deviation from perfect flatness on the part of any one plane could be matched by a contrary deviation on one of the others, when each was ground against the third, the two deviations would cancel out. Mathematically, we are dealing with continuous one-one mappings that are reflections in a plane; any deviation of magnitude x is carried into a deviation of $-x$; and whereas $(-) \times (-) = +$, $(-) \times (-) \times (-) \neq +$ generally, and $-(-(-x)) = x$ only if $x = 0$.

Straight lines lead naturally to projective geometry, but Desargue's theorem in not a theorem in two-dimensional plane geometry, and although non-Desarguian projective geometries are consistent, they are are not nice, and a logicist, as opposed to a formalist, programme would want to exclude them. Rather than postulate Desargue's theorem as an extra axiom, it would be better to embed two-dimensional plane geometry in a three-dimensional space, where Desargue's theorem *is* a theorem.

Further specification of affine and Euclidean geometry re possible without postulation. Whitehead's empiricist bent might have led him to pick out Euclidean geometry as the one that was invariant under translation, rotation and reflection. Events then could be recognised from different points of view, and could occur in different locations. A more abstract characterization depends on the insight of Wallis and Saccheri that only in Euclidean geometry does size vary independently of shape. Euclidean geometry gives us the greatest freedom to fit it to actual facts on the one hand and pragmatic aims on the other.

But these are mere speculations. We cannot say how, in all its painful detail, volume IV of *Principia Mathematica* might have been written a hundred years ago, nor need we elaborate how it should be written now. It is enough to indicate how a logicist programme to base mathematics on a few simple logical principles was, and is, feasible.