Chapter 9B
Transitive Relations

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§9B.1 Relations

Once predicate calculus is allowed to be part of logic, the study of relations, which are symbolized by dyadic and polyadic predicates, must be recognised as part of logic too. In particular, equivalence relations, ordering relations, one-one, and many-one, relations are of central importance. Frege made use of the relation of equinumerosity to show that different numerical expressions referred to the same thing, the same natural number.[1] Dedekind characterized the counting numbers in terms of their order.[2]

Relations can be characterized in different ways. Many-one and one-one relations give rise to functions, and in particular to mappings, themselves establishing homomorphisms and isomorphisms. These will be discussed in §9.3, after considering the equivalence relations they characteristically generate. **Equivalence** relations are **transitive** and **symmetric**, and it is in terms of these features that relations are most illuminatingly classified. We can have relations that are symmetric but not transitive, such as *spouse*, *other than*, and *different from*; but of greater interest are the **ordering** relations, that is those that are transitive and asymmetric.

Query

Could there be a transitive, non-symmetric, *non*-reflexive relation?

[1] See above, §4.7.

[2] See above, §5.2.

We can further disjoin an equivalence relation with an ordering relation to form an **antisymmetric** relation, such as *being the same age as or older than.* Antisymmetric relations are transitive and non-symmetric. They are characterized by the three axioms,

1. $(x \lesssim y \wedge y \lesssim z) \rightarrow x \lesssim z$
2. $x \lesssim x$
3. $(x \lesssim y \wedge y \lesssim x) \rightarrow x \approx y$

Conversely, we can define both equivalence and ordering relations in terms of antisymmetric relations. It is tempting to do so, both for the sake of economy, and because the logicians' $\rightarrow$ and entailment, $\vdash$, are antisymmetric.[3] Nevertheless, it would be an unwise course. We should be wary of a logical economy that is not reflected in our ordinary understanding. The officialese *on or before* sounds awkward because we do not naturally disjoin the two types of relation. For the sake of clarity, too, it is better to distinguish them. It is dangerously easy to confuse "strict" orderings, characterized by means of asymmetric irreflexive relations, with "quasi" orderings, characterized by means of antisymmetric relations. And finally, in particular and conclusively, we need to take every reasonable step to secure ourselves against confusing identity, =, with the more general equivalence relation $\approx$.[4] For these reasons we shall, where possible, avoid antisymmetric relations in favour of equivalence relations and ordering relations, properly so called.

For a typology of possible relations see Appendix III, p.270

[3] It is unfortunate that the standard logical symbolism, $\rightarrow, \supset$ and $\vdash$ obscure this, and half suggest that they are asymmetric. But it would be too cumbersome and finicky to write $\vec{\sim}$, or $\vec{-}$, *etc.* in general, though we shall in §9.11 where confusion is particularly likely.

[4] See below, §9.2.

§9B.2 Equivalence Relations

Equivalence relations are much used in modern mathematics. They are used to introduce new, abstract entities, as they were in Chapter Three to explicate rational and negative numbers in terms of ordered pairs of natural numbers.[5] The theory of groups can be seen as a natural generalisation of equivalence relations, as also, in a different way, category theory. Much of the theory of measurement depends on equivalence classes, and they will be needed when we develop an account of magnitudes.

Equivalence relations are transitive. They differ from ordering relations in that whereas an ordering relation is transitive, asymmetric and irreflexive, an equivalence relation is transitive, symmetric and reflexive. More formally we may define a relation R as being an *equivalence relation* iff

1. If xRy and yRz then xRz, and
2. if xRy then yRx.

It follows, in any domain in which an equivalence relation holds at all, that it is reflexive; that is

3. xRx.

An equivalence relation is often written $x \sim y$, or $x \approx y$, or $x \equiv y$.

There are many equivalence relations: *contemporary, colleague, compatriot, co-religionist, comrade*; that is, *being the same age as, being in the same institution as, being a citizen of the same country as, having the same religious affiliation as, being in the same situation as*. Equivalence relations capture the sense of 'being the same as', 'being similar to', 'being like'. Often there is a special word, usually constructed from the Latin, *co-*, *com-*, or *con-*, but we can make any equivalence relation we want by specifying the respect in which we are the same: I can ask whether I am the same height as you, or the same weight as you, although, since these are not scientifically, socially, morally, or emotionally, important concepts, we do not have special words to say that I am co-altitudinous, or componderous, with you.

It follows that we can treat equivalence relations in two ways. We can treat each one individually, call it R, or S, or T, and say that R, or S, or T, is a dyadic two-term relation which happens to be an equivalence relation, *i.e.* it happens to be the case that R, or S, or T, is transitive, symmetric and reflexive, but this is, so to

[5] §3.7.

speak, a one-off property of the particular relation under consideration. Alternatively we can consider the whole family of equivalence relations, recognising that there are particular members, such as R and S and T, but concentrating our attention on the family characteristics. In that case, rather than pick on a particular R, S, or T, we use some generic symbol, such as $=, \sim, \approx, \equiv$. Unfortunately logicians often forget that $\approx$, or any other sign of equivalence, is an incomplete symbol, unless we indicate, either contextually or explicitly, which equivalence relation is under consideration. The same is true of *‘ is the same as ’*. If you are asked if you are the same as me, you do not know how to answer. You say ‘Am I the same *what*?’. Once you are told—‘the same age’, ‘the same height’, ‘the same sex’—you can answer the question, but if the respect in which we might be the same is left unspecified, you can only say ‘I am the same as him in some respects, and different from him in others’. We can express this by saying ‘— is the same as . . . ’ is not, as it seems, a dyadic, two-term relation, but a triadic, three-term one: ‘— is the same as . . . in respect of - - -’. Similarly, when we formalise, we should not say simply

$$x \approx y$$

but

$$x \approx_r y$$

where the subscript $_r$ indicates the respect in which x and y are said to be equivalent. We use a lower-case letter to indicate that it is referring to a respect, and not directly to a relation. That is to say,

$$x \approx_r y$$

is to mean the same as

$$xRy$$

where R is an equivalence relation; and although it might be tempting to write the former

$$x \approx_R y,$$

to do so would slur over the distinction between relations and respects. If R stands for the particular equivalence relation ‘— is contemporary with . . . ’ then r stands for ‘age’: if I am contemporary with you, I am the same age as you; but ‘age’ is of a different logical type from ‘contemporary with’.

The fact that equivalence relations are triadic seems obvious enough, but we often overlook it, and then are easily misled. If the question is asked whether I am the same height as you, it is not enough to consider you and me and whether we are the same or not: the question is whether we are the same *height*, and not whether we are the same weight, same age, or same nationality. Once the respect is specified, we have a particular equivalence relation which is dyadic, transitive and symmetrical, and can be handled quite easily. But if it is not specified, confusion abounds. In political discourse egalitarians often argue that since all men are equal in some respects—we all are men, we all are mortal, death is the great equaliser—they are equal without qualification, and so ought to be treated the same in every respect, and be given the same income, the same education, the same opportunities, the same everything. If we always specify respects, such fallacies can be avoided. The facts that we are all men, all mortal, all sentient beings, all rational agents, are relevant facts, and are grounds for serious political argument, but the argument as normally stated or implied is invalid because it assumes what is really a triadic, three-term relation to be merely a dyadic, two-term one. Arguments are needed (which sometimes can be given) why sameness in one respect is evidence, or a ground, for sameness in some other respect. The only general thesis in politics we can properly propound is that any two people are the same in some respects, different in others.

Mathematicians are more careful. But it is useful even for them to distinguish different sorts, and even different degrees, of sameness. On the whole we distinguish being the same as from being similar to by using the latter where the points of resemblance are few and the former where they are many. Similar triangles have their angles the same but not their sides. If their sides also are the same, and their orientation too, we use a stronger word, in this case 'congruent', with the slightly weaker 'counterpart' for two triangles with their angles the same and the lengths of their sides the same but of opposite parity, that is with one being the mirror image of the other. The extreme case of being the same in all respects, is identity. But although we can be happy to say that a is identical with a, we have difficulty as soon as we use different names for one and the same thing, for someone might then know one of them but not the other, and not know, for example, that Tully was identical with Cicero; in which case there would be some property of Tully—*e.g.* being known as the author of Tully's *Offices*—not possessed by Cicero. Clearly something has gone wrong, and we exclude such "referentially opaque" properties. But it is difficult to draw the line, and often we say that a is identical with b, in the weaker sense of qualitative identity rather than strict numerical identity. We need, therefore, to be careful in saying that a is the same as b in *all* respects, for there is considerable unclarity as to what exactly constitutes a respect, and therefore how far 'all respects' extends. For the present it is enough to require that the respects be specified, and to note that for the most part 'being similar to' implies that there are fewer respects in which resemblance is claimed than 'being the same as'.

Further snares lie ahead. Perhaps I went to the same school as you, and you went to the same school as Peter: it does not follow that I went to the same school as Peter. You went to different schools, an early nursery school where you romped with me, and a big proper school where you got to know Peter and other people who would be useful to you in later life. Schools are not like weight or height. Mathematical physicists sometimes are confused because they think that simultaneity is like being the same weight as, instead of seeing that being simultaneous in one frame of reference is like possessing characteristics shared with fellow-Wykehamists, and being simultaneous in another frame of reference is like possessing characteristics shared with fellow-Etonians. 'At the same school as', or 'at the same time as', denotes not one but a whole family of equivalence relations, each picking out a different equivalence class manifesting different common characteristics.[6]

There are many, many possible equivalence relations. At one extreme we have identity (normally written =, though we could express it as $\approx_i$) which each individual bears to itself alone. At the other we have the universal relation, which each individual bears

[6] For fallacious arguments involving the concept of simultaneity in the Special Theory, see H.Putnam, "Time and Physical Geometry", *Journal of Philosophy*, **64**, 1967, pp. 240-247; reprinted in H.Putnam, *Mathematics, Matter and Method. Philosophical Papers*, I, Cambridge, 1979, pp.198-205; C.W.Rietdijk, "A Rigorous Proof of Determinism Derived from the Special Theory of Relativity", *Philosophy of Science*, **33**, 1966, 341-344, and "Special Relativity and Determinism", *Philosophy of Science*, **43**, 1976, pp. 598-609; John W. Lango, "The Logic of Simultaneity", *Journal of Philosophy*, **66**, 1969, pp.340-350. See further fn.38 in §9.8 below

to every other. We can use a quality, expressed by a monadic predicate, to generate an equivalence relation. If $Q(x)$ and $Q(y)$, then we can say $x \approx_q y$, that is, that x and y resemble each other in both possessing the property Q.[7] Normally we use a determinable, sex, colour, shape, and say that I am the same sex as you, rather than that I am the same as you in that I am male, but that locution is possible, and is sometimes insisted on. Indeed, given any class of individuals, each of which possesses some property in common, we can define a corresponding equivalence relation. This, in effect, is what is done when it is argued that all men are equal because they all are men. They are all equal to one another *in respect of* humanity. Given any class, say that of prime numbers, we can characterize the members as bearing an equivalence relation, that of "equi-primeness" to one another. We can express this by saying $x \approx_p y$ iff x is prime and y is prime, and there is no objection to our doing so, provided we take care not to drop the subscript p, and assume that since x is equi-prime with y, x is equal to y. In this case there is only one equivalence class. We could, at the cost of considerable artificiality, take several, quite different classes, and provided they were mutually disjoint and jointly exhaustive, construct an equivalence relation which would generate just that partition. In the domain of natural numbers we might partition the composite numbers by reference to their smallest factor—thus having all the even numbers except two, then all the odd numbers divisible by three except three itself, then all the remaining numbers whose least divisor was five, except five itself, and so on. It is an instructive exercise to define the corresponding equivalence relation. Other exercises can be constructed, but soon become tedious.

Given any equivalence relation, $\approx_a$, *being contemporary with* we can (in this case only roughly, with considerable ambiguity over marginal cases) separate a population into "age-groups", that is, people who are all contemporary with one another. These age-groups are equivalence classes. All the members of a particular age-group have something in common—namely their age. Age is just that which all members of a given age-group have in common. It is the common property of each equivalence class. Similarly with weight, height, and a whole range of other abstract entities. Weight

[7] P. Simons, *Parts*, Oxford, 1987, ch.9, §9.3, p.335, uses the symbolism $xFFy$ if their common bond is the fact that $Fx \wedge Fy$

depends on balancing. Balancing, if done properly, is a symmetric, and within reasonable limits, a transitive relation. It therefore can separate out equivalence classes of bodies which balance against one another, and each such class is said to be a class of things "having the same weight".

In this way age, or height, or weight, or rational number, can be generally defined. If we have a particular equivalence relation $\approx_r$, then the equivalence relation will divide up any domain, X, of individuals into a "partition", as it is called, of mutually disjoint and jointly exhaustive equivalence classes: $A, B, C, ...,$ *etc*. So that

if x is a member of A, then y is a member of A iff $x \approx_r y$.

Instead of introducing new letters, $A, B, C, ..,$ *etc*., we can define the equivalence class of any particular individual, say x, which we write $[x]_r$ and say that y is a member of $[x]_r$ iff $x \approx_r y$. Thus in ordinary speech instead of saying that Bernard is *contemporary with* me, I can say that he is *a contemporary of mine*. Once we have partitioned a domain into equivalence classes, we can distinguish the generic respect from particular specifications of it. If 'is contemporary with' is given by the relation T, then 'being the same age as' will be expressed

$$x \approx_t y$$

and there will be as many distinguishable age-groups when the domain of human beings is partitioned by the relation $\approx_t$ into equivalence classes. So t would be age generally, and particular ages would be indicated by $[x]_t$ where $[x]_t$ and $[y]_t$ refer to the same particular age if $x \approx_t y$, and to different ages if $\neg(x \approx_t y)$.

Each specific equivalence relation generates a set of mutually disjoint and jointly exhaustive equivalence classes, each one of which can be seen as having some property in common, and the whole family of equivalence classes as being particular specifications of some generic respect. We can then consider the set of equivalence classes as a set in its own right, which we can write $\langle X/_{\approx}\rangle$. Such a class is called a "quotient class". When we come to measure physical magnitudes in Chapter Eleven, we shall need to establish some underlying equivalence relations which tell us what things are equal, as regards weight, or duration, or length, or angle, and then form the quotient class, and establish a linear order between different weights, or durations, or lengths, or angles. We shall look at the conditions under which this can be done in §9.8.

§9B.3 Functions

Functions are normally regarded as definitely mathematical, rather than logical, concepts. But they can be explicated in terms of many-one relations, and thus have logical roots. A function assigns to each value of the "argument" a unique resultant value. Usually both the arguments and the resultant value are numbers, but they do not have to be: truth-functions map truth-values into truth-values.

A function corresponds to a relation between the argument(s) and the resultant value, with the proviso that the resultant must be unique. That is, if xRy and xRz, then $y = z$. There is no similar restriction on the arguments: if xRy and zRy, then it does not follow that $x = z$. Often there is more than one argument: we can easily have a function from the ordered pair $(x; y)$ to some resultant value. The resultant value can also be an ordered pair, but that is less common.

Functions are essentially many-one relations, and many includes one: that is, we consider as functions those relations, where not only if xRy and xRz, then $y = z$, but the other implication, if xRy and zRy, then $x = z$, does also hold. These are one-one, or "bi-unique", functions. They evidently have inverses, since if y bears the relation R to x, x bears the converse relation R^{-1} to y, and is unique in doing so.

Functions are transitive. If x bears a many-one relation R to y, and y bears a many-one relation S to z, then x bears some many-one relation to z, since to each x there is a unique y, and to each y there is a unique z, so that to each x there is a unique z.

Most naturally we think of functions of functions as constituting a discrete relationship,[8] but that does not have to be so, and we can have families of functions, $f_n(x)$, where n is a dense or continuous parameter. The best example is the causal transformation over time t, of a system of particles, where the nth particle has position q_{nx}, q_{ny}, q_{nz}, and momentum p_{nx}, p_{ny}, p_{nz}, and there is a general law of evolution, giving the position and momentum of each after time t, granted that of them all initially. This can be seen as a function of phase space into phase space, but is parametrized continuously rather than discretely.

Functions give rise to "morphisms". The simplest case is Frege's *gleichzahlig*, 'equinumerous'. In that case we are concerned only

[8] See below, §9.6.

with quotity, the question 'How many?', and know that if a one–one function holds between one set and another, the answer must in each case be the same. More generally, we consider not only a set of individuals, X, but various properties, monadic—qualities—and polyadic—relations—that they have. Two "relational structures", $\langle X, Q, R\ \rangle$ and $\langle X', Q', R'\ \rangle$, are "isomorphic" if and only if there is some one–one function f from X onto X' such that if $x' = f(x)$, and $y' = f(y)$, then $Q'(x')$ if and only if $Q(x)$, and $R'(x', y')$ if and only if $R(x, y)$. The definition can be generalised to have more than one quality Q, more than one relation R, and further relations which are not merely dyadic, but obtaining between three or more individuals; it is convenient also to include functions separately from relations, so that we consider $\langle X; Q_1, Q_2, \ldots; R_1, R_2, \ldots; f_1, f_2, \ldots \rangle$ being isomorphic to $\langle X'; Q'_1, Q'_2, \ldots; R'_1, R'_2, \ldots; f'_1, f'_2, \ldots \rangle$. The underlying idea is that the one–one function not only correlates individuals, but maps features, so that if any two individuals share a feature, or stand in some particular relation to each other, then their opposite numbers will also share some corresponding feature, or stand in some corresponding relation to each other.

Besides isomorphisms, we can consider the particular case where the function is from a set onto itself: this is called an automorphism. We also consider, the more general case, with many–one, instead of one–one, functions. Such functions do not have inverses, and the morphisms they give rise to are called homomorphisms instead of isomorphisms. The definition of a homomorphism is like that of an isomorphism, except that the 'if and only if' is weakened to an 'only if'. Two relational structures, $\langle X, Q, R\ \rangle$ and $\langle X', Q', R'\ \rangle$, are homomorphic if and only if there is some many–one function f from X onto X' such that if $x' = f(x)$, and $y' = f(y)$, then $Q(x)$ only if $Q'(x')$, and $R(x, y)$ only if $R'(x', y')$.

Every relational structure is isomorphic with itself, but typically morphisms hold between relational structures that are not identical, and then the question arises as to what the interrelation between function and sameness is: what functions preserve what samenesses, and *vice versa*? We can see relational structures as the natural generalisation of sets, and isomorphisms as the generalisation of cardinal number. Morphisms pick out deep similarities between relational structures, and these can be regarded as the underlying object of mathematical concern.

Category theory adopts this very abstract approach, dealing with very general features, and very general classes of functions and transformations.[9] On this approach the fundamental objects of mathematical interest are structures, rather than objects. Similarity of structure is often elusive and difficult to pin down—like covariance rather than simple invariance: category theory encourages us to view mathematics not as about abstract entities, but as revealing the interconnectedness of things. "Category theory is like a language in which the 'verbs' are on an equal footing with the 'nouns'."[10] In modern times we can see Dedekind, with his ordinal characterization of the natural numbers, as a precursor;[11] though the thought that mathematics is a science of structures goes back to Plato, with his study of παραδείγματα (*paradeigmata*), ideal patterns.[12]

§9B.4 Identity in Difference

The search for underlying samenesses raises problems. Equivalence relations generate equivalence classes all of whose members are, in the relevant respect, the same, and thus seem to obliterate individuality. We therefore need to retain identity, and consider the interplay of *two* equivalence relations, combining a general $\approx_r$ with the identity $=$. We may, for example, think of the ways in which a regular shape, say an ice crystal or hexagon, can be moved, rotated, or reflected so as to look exactly the same.

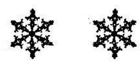


Figure 9.4.1 The figure on the left looks the same as the figure on the right

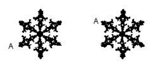


Figure 9.4.2 The figure on the left is not identically the same as the figure on the right

[9] For a brief account, see J.L.Bell, "Categories, Toposes and Sets", *Synthese*, **51**, 1982, pp. 293-337; or, even briefer, his *Toposes and Local Set Theories*, Oxford, 1988, ch.8, Epilogue, pp. 235ff.

[10] J.L.Bell, *Toposes and Local Set Theories*, Oxford, 1988, p.236

[11] See above, ch.5, esp. §5.2.

[12] See further below, §14.3, §15.6.

Thus the figure on the left in Figure 9.4.1 above looks the same as the figure on the right and under the relation $\approx_l$ would be placed in the same l-equivalence (looks-equivalence) class. But if we also have the relation of identity, and can identify the individual elements of the structure, we may distinguish them. This is why, in giving our account of geometries in Chapter Two, it was not enough to characterize them in terms of the Euclidean or some other group (§2.7), or define them only implicitly by a set of axioms (§2.3), but needed also to give some independent characterization of points and lines (§2.5).

In this way we are preserving individual identity in the l-equivalence class, and can distinguish between different members that are l-equivalent. In the theory of groups, the operators of any group always combine to form another operation of that group; this ensures that taken as a whole they are transitive: to every operator there is an inverse; this ensures that they are symmetric: and, in consequence, there is always an identity operator in the group; this secures reflexivity. Any set of functions, transformations, operations, or mappings, which is such that

1. any two can be compounded to form a third,
2. the associative law holds,
3. there is an identity,
4. to each one there is an inverse,

is called a "group". We can thus view the theory of groups as being a natural elaboration of our theory of equivalence relations, in which we have both a general equivalence relation and the most exclusive one of identity. A Hegelian would describe it as a development of the theme of identity in difference. A physicist would say it was giving the "fine-structure" of an equivalence relation. The equivalence relation expresses what is common to all those things that can be transformed into one another by transformations of the group: and each transformation of the group characterizes what is peculiar to the relation of one thing to some particular other thing. Thus the theory of groups should be seen as grounded in the logic of sameness and difference rather than in that of the quotifiers. Of course group theorists use numbers, and have special theorems about groups with a prime number of elements, and the sub-groups of groups with a divisible number of elements, and classify infinite groups separately from finite ones. But their chief concern is with distinguishable differences within some over-arching similarities. Plato might have portrayed the theory of groups as the progeny of the Form of the Like and Form of the Unlike.

Difference is more difficult than sameness. *Being different from* is a symmetric relation, like being the same as, but is non-transitive and irreflexive. As with equivalence relations, we can consider either particular difference relations, which are dyadic, two-term, or a general difference relation $\neq$, which is a triadic, three-term relation, and needs to have the respect in which the difference obtains specified, $\neq_r$ *etc.* But because they are non-transitive, difference relations are more difficult to treat generally than equivalence relations, and we concentrate on a subclass of them, those that are in fact transitive, that is to say, the ordering relations, to which we now turn.

§9B.5 Ordering Relations

Ordering relations are, like equivalence relations, fundamental. Dedekind used a special type of discrete ordering relation in his ordinal characterization of the natural numbers, and the magnitudes we measure are inherently ordered. The concepts of *finite* and *infinite*, as we saw,[13] and of *limit* and *boundary*, as we shall see, are to be elucidated in terms of some ordering relation, and so too are the neighbourhoods that form the basis of topology.[14]

Ordering relations are, like equivalence relations, transitive, but asymmetric and irreflexive, instead of symmetric and reflexive. All English phrases ' —er than', express relations that are irreflexive, and, with one exception, asymmetric and transitive. [The one exception is 'other than', which shows that the fundamental force of 'than' is irreflexive; so too the use of 'than' with 'else', and the Americans' urge to say 'different than'; the Greek ἤ (*e*) is the same, being used both after comparatives and after ἄλλος (*allos*).] The transitive asymmetric irreflexive relations expressed in the —er form are the standard, though not the only, way of expressing ordering relations. Either of the latter conditions implies the other. That is, an ordering relation may be defined as one that is transitive and irreflexive, and therefore asymmetric, or preferably, since more systematically, as one that is transitive and asymmetric, and hence irreflexive; more formally,

a relation R is an *ordering relation* iff

1. If xRy and yRz then xRz, and
2. if xRy then $\neg yRx$.

[13] §7.3.

[14] In ch.10.

It follows, in any domain in which an ordering relation holds at all, that it is irreflexive; that is

3. $\neg xRx$.

An ordering relation is often written $<$, or $>$, and read 'less than', or 'greater than'.[15]

Ordering relations are, like equivalence relations, triadic, and we need to specify, either implicitly in the context or explicitly by a subscript, the respect in which something is greater than or less than another—am I greater than you in weight, in height, or in age? So we should write

$$>_w, >_h, >_t, <_w, <_h, <_t, \ldots \textit{etc.}$$

Fortunately, however, we are better protected by linguistic usage from overlooking the respect in which things are ordered than the respect in which things are the same: fewer confusions arise, and so we shall be readier to drop the subscript. Where we need to distinguish only a few ordering relations, besides $\prec$, we can use $\ll$, $\prec\prec$, or $\subset$, to supplement $<$.

The converse of an ordering relation, unlike that of an equivalence relation, is a different relation. It is usually expressed by $>$, $\succ$, $\gg$, $\succ\succ$, or $\supset$. For the sake of uniformity, we mostly use the *less than* form. The converse of an ordering relation is necessarily not the same as the ordering relation itself, but it is often very similar, and has the "mirror image property",[16] whereby general features of $<$ are also features of $>$.

[15] E.V.Huntington, "A Set of Postulates for Real Algebra, Comprising Postulates for a One-Dimensional Continuum and for the Theory of Groups", *Transactions of the American Mathematical Society*, **6**, 1905, p.18, n., points out that *greater than*, and *less than* have connotations of quantity which are inappropriate here, and suggests that *below* and *above*, or *before* and *after*, are preferable. The point is well taken, though the spatial and temporal connotations of *below* and *before* can also mislead: *precedes*, symbolized by $\prec$, is perhaps the least loaded term. In this section we shall continue to use the familiar $<$, but hereafter, as we come to need the distinction between metrical and purely ordering concepts, we shall tend to use $\prec$.

[16] I take this term from C.L.Hamblin, cited by A.N.Prior, *Past, Present and Future*, Oxford, 1967, p.35.

Although grammatically the comparative is formed from the positive, the logical dependence is the other way about. The positive adjectives, such as ‘great’, ‘big’, ‘large’, ‘tall’, ‘old’, ‘fast’, ‘small’, ‘little’, ‘short’, ‘young’ and ‘slow’, are the simplest, and the comparatives, ‘greater’, ‘bigger’ *etc.*, are formed from them. Logically, however, the comparative is the basic form, and both the positive and the superlative are formed from the comparative.

Plato, when he first put forward the theory of forms, thought that every adjective denoted some abstract universal. ‘White’ denoted the colour white, ‘square’ denoted the square shape, and similarly ‘great’ denoted the property of greatness, ‘big’ of bigness, ‘little’ of littleness. But then there was a paradox. Simmias was big in comparison with Socrates, little in comparison with Phaedo: my fourth finger is big in comparison with my little finger, little in comparison with my middle finger: a very beautiful ape would be hideous as a human. Plato at first thought these puzzles showed the unreliability of the senses, which could easily be bamboozled into seeing inconsistent properties present in the same thing at the same time: but later realised his mistake, and that the puzzles were puerile, and evaporated once we recognised that the positives were back-formations from the logically more primitive comparatives. Words like ‘great’, ‘big’ and ‘little’ do not denote qualities or ascribe properties in the way that words like ‘white’ or ‘square’ do. To say that the middle finger is big is to say that it is *bigger than* most of the others: to say that Socrates is short is to say that he is shorter than most men. Only if the range of comparison is explicitly stated or evident from the context do we know what is meant by saying Socrates is short. He is short *for* a man, but not short for a boy: indeed he is taller than most boys. How much shorter than the average man or taller than the average boy someone needs to be in order to be short or tall is not clear. A man who was only very slightly shorter than the mean, or median, man in his group of comparison, would not be accounted short or tall. Statisticians might reckon that an individual needs to be more than one standard deviation beyond the mean to be accounted short or tall. But this is to impute more precision than ordinary men normally will admit.

Superlatives are relevant to ordering relations, and, more obviously than positives, derive from the comparative. The first, the last, the greatest, the least, the most, and in particular the next, play a crucial role in distinguishing the various global and local structures that different sorts of ordering have.

§9B.6 The Marriage of Equivalence with Order

We often have to consider ordering and equivalence relations in the same logical breath. Indeed, it is only when we could intelligibly speak of something's being greater or less than another, that we may properly speak of their being equal. In developing a theory of measurement, we characteristically use arguments of the form x balances against y, y is heavier than z, so x is heavier than z. We take *balancing against* to be an equivalence relation, and assume that it must be compatible with *being heavier than*, writing the one as $x \approx_w y$, and the other as $y >_w z$. But this is not in all cases so: proper set inclusion, we have already seen,[17] is not compatible with equinumerosity, and $\subset$ has seemed insignificant and has given way to $\subseteq$ in transfinite set theory. *Per contra*, if we are to retain some sense of *being less than* among equinumerous infinites, so that we shall be able to say that one is a proper part of the other, we must, as Gregory of Rimini observed,[18] distinguish a different sense of *being less than*, in which a part is, indeed, less than that of which it is a part; and this in turn will determine what sort of equivalence relations are compatible with it.[19] We need to think explicitly about the conditions under which the ordering and the equivalence relation involved are compatible with each other.

Equivalence relations can be based on order, just as they can on qualities, represented by one-place predicates.[20] In general, x and y are R-equivalent iff everything that stands in the relation R to x stands in the relation R to y, and *vice versa*. If R is an ordering relation, $<$, we can express it formally,

$$x \approx_< y \text{ iff } (\mathrm{A}z)((z < x) \leftrightarrow (z < y)).$$

This is evidently a symmetric relation, and easily shown to be transitive. The equivalence relation defined by $<$ is not necessarily the same as that defined by $>$. I have the same descendants as my wife, the same ancestors as my brother and sisters. Evidently

17 §7.2.

18 Gregory of Rimini, *Comm. Sent*, Lib 1, dist 42-44.Q.4:f.173^v; cited and translated in Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., *Cambridge History of Later Medieval Philosophy*, p.572.

19 See further below, §11.3 and §11.4.

20 See above, §9.2.

then $\approx_<$ is not the same as $\approx_>$ (which could also be written $\approx^<$). But, although the equivalence relation generated by a relation is not necessarily the same as that generated by its converse, very often it is, and we need the two to be the same if the equivalence relation is to be generally compatible with the ordering relation. Granted that

$$x \approx y \;\text{ iff }\; (\mathrm{A}z)((z < x) \leftrightarrow (z < y)) \;\text{ iff }\; (\mathrm{A}z)((z > x) \leftrightarrow (z > y)),$$

it is straightforward to show that

$$x \approx y \;\wedge\; y < z \rightarrow x < z$$

and

$$y < x \;\wedge\; x \approx y \rightarrow y < z.$$

The equivalence relation generated from an ordering relation is not the only compatible one. The identity relation, =, is always compatible with any ordering relation. It represents the limiting case, when the equivalence relation generated by an ordering relation is the most stringent possible; evidently it will be the same for both the relation and its converse, and fully compatible with both. Often there are other compatible equivalence relations: they play an important role in the theory of measurement, which we shall discuss in Chapter Eleven.

Granted a compatible equivalence relation, we can operate with a modified law of trichotomy. If I am neither older nor younger than you, it follows that I am the same age, but not that I am identically the same person. With age and height and weight and many other magnitudes, we do not have the strict law of trichotomy[21]

$$(\mathrm{A}x)(\mathrm{A}y)((x = y) \vee (x \prec y) \vee (x \succ y)),$$

but the modified one,

$$(\mathrm{A}x)(\mathrm{A}y)((x \approx y) \vee (x \prec y) \vee (x \succ y)).$$

The modified law of trichotomy does not always obtain, and errors arise from assuming that it does. It is illuminating to take an example from social life, because our emotions are involved, and we

[21] See above, §9.6.

see therefore in sharper relief how they are perceived. Many people are class-conscious, and classes are equivalence classes between which there is some ordering relation. At first we are inclined to think that the ordering relation is linear, that of any two people one must be superior to the other unless they are both of the same class. But are you the social superior, the social equal, or the social inferior of an American Congressman? Both you and he are the social inferior of the Queen, and the social superior of a taxi-driver at London Airport. But you would hesitate to claim or concede social superiority, and yet would also hesitate to admit him as social equal because social equals have something in common, and you have nothing in common with him. We are torn. One definition of equality is based on order and negative—not being either superior or inferior—the other is in terms of some positive feature shared by all who are equal. Some egalitarians are moved negatively, by a dislike of anyone's being better than anybody else, others are inspired positively by a sense of having much in common with others. A very similar confusion occurs in the Special Theory of Relativity. Instead of shifting from simultaneity in one frame of reference to simultaneity in another, a new concept of topological simultaneity is defined: two events which are neither before nor after one another—neither is in the light-cone of the other—are said to be "topologically simultaneous", and from this definition and the assumption that topological simultaneity is an equivalence relation, it is easy to derive bizarre conclusions about time, necessity and free will.[22]

Where the modified law of trichotomy does obtain, we can reduce it to the strict one by "factoring out" the equivalence relation $\approx$ and the equivalence classes it generates to form a quotient

[22] See references in §9.2, above, fn.6. Similar fallacies have been based on a concept of ET-simultaneity; see E.Stump and N.Kretzmann, "Eternity", *Journal of Philosophy*, **78**, 1981, pp.428-458; reprinted in T.V.Morris, ed., *The Concept of God*, Oxford, 1987, pp.219-252. For further discussion, see Howard Stein, "On Einstein-Minkowski space-time", *Journal of Philosophy*, **65**, 1968, pp.5-23; and "A note on time and Relativity Theory", *Journal of Philosophy*, **67**, 1970, pp.289-294; see also R.Sorabji, *Necessity, Cause and Blame*, London, 1980, pp.114-119; R.Torretti, *Relativity and Geometry*, Oxford, 1983, §7.3, pp.75-89; and J.R.Lucas and P.E.Hodgson, *Spacetime and Electromagnetism*, Oxford, 1990, §2.9, pp.65-67. See also below, §11.3.

class.[23] Clearly, if two members of the original set $\langle X \rangle$ are ordered by the relation $<_t$, their respective equivalence classes, $[x]_t \ldots$, will be ordered in a similar way by $>_{\mathbf{t}}$ (with a bold subscript t to indicate that the relation, holding in a different domain, is formally a different relation). Formally,

$$[x]_t <_{\mathbf{t}} [z]_t \text{ iff } x <_t z.$$

What we have done is to alter the domain of our relation from that of people, $\langle X \rangle$, to that of age-groups, the abstract entity common to all those who are the same age as one another, which we should then express as $\langle X / \approx_t \rangle$. People cannot be strictly ordered by age, because I may, besides being either older or younger than you, be the same age: ages, by contrast, are strictly ordered; the time of my birth was either earlier or later than the time of your birth, or else exactly the same time. In this way, then, the modified law of trichotomy can be reduced to a strict one. Where the quotient class obtained by factoring out the compatible equivalence relation is itself a linear ordering, the modified law of trichotomy holds, and no confusion will result from defining the equivalence relation negatively. It is very obvious; so obvious, indeed, that we can slip into confusion as we slide from the quasi-linear ordering of weighty material objects to the genuinely linear ordering of weights.

[23] See above, §9.2.

§9B.7 Converse Transitivity

In the previous section an equivalence relation was defined in terms of a strict ordering relation. This suggests a comparable definition of a further quasi-ordering relation again in terms of a strict ordering relation. We can generate from one ordering relation $<$ a further quasi-ordering relation $\precsim_{\prec}$:

$$x \lesssim_{\prec} y \ \text{ iff } \ (\mathrm{A}z)(z \prec x \rightarrow z \prec y).$$

The new relation $\precsim_{\prec}$ cannot be precisely the same as the relation $<$ which generated it, since $\precsim_{\prec}$ is antisymmetric, while $\prec$ is asymmetric, but we can compare it with $\lesssim$, and in some cases, it will indeed be the case that

$$(x \precsim_{\prec} y) \leftrightarrow (x \precsim y),$$

that is,

$$x \precsim y \text{ iff } (\mathrm{A}z)(z \prec x \rightarrow z \prec y).$$

This condition parallels the ordinary transitivity condition

$$(x \prec y) \ \wedge \ (z \prec x) \rightarrow (z \prec y);$$

we might call it "converse transitivity", and in this case, "converse transitivity from below",[24]. to distinguish it from the comparable condition

$$x \succsim y \text{ iff } (\mathrm{A}z)(z \succ x \rightarrow z \succ y),$$

which we might call "converse transitivity from above". Converse transitivity from below and converse transitivity from above are not necessarily the same, as the spouse/sibling example shows,[25] and it is possible to construct other examples in which one or the other or both conditions do not hold.

[24] P. Simons, *Parts*, Oxford, 1987, ch.1, §1.4, p.28, calls this the *Proper Parts Principle*

[25] See above, §9.8.

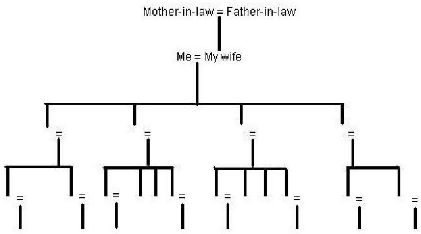


Figure 9.9.1 If I am strictly monogamous, all my descendants are descendants of my father-in-law, but I am not descended from him.

Nevertheless, there is a certain naturalness about them. Although ordering relations are asymmetric, they often have the mirror-image property, so that everything true of $\prec$ is, *mutatis mutandis*, true of $\succ$. In that case, $\approx_{\prec}$ is the same as $\approx_{\succ}$, and then converse transitivity from below and above must also hold. For paradigmatic orderings, therefore, we stipulate that they generate a single equivalence relation, which is factored down to identity, and they are conversely transitive.

§9B.8 Microstructure and Macrostructure

Orderings may be classified by reference to their local, or their global, structure. As regards their local, or micro-, structure, some orderings are discrete, while others are dense, some being not merely dense but continuous. And as regards their global, macro-, structure, they may be characterized first as either having extremal elements or as being serial; and secondly they may also be characterized as either linear or partial, and among partial orderings some are lattices, and others trees. (These strict orderings generated by the ordering relations properly so called, need to be distinguished from the quasi-orderings generated by reflexive antisymmetric relations—which here will be done by avoiding the latter altogether for the reasons given in §9B.1)[26]

If every element of an ordering $\langle X, \prec \rangle$ has a next, the ordering is *discrete*. Examples are the natural numbers, $0, 1, 2, 3 \ldots$, with order-type ω, and the integers $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$, with order-type $\omega^* + \omega$. If the contrary condition holds, that every element is to have no next, that is to say, if it is always the case

[26] [CHECK that this is done]

that between any two elements of an ordering $\langle X, \prec\ \rangle$ there is a third, that is,

$$(\mathrm{A}x)(\mathrm{A}y)(x \prec y \rightarrow (\mathrm{V}z)(x \prec z \wedge z \prec y)),$$

the ordering is called "dense". An example is the rational numbers, with the order-type η.[27] The classification is not exhaustive. An ordering could be in some parts dense, in others discrete. Nevertheless, all the orders we shall be concerned with are either dense throughout or else altogether discrete.

The word 'next' is a superlative. Superlatives are relevant to ordering relations in two ways: not only whether within the ordering there are next elements, but whether there are extreme elements that are the first or the last, the highest or the lowest, with respect to the order in question. In ordinary usage the greatest is the one which is greater than all the others, the biggest the one which is bigger than all the others, the least the one which is less than all the others. On reflection we recognise that sometimes there may not be a single one that outstrips all the others, as when two boys come first-equal in a form order. It is useful to distinguish a strict maxim**um** which alone is greater than everything else from those elements in an ordering that are merely maxim**al**, that is to say that each of them is not exceeded by any other element, even though it does not itself exceed the other maximal elements. In some orderings no element is maximal, and however far we go, there is always another element that is more than the one in question. Such an ordering is called *serial*: formally, an ordering, $\langle X, \prec\ \rangle$ is *serial* iff $(\mathrm{A}x)(\mathrm{V}y)(y \prec x)$. It is possible for an ordering to be serial, with no maximum element in one direction, but not in the other. There is no greatest, or last, natural number, but there is a least, or first,[28] one. The integers, $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$, with the order-type $\omega^* + \omega$, have neither a greatest nor a least element, and thus constitute a serial ordering for both $\prec$ and $\succ$. Our definition of an ordering being *discrete* was in terms of there always

[27] E.V.Huntington, *The Continuum and Other Types of Serial Order*, Dover, 1955, ch.III, §24, p.21, §26, p.22, ch.IV, §44, p.35.

[28] It is customary, though not mandatory, to confine the terms 'first' and 'last' to linear orderings, and use 'greatest' and 'least' more generally for partial orderings.

being a next, that is to say an immediate successor; but this example suggests that equally, if every element of an ordering has an immediate predecessor, it is *discrete*: but we need to be careful; these two conditions are not quite the same: if we consider all the even numbers followed by all the odd numbers, every number has a next number, but 1 has no immediate predecessor.[29] Ordinals are another type of discrete ordering; they are a generalisation of Dedekind's progression ω.[30] An ordinal is an ordering $\langle X, \prec \rangle$ in which every (proper or improper) subset has a *first* element, and the elements of such an ordering are said to be **well-ordered** by the ordering relation. Well-ordered sets are easily shown to be linear. They are almost the same as those that always have a next element, but the latter condition is satisfied by the integers, with order-type $\omega^* + \omega$, which is not an ordinal, since it does not have a first member. Well-orderings lack the mirror-image property.[31]

An ordering $\langle X, \prec \rangle$, consisting of the relation $\prec$ on a set X, is linear[32] if the relation is "connected", that is, if the ordering relation always holds between any two distinct elements. Provided x and y are distinct, either $x \prec y$ or $y \prec x$; which is often reformulated as the "law of trichotomy":

$$(\mathrm{A}x)(\mathrm{A}y)((x = y) \vee (x \prec y) \vee (y \prec x)) \text{ holds.}$$

Given any two elements, either the one is higher than the other, or the other than the one, or they are boththe same. Either my birthday is before yours, or yours is before mine, or they are both the same.

Linear orderings are contrasted with partial orderings. There are many examples of partial orderings: the relation '— is divisible by ...', or in the Special Theory of Relativity the relation of *being after*, or *causal influenceability* as it could more accurately

[29] For a careful definition, see E.V.Huntington, *The Continuum and Other Types of Serial Order*, Dover, 1955, ch.III, §21, p.19.

[30] §5.2.

[31] See the previous section, §9.5.

[32] The word 'linear' is used in many different, though related, senses by mathematicians. Some readers may prefer to use 'connected' of orderings, in order that they may not be confused with linear transformations.

be termed. Partial orderings may be characterized, by an axiom of Robb's:[33]

For every element x, there is another element y such that neither $x > y$ nor $y > x$.

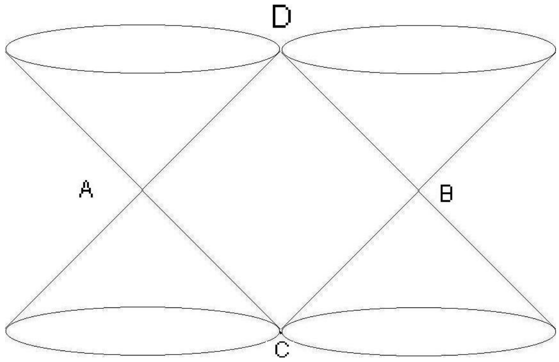


Figure 9.6.1 In the Special Theory of Relativity two events may be neither before nor after each other: A is neither before nor after B, though both A and B are before D and after C.

Robb's axiom, however, requires also that the ordering be non-serial, and secures only "global non-linearity", being satisfied by orderings that are not "locally non-linear".

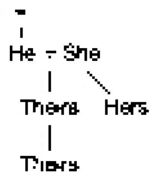


Figure 9.6.2 If an only child marries a single parent and they then have one child, who in turn has an only child, the order generated by *descendant* is globally non-linear, but with locally linear sub-orderings.

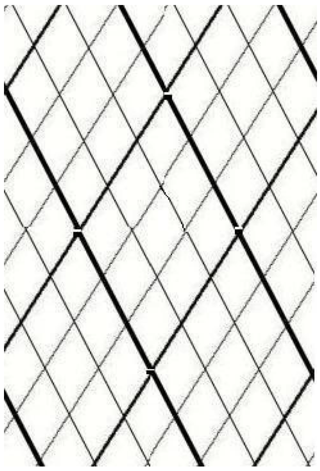
[33] A.A.Robb, *The Absolute Relations of Time and Space*, Cambridge, 1921, Postulate V, p.17; or *Geometry of Time and Space*, Cambridge, 1914 and 1936, p.27. It is worthy of note, though not for discussion here, that with the aid of very few other axioms Robb is able to derive the Special Theory of Relativity as the instantiation of his "conical order".

We need to strengthen Robb's axiom by requiring that it apply within the range of predecessors of each element.

$$(\mathrm{A}x)(\mathrm{A}y)(\mathrm{V}z)(y \prec x \rightarrow z \prec x \wedge \neg(z = y \vee z \prec y \vee y \prec z))$$

The partial orderings hereafter considered will all be locally non-linear.

Partial orderings can be divided into further types, according as to whether any two elements have some third element higher than (or preceded by) both or some fourth element lower than (or that precedes) both. In the former case it is said to be "upper-directed": in the latter case it is said to be "lower-directed". If both conditions hold, the ordering is said to be *directed*, or to possess the "Moore-Smith property".[34]



Figuure 9.10.1 Hasse diagram representing a directed ordering (one possessing the "Moore-Smith property").

If the ordering is directed and not serial, and has both a maximum and a minimum element, it is a **lattice**.

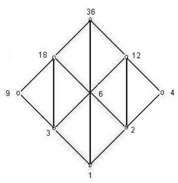


Figure 9.10.2 A lattice—the factors of 36.

[34] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon, Oxford, 1962, pp.1ff.

If it is only one-way directed—if there is just either always an element higher than (preceded by) any given two elements, or always one lower than (that precedes) each of them, but not always both—then the ordering is a tree.

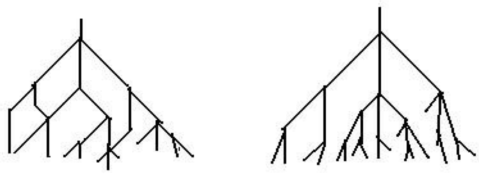


Figure 9.10.3 Directed Orderings and Trees. Note that the definition given does not exclude branches subsequently reuniting, as occasionally happens with lime trees, illustrated in the left-hand figure.

A tree may have a maximum and minimal (or a minimum and maximal) elements, but need not; and similarly, it may be dense (and even continuous), but may also be discrete. It may be that two elements may be neither the one higher than the other nor both lower than some third element nor both higher than some fourth one, although either or both of these conditions may hold for other pairs. We might call such an ordering a "thicket".

§9B.9 Lattices and Set Theory

Lattices are paradigm orderings, and, apart from linear orderings, the simplest. If there is always *an* element lower-than-or-identical-with any two elements, there cannot be a number of minimal elements, and it is natural to posit a single *lowest* or a *least*. Similarly, if there is always *an* element higher (or more)-than-or-identical-with any two elements, it is natural to posit a *highest* or a *most*. Hence, if we are to take the directed option, it is natural to stipulate that such orderings should not have merely extremal elements, and should not be serial, but should be complete lattices, and have a maximum and a minimum element, and to add to this the further superlative excellence of discreteness. Granted discreteness, it is unproblematic to identify a *lowest* upper bound and *highest* lower bound to every pair of elements in the lattice. The least upper bound of x and y is called the **join** of the two elements, and the highest lower bound is called the **meet**. They are represented by $x \cup y$ and $x \cap y$ respectively.

Instead of defining the join and meet, $\cup$ and $\cap$, in terms of an ordering relation, we can characterize them in terms of their mutual relations, and then define the antisymmetric relation, $\preceq$, giving rise to a quasi-ordering. We could equally well define its converse, $\succeq$; lattices can have the mirror-image property. The following four axioms characterize a lattice:

L1 $\quad x \cup x = x \quad \text{and} \quad x \cap x = x$

L2 $\quad x \cup y = y \cup x \quad \text{and} \quad x \cap y = y \cap x$

L3 $\quad (x \cup y) \cup z = x \cup (y \cup z) \quad \text{and} \quad (x \cap y) \cap z = x \cap (y \cap z)$

L4 $\quad (x \cup y) \cap x = x \quad \text{and} \quad (x \cap y) \cup x = x$

We can then define $x \preceq y$:

$x \preceq y$ iff $x \cap y = x$

Paradigm lattices have two further properties: they are distributive, and they can be complemented. The meet of one set with the join of two others is the same as the join of the meet of the one set with one of the others and the meet of the one set with the other: in symbols

L5 $\quad x \cap (y \cup z) = (x \cap y) \cup (x \cap z).$

And similarly the join of one set with the meet of two others is the same as the meet of the join of the one set with one of the others and the join of the one set with the other:

L5′ $\quad x \cup (y \cap z) = (x \cup y) \cap (x \cup z).$

Not all lattices are distributive: the lattice of Hilbert spaces under the relation *subspace of* is not distributive, and much effort has been expended on trying to devise a "quantum logic" to mirror this feature.

If a lattice has a maximum element **I**, and a minimium element **O**, they will act as identity and universal elements for the operations $x \cap x$ and $x \cup x$:

$$\mathbf{I} \cap x = x \quad \text{and} \quad \mathbf{O} \cup x = x$$
$$\mathbf{I} \cup x = \mathbf{I} \quad \text{and} \quad \mathbf{O} \cap x = \mathbf{O}$$

Complementation can be defined in terms of maximum and minimum elements. Each set, x, has a complement, x', such that

$$x \cup x' = \mathbf{I},$$

and

$$x \cap x' = \mathbf{O}.$$

A lattice that is complemented and also distributive is "orthocomplemented", that is, its complement is unique, and satisfies the further two conditions:

$$(x')' = x$$

$$x < y \leftrightarrow y' < x'.$$

Complementation is like negation. Essentially, we have defined complementation in terms of $\cap$ and $\cup$, and these two symbols in terms of the quasi-ordering relation $\preceq$ and the word 'and' of ordinary English.[35]

A complemented distributive lattice is called a "Boolean lattice". Set theory is the paradigm of Boolean lattices. It orders sets by means of an irreflexive subset relation, $\subset$, or quasi-orders them by means of an antisymmetric subset relation, $\subseteq$. There is a minimum set, the null set, Λ,[36] and, with some qualms, a maximum, $\mathbf{V}$.[37] The null set exists by stipulation rather than as an expression of our untutored thought. In our natural way of thinking we should say that the sets of Roman Catholic priests and matriarchs had no intersection, that there was no set which was a subset of them both. We could have a workable set theory in which there was no null set, and we used only the fused phrase 'is null' of the intersection of two sets when they had no non-empty intersection. As it is, we *postulate* the *existence* of the null set, and deem it to be a subset of every set, thereby securing the lattice property, that every two sets have a meet, that is, a set less-than-or-identical-with them both.

Since Peano, we have been careful to distinguish the proper subset relation, $\subset$, from the relation of set membership, $\in$, and

[35] See further, J.L.Bell and M.Machover, *A Course in Mathematical Logic*, North Holland, 1977, ch.4, §1, pp.125-129.

[36] There are many different representations of the null set, or empty set: $\emptyset$ is typographically close to 0, but sometimes mistaken for a Danish ø, or a Greek ϕ. Here we align our symbol with $\wedge$, $\cap$, $(\mathrm{A}x)$, and $\bigcap$, and think of Λ as the intersection of all sets, and the universal set, $\mathbf{V}$, as the union of all sets, analogously with $\vee$, $\cup$, $(\mathrm{V}x)$, and $\bigcup$. Sometimes, instead of Λ, $\bigwedge$ is used, and instead of $\mathbf{V}$, $\bigvee$, which is logical, but typographically awkward.

[37] See above, §7.11.

individual members from singleton sets. But the distinction is not intuitively clear, and in transitive set theory, set membership is transitive as well as the subset relation. If we work with transitive set theory, we often lay down an axiom of regularity, or an axiom of foundation, which secures, in the absence of a null set, the existence of minimal elements. We are then working with a discrete, non-serial tree, rather than a lattice. From the present point of view, however, we do not need to consider set membership at all. We lay down a lattice structure, with the null set, Λ, as the minimum element, and the singleton sets next above it, being those sets whose only subsets are themselves and the null set.

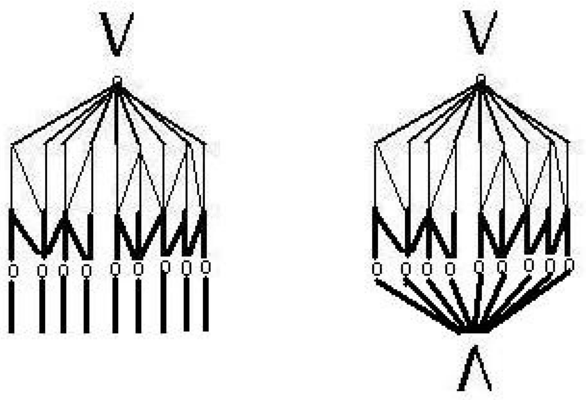


Figure 9.11.1 On left: Transitive Set Theory, with Axiom of Regularity (Axiom of Foundation) but no Null Set. On right: Standard Set Theory quasi-ordered by $\subseteq$, showing the singleton sets on the level above the null set.

Set inclusion is conversely transitive from below and above. From this it follows that if we start with the strict ordering relation of proper set inclusion, $\subset$, we can define identity by means of the equivalence relation generated by $\subset$. Equivalence from below is expressed by the Axiom of Extensionality. Normally we formulate the axiom in terms of set membership, $\in$, and lay down

$$x = y \leftrightarrow (\mathrm{A}z)(z \in x \leftrightarrow z \in y),$$

but if, instead, we consider proper set inclusion, $\subset$, we have

$$x = y \leftrightarrow (\mathrm{A}z)(z \subset x \leftrightarrow z \subset y).$$

Similarly, equivalence generated from above yields

$$x = y \leftrightarrow (\mathrm{A}z)(x \subset z \leftrightarrow y \subset z).$$

On the face of it, this is a relational thesis: but relations can be construed monadically, though at the cost of much reduplication, and then equivalence from above is seen as a version of Leibniz' Law of the Identity of Indiscernibles

$$x = y \leftrightarrow (\mathrm{A}F)(F(x) \leftrightarrow F(y)),$$

with $F(*)$ in place of $* \in z$.

§9B.10 Trees and Mereology

The logic of the relation *is a part of* has been studied, sometimes under the title "Mereology" (from the Greek *μέρος* (*meros*), a part), sometimes under that of "The Calculus of Individuals".[38] Often the questions addressed have been ontological, and many different systems have been developed, to take account of different metaphysical intimations. We can approach mereology in two ways. We can idealise the transitive irreflexive relations we have been discussing so as to produce a paradigm of ordering, or we can articulate our everyday concept of being a part of, as exhibited in the ordinary use of the word 'part', itself reflecting the metaphysical assumptions we, often unconsciously, have been making. The approaches are not exclusive. Our ordinary use is influenced by ideas of what ideally ought to be the case, and our logical paradigms can be developed in different directions to satisfy alternative *desiderata*. Such concerns we shall at first leave on one side, developing a formal paradigm to exemplify the comparative ideal, so as to be

[38] Much of the early work was done in Poland, notably by Leśniewski, Lejewski and Tarski. See also N.Goodman and H.S.Leonard, "The Calculus of Individuals and its Uses", *Journal of Symbolic Logic*, **5**, 1940, pp.45-55. A.N.Whitehead, *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge, 1919, §27.2, pp.101-2; his line of thought has been developed by T. de Laguna, "Point, Line and Surface, as Sets of Solids", *Journal of Philosophy*, **19**, 1922, pp.449-461; David Bostock, *Logic and Arithmetic*, vol.2, Oxford, 1979, ch.2, §4, pp.112-130; and J.E.Tiles, *Things that Happen*, Aberdeen, 1981, §8. An excellent survey of the different mereological systems is to be found in P. Simons, *Parts*, Oxford, 1987, ch.2, pp.46-100.

able, in the next two chapters, to consider how far it is possible to ground the concepts of topology and a theory of measurement in ordering relations.

Set theory operates with a partial ordering, conversely transitive from above and below, and has a maximum and a minimum, thus being a lattice, and is distributive and complemented. The opposite paradigm is likewise a partial ordering, everywhere nonlinear, and conversely transitive from above and below, having the mirror-image property, but, instead of being discrete, we want it to be at least dense, and perhaps continuous, and instead of having a maximum and minimum, we want its ordering relation, together with its converse, to be serial. It is natural to go further, and deny that it is directed; if the ordering is to have the mirror-image property, it will be neither upper- nor lower-directed. Distributivity, therefore, becomes a hypothetical issue, and complementation likewise does not feature in the comparative paradigm.

We consider a relational structure, $\langle X, \prec \rangle$, where $\prec$ is intended to capture the sense of *proper part of*,[39] which together with its converse, *having as a proper part*, or in Whitehead's terminology, *extends over*, ranges over a field of portions, parts, regions, or extensions.[40] As with set theory, we stipulate that there shall be

[39] Hereafter *is a part of* will be taken to mean *is a proper part of*. Although, for the reasons given in §9.1, I take the strict ordering generated by the asymmetric $\prec$ as fundamental, it is often convenient to work with the derived antisymmetric relation $\preceq$, like $\subseteq$ in set theory. And exceptionally, I shall sometimes write $\overset{\rightarrow}{-}$, instead of $\rightarrow$, in order to bring out the antisymmetry of implication. The reader should also note that Bostock works with an antisymmetric *proper or improper part of* relation, which he writes as $\subset$. In quoting him, I have rewritten this as $\preceq$. Simons uses an irreflexive relation, $\ll$, which I rewrite as $\prec$, and an antisymmetric $<$, which I rewrite as $\leq$.

[40] No term is altogether apposite: 'portion' is reminiscent of canteens; 'part' is idiomatically unexceptionable, but awkward in the context of the relation *part of*; S.Leśniewski, "Podstawy ogólnej teoryi mnogości", *Prace Polskiego Kola Naukowe w Moskwie*, Moscow, 1916, used *przedmiot*, 'object'; 'lump' (used by Menger), 'chunk' (used by Veblen) and 'piece' (used by Moore) have too materialistic connotations; many English writers have used 'individual'; 'extension' would be the best term for *res extensa* and extended magnitudes, but that carries with it connotations of connectedness we have not as yet justified, and invites confusion with the Axiom of Extension in set theory, and so later I shall use 'region' as the least misleading term I can think of. In this chapter I shall use 'portions'.

converse transitivity from below and above:

$$x \preceq y \text{ iff } (\mathrm{A}z)(z \prec x \overset{\rightarrow}{-} z \prec y),$$

and

$$x \preceq y \text{ iff } (\mathrm{A}z)(x \prec z \overset{\rightarrow}{-} y \prec z).$$

> Compare, in set theory,
> $x \subseteq y \text{ iff } (\mathrm{A}z)(z \subset x \overset{\rightarrow}{-} z \subset y)$
> $x \subseteq y \text{ iff } (\mathrm{A}z)(x \subset z \overset{\rightarrow}{-} y \subset z)$

We go further and stipulate that the equivalence relations generated by $\prec$ shall both be just the identity, that is,

$$((\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)(z \prec x \leftrightarrow z \prec y) \rightarrow x = y)$$

and

$$((\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)(x \prec z \leftrightarrow y \prec z) \rightarrow x = y),$$

which is to say $\approx_{\prec}$ and $\approx^{\prec}$ are both the identity relation, $=$.
We may call these the Axiom of Constituents and the Theorem of Envelopes,[41] regarding them as analogies with the Axiom of Extensionality and the Identity of Indiscernibles in set theory. The Axiom of Constituents has a materialist tinge. Things are just what they are made up of, with no allowance for supervenient graces. But that is the logic of portions. Considered as a portion, I am just so much carbon, so much oxygen, so much hydrogen, so much nitrogen, *etc.* No reckoning is made of my social accomplishments, my spiritual aspirations, or my political correctitude. The airline operator is not interested in such things: only my weight might concern him, and minds and souls have no weight. The Axiom of Envelopes likewise focuses attention away from organic wholes and relational structures, and considers a portion simply as that, and not with respect to its significance in some wider scheme of things.

[41] Only one of these is an axiom, the other being a theorem, granted that the equivalence relation has the mirror-image property. But for expository purposes, it may be easier to construe them both as mirror-image axioms.

My liver may play a key role in my metabolism, but is counted only in respect of its contribution to my overall weight.

Converse transitivity from below and above offer a justification for these axioms, and seem desirable features in their own right, again suggesting a certain metaphysical slimness characteristic of the *part of* relation. Later, however, their absence will be a crucial feature of a stronger, topological, relation.[42]

The *part of* relation is dense:

$$(\mathrm{A}x)(\mathrm{A}y)(x \prec y \rightarrow (\mathrm{V}z)(x \prec z \wedge z \prec y)).$$

The *not discrete—no minimum—not a lattice* inference is reasonable, though not watertight, and guides us to lay down that $\prec$ shall be serial:

$$(\mathrm{A}x)(\mathrm{V}y)(y \prec x).$$

Every part, no matter how small, has a proper part that is smaller: there are no minimal elements—no atoms. The mirror-image property requires that $\succ$, too, shall be serial, that is to say, that there shall be no maximal portions. In order not to be a lattice, a partial ordering must be not both upper- and lower-directed. If we deny the Big Bang, and the possibility of a Big Crunch, Robb's conical ordering for the Special Theory would be directed, though serial in both directions, and thus without extremal elements. Simons offers a stronger version that Robb's axiom of non-linearity, which also secures that the partial ordering be not lower-directed.[43] It stipulates that if one element is below another, then there is a third element also below it, without there being any fourth one below them all:

$$(\mathrm{A}x)(\mathrm{A}y)(x \prec y \rightarrow (\mathrm{V}z)(z \prec y \wedge \neg(\mathrm{V}w)(w \prec z \wedge w \prec x))).$$

The antecedent of the conditional is always satisfied, since the ordering is serial in the upward direction and so we are saying that for every portion, there is another that is "disjoint" from it, which is to say that it does not overlap it at all.[44] The mirror-image

[42] §10.8.

[43] P.Simons, *Parts*, Oxford, 1987, p.28: SA3 "The Weak Supplementation Principle".

[44] David Bostock, in his *Logic and Arithmetic*, vol.2, Oxford, 1979, p.113, uses the word 'discrete', as do others who write in this field; but that word is standardly used in a different sense in mathematical logic.

requirement stipulates that the ordering should also be not upper-directed. In many presentations, however, though two elements do not always have a meet, they do always have a join, so that the system is upper-directed, and lacks the mirror-image property. Nevertheless, for the sake of purity, we stipulate the mirror image of Simons' axiom, but leaving out the antecedent (already secured by upward seriality):

$$(\mathrm{A}y)(\mathrm{V}z)(y \prec\ z \wedge (\mathrm{V}w)(y \prec\ w \wedge \neg(w = z \vee w \prec\ z \vee z \prec\ w))).$$

It is convenient to have formal definitions of overlapping and being disjoint. Let us symbolize *overlaps* by $\circ$, and say:

$$x \circ y \text{ iff } (\mathrm{V}z)((z \prec x) \wedge (z \prec y)).$$

Let us symbolize being disjoint by $|$, and say

$$x \mid y \text{ iff } \neg(x \circ y),$$

that is,

$$x \mid y \text{ iff } \neg(\mathrm{V}z)((z \prec x) \wedge (z \prec y)).$$

These permit more succinct and perspicacious formulation of some of the axioms, further economy often being possible, by reason of the presence of other axioms, such as seriality and denseness. With their aid the pure paradigm of a partial ordering that has the mirror-image property is dense, serial, everywhere non-linear, conversely transitive from above and below, and not necessarily directed, is expressed by the following axioms:

Our everyday concept of being a part of does not fully conform to the ideal paradigm, and has been articulated in different directions. Some mereologies are lattices.[45] Others differ from set theory only in having no minimum. They are perfectly possible formal systems—we may wish to avoid a minimum, null element for other reasons than a dislike of superlatives. Many mereologies have minim*al* null portions. Pythagoras' definition of a point,

[45] M.Bunge, "On Null Individuals", *Journal of Philosophy*, **63**, 1966, pp.776-778, propounds a lattice mereology in which there is a null portion (*cf.* the null set) that is a part of every portion.

Pure Paradigm

'$\prec$' primitive; '$=$', '$\preceq$', '$\circ$', '$|$' defined

A 1 $x \prec y \wedge y \prec z \rightarrow x \prec z$ Transitive

A 2 $\neg(x \prec x)$ Irreflexive

A 3 $x \prec y \rightarrow (\mathrm{V}z)(x \prec z \wedge z \prec y)$ Dense

A 4 (i) $(\mathrm{A}x)(\mathrm{V}y)(x \prec y)$ Upwards Serial

A 4 (ii) $(\mathrm{A}x)(\mathrm{V}y)(y \prec x)$ Downwards Serial

A 5 $(\mathrm{A}x)(\mathrm{A}y)((\mathrm{A}z)(z \prec x \leftrightarrow z \prec y) \leftrightarrow (\mathrm{A}z)(x \prec z \leftrightarrow y \prec z))$
Mirror Equivalence

D 1 $x = y$ for $((\mathrm{A}z)(z \prec x \leftrightarrow z \prec y)$

D 2 $x \circ y$ for $(\mathrm{V}z)(z \prec x \wedge z \prec y)$

D 3 $x \mid y$ for $\neg(x \circ y)$, *i.e.* $\neg(\mathrm{V}z)(z \prec x \wedge z \prec y)$

D 4 $x \preceq y$ for $x \prec y \vee x = y$

A 6 $(\mathrm{A}x)(\mathrm{A}y)(y \prec x \rightarrow (\mathrm{V}z)(z \prec x \wedge z \mid y))$
Locally Non-linear and Not Downward Directed

A 7 $(\mathrm{A}x)(\mathrm{A}y)(\mathrm{V}z)(\mathrm{A}w)(x \prec y \rightarrow (x \prec z \wedge \neg(y \prec w \wedge z \prec w)))$
Not Upward Directed

A 8 (i) $(\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)((z \prec x \rightarrow z \prec y) \rightarrow x \preceq y)$
Conversely Transitive from Below

A 8 (ii) $(\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)((x \prec z \rightarrow y \prec z) \rightarrow y \preceq x)$
Conversely Transitive from Above

given in §2.5, is that of a minimal portion that itself has no parts.[46] In metaphysics atomism requires such a mereology. But atoms, in spite of their name, have an uncomfortable tendency to be split. Even if, in the current state of particle physics, there are quarks or quarklets which owing to a lack of funds have not yet been split, physicists are able conceptually to unglue them, and imagine their being split if only high enough energy were available. Logical atomism is not a contradiction in terms, but is discrete, not dense, and in not being serial runs counter to a human urge ever to go further in the search for a deeper understanding.

[46] For a clear account of such a "classical extensional mereology", see P. Simons, *Parts*, Oxford, 1987, ch.1, §1.5, pp.37-41. See earlier, A.Tarski, *Logic, Semantics, Metamathematics*, Oxford, 1956, pp.333-334n.

The argument from non-discreteness to non-seriality has seemed less compelling in the upward direction; we often have intimations of there being some whole, some universe, which comprehends everything.[47] Even without that assumption, it is often laid down that the ordering be upper-directed.[48] In either case we may call the structure tree-like, though we must be careful to allow lime-like trees, in which portions, though different, may yet overlap. In such cases the *part of* relation lacks the mirror-image property. But in our ordinary thinking we are much more concerned whether two portions overlap or are disjoint than whether or not there is some further portion which they both are parts of. Only in special circumstances do we ask whether they are both part of the same portion—only when they are mutually disjoint and jointly exhaustive of it, as well as satisfying some further condition of togetherness, yet to be elucidated.

Mereologists are as much interested in portions as in the *part of* relation, and want to be able to identify particular portions. It is not enough to know that two portions overlap, they want to be able to talk about *the* overlap. They supplement the comparative language of *part of* with the superlatives 'greatest' and 'least'. The "product", $x \bullet y$, of two portions, x and y, that overlap, is the highest lower bound of x and y,

$$z = x \bullet y \quad \text{if and only if} \quad (\mathrm{A}w)(w \prec z \leftrightarrow (w \prec x \wedge w \prec y)).$$

With highest lower bounds and lowest upper bounds, further economies of axiomatization are possible. With some unfairness to each, it is possible to view Whitehead, Bostock and Simons, as developing a canonical mereology, which takes more account of our actual concept of being a part of than the idealized paradigm, while still offering logical coherence and clarity. It does not have the mirror-image property, but is dense, serial, everywhere non-linear, conversely transitive from above and below, and not downward directed.

[47] See further, M.K.Munitz, *Existence and Logic*, New York, 1974, pp.191ff.

[48] A.N.Whitehead, *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge, 1919, §27.2 (vi), p.101.

Whitehead's presentation is deliberately informal.[49] He worked with a transitive irreflexive relation K, 'covers', equivalent to $\succ$. Bostock works with an antisymmetric relation, and combines the axioms expressing the transitivity, antisymmetry and converse transitivity from below of $\preceq$ into a single axiom,

P1 $$x \preceq y \leftrightarrow (\mathrm{A}z)(y \mid z \rightarrow x \mid z).$$

> Compare, in set theory,
> $x \subseteq y \leftrightarrow (\mathrm{A}z)((z \cap y = \Lambda) \rightarrow (z \cap x = \Lambda))$

Bostock strengthens Whitehead's axiom of upper-directedness by a further one:

P2 $$(\mathrm{V}x)(Fx) \wedge (\mathrm{V}y)(\mathrm{A}x)(Fx \rightarrow x \preceq y) \rightarrow (\mathrm{V}y)(\mathrm{A}z)(y \mid z \leftrightarrow (\mathrm{A}x)(Fx \rightarrow x \mid z)).$$

This axiom posits a Lowest Upper Bound property. Provided we have a non-empty family, F, of portions, all of which are proper or improper parts of some portion y (so that in effect y is an upper bound, so far as the *part of* relation is concerned), then there is a maximum portion z, which is disjoint from all but only those portions which do not overlap any member of the family F. Whitehead's axiom secures that for any two branches of the tree, there is a bigger branch they are both branches of: Bostock's secures that there is a node, a final point of branching. More formally, Whitehead's posits the existence of an upper bound, Bostock's of a lowest upper one (and for any number, not just two).

Simons separates out his axioms for transitivity and irreflexivity, and then secures local non-linearity by his Weak Supplementation Principle:

SA3 $$x \prec y \rightarrow (\mathrm{V}z)(z \prec y \wedge z \mid x)$$

[49] Whitehead has been much criticized for being slipshod. See V.F.Sinisi, "Leśniewski's Analysis of Whitehead's Theory of Events", *Notre Dame Journal of Formal Logic*, **7**, 1966, pp.323-327. But Whitehead was not concerned to give a rigorous account—something that the author of *Principia Mathematica* could have done perfectly well, if he had wanted to. Rather, he was concerned to convey the general lines of an approach. There is a trade-off between rigour and intelligibility. Both are good. But it is captious to criticize Whitehead when he seeks to secure the latter, on the grounds that he has failed to secure the former.

He is then able to show, by simple but ingenious derivations, that both local non-linearity and converse transitivity from below follow from an axiom stating that if two portions overlap, then there is a unique product.

$$\text{SA6} \qquad x \circ y \to (\mathrm{V}z)(z = x \bullet y).$$

This he calls *Minimal Extensional Mereology.*

Applied Mereology

Whitehead-Bostock-Simons

'$\prec$', '$=$' primitive; '$\preceq$', '$\circ$', '$|$' defined

A 1	$x \prec y \wedge y \prec z \to x \prec z$	Transitive
A 2	$\neg(x \prec x)$	Irreflexive
A 3	$x \prec y \to (\mathrm{V}z)(x \prec z \wedge z \prec y)$	Dense
A 4 (i)	$(\mathrm{A}x)(\mathrm{V}y)(x \prec y)$	Upwards Serial
A 4 (ii)	$(\mathrm{A}x)(\mathrm{V}y)(y \prec x)$	Downwards Serial
A 5	$(\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)((z \prec x \leftrightarrow z \prec y) \leftrightarrow x = y)$	Axiom of Constituents
D 1	$x \preceq y$ for $x \prec y \vee x = y$	
D 2	$x \circ y$ for $(\mathrm{V}z)(z \prec x \wedge z \prec y)$	
D 3	$x \mid y$ for $\neg(x \circ y)$, *i.e.* $\neg(\mathrm{V}z)(z \prec x \wedge z \prec y)$	
A 6	$(\mathrm{A}x)(\mathrm{A}y)(y \prec x \to (\mathrm{V}z)(z \prec x \wedge z \mid y))$	Locally Non-linear and Not Downward-directed
A 7	$(\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)((z \prec x \to z \prec y) \to x \preceq y)$	Conversely Transitive from Below
A 8	$(\mathrm{A}x)(\mathrm{A}y)(\mathrm{A}z)((x \prec z \to y \prec z) \to y \preceq x)$	Conversely Transitive from Above
A 9	$(\mathrm{A}x)(\mathrm{A}y)(\mathrm{V}z)(x \prec z \wedge y \prec z)$	Upward-directed
A 10	$(\mathrm{A}x)(\mathrm{A}y)(x \circ y \to (\mathrm{V}z)(z \prec x \wedge z \prec y \wedge (\mathrm{A}w)(w \prec z \leftrightarrow w \prec x \wedge w \prec y)))$	Unique Product

The axioms in the box on the opposite page could be formulated differently. We could, for instance define identity in terms of equivalence from below; but that would weaken the analogy with the Axiom of Extensionality in set theory. Bostock's and Simons' own formulations are more elegant and economical, but make it less evident where they are departing from the pure paradigm. There are many other axiomatizations, not all of them equivalent. They articulate different understandings of the concept of *portion* and *being a part of.* These concepts, like that of a set, are not clear-cut, and it is by considering different axiomatizations that we are helped to decide what we ought to mean when we use these terms.

The full strength of *Classical Extensional Mereology* is achieved by replacing SA6 by the *General Sum Principle*:

SA24 $(\mathrm{V}x)(Fx) \rightarrow (\mathrm{V}w)(\mathrm{A}y)(y \circ w \leftrightarrow (\mathrm{V}z)(Fz \wedge y \circ z));$

but this is inconsistent with the partial ordering being upwardly serial. All that Whitehead would allow is a *Binary Sum Principle*, guaranteeing not only the existence for any two portions of a node, that is, a minimal portion of which they were each a part, but one that had no other parts disjoint from those two.

If we posit the existence of a lowest upper bound and a sum, we are securing the mirror image of there always being a maximum product, if two portions overlap at all. But neither thesis is to be taken for granted if we are dealing with some sorts of portions. Two areas may be entirely separate, and not combine into a single region, and even if they intersect, their common portions may be themselves separate.[50]

With the Lowest Upper Bound and Highest Lower Bound, mereology verges towards a discipline concerned with limits and boundaries, where we deal not merely with the *portions* that are parts of some whole, but the *regions*, or *extensions*, which, sharing a boundary, are unseparated parts and together constitute one continuous extent. The study of these constitutes the discipline of topology; and also is a precondition of a satisfactory account of measurement, in which we systematically assign numbers to portions of various sorts.

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[50] See below, §10.6.

A Typology of Relations