Chapter 4*
Numbers: The Cardinal Approach

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§4*.1 Etymology

Linguistically, numerals are odd. Often in inflected languages, though not in Russian, they are, with the exception of the smaller ones, indeclinable. In Greek εἷς, δύο, τρεῖς, τέσσαρες (*heis, duo, treis, tessares*) decline, but the rest not. In Latin *unus, duo, tres* decline, but the rest not. In French and German indeclinability begins with *deux* and *zwei*, with only *un, une* and *ein, eine* behaving like adjectives, as in almost every other major language where the words for none and one are declinable, and agree with their noun. In English all the numerals are indeclinable; in German there is a distinction between the use of *ein, eine* as a number asking, or anticipating the question 'How many?', in which use it declines and agrees with the noun, and the use of *eins* as a counting number, where it is indeclinable. Indeclinability is on the increase. The older the language, the more numerals are wedded to their nouns, and have to agree with them, like adjectives; but they are gradually shaking off their etymological shackles, and establishing themselves as free-standing self-subsistent citizens.

Most languages are unsystematic below ten, with the exception of Basque, where the word for nine is *bederatzi* meaning 'one lower', in the same way as *undeviginti* means nineteen in Latin. Nine, ἐννέα (*ennea*), *novem, neuf, neun*, was the new number, and perhaps goes back to a time when our ancestors were beginning to go beyond the numbers they all could see at a glance, and were able to answer 'Nine' when asked 'How Many?' only by resorting to the new method of counting. Eight, ὀκτώ, *octo*, was originally a dual, perhaps expressed by the two hands being held up showing four fingers apiece.

After ten, and especially after twenty, the names become systematic.[1] In Latin and Greek, *undecim, duodecim*, *ἕνδεκα*, *δώδεκα* are already on a decimal scheme. Our 'eleven', the German *elf*, is still irregular, but 'twelve' already shows signs of twoness, and thirteen, fourteen, *etc.*, are obviously systematic. After twenty almost every language is regular. The only exception is French, where seventy, eighty, and ninety are expressed periphrastically, like our 'three score years and ten'. The metropolitan French despise the Belgians who say *septante, huitante*, and *nonante*, but it is the metropolitan version that has regressed, while the French spoken on the peripheries, in Belgium, in Switzerland, and in Quebec, has kept the logic that is the pride of the Gallic tongue.

The ordinals are also instructive. 'First', *primus*, *πρῶτος* (*protos*), are all superlative in form. *Secundus* in Latin and 'second' in English are not formed from the cardinals *'duo'* and 'two', but from *sequor*, I follow. In Greek *δεύτερος* (*deuteros*), is connected with *duo* but is comparative in form, as is the Latin *alter* which is often used for the "twoth" member of a sequence. 'Third' and *tertius* are not quite standard in form, but from 'fourth' onwards the ordinals are formed systematically from the cardinals. It is noteworthy, and of considerable significance, that although we can use an ordinal form for rather large numbers—'The Old Hundredth', 'One hundred and fourthly', 'The Hundred and Twenty First Psalm'—we tend to use the simple form 'Ps. 100', '104.)' or 'Ps. 121' in which we use cardinal numbers in an ordinal way as counting numbers. Language suggests that smaller numbers have highly individual names and present themselves as predominantly cardinal, large numbers are named systematically and are much more ordinal in feel.

There are many other numerals besides the cardinals and ordinals:

once, twice, thrice, four times, ...
whole, half, third, quarter, fifth, ...
single, double, treble
single, couple, triple, quadruple, quintuple, ... n-tuple,
twins, triplets, quads, quins, sextuplets
twosome, threesome, foursome
solo, duo/duet, trio, quartet, quintet, ..., octet.

[1] Most languages are based on the decimal system, although shepherds in the North of England used to count in a Celtic dialect to the base five.

primary, secondary, tertiary, quaternary.

We should also notice the contrast simple/multiple, simplicity/duplicity. Besides the word 'couple' we have 'pair' and 'brace'. Often the appropriate word depends on the social or linguistic context—we play doubles in tennis but row in pairs; we have a couple of hounds, a pair of turtle doves, and a brace of pheasants. In Japanese the principle is extended very far, and the numerals vary systematically with what is being numbered.[2] In English we do not go beyond two in exact numbering, but some of our inexact collective words vary in the same way—a flock of sheep, a herd of pigs, a covey of partridges, a gaggle of geese, a drove of cattle, a team of oxen, a hatch of flies, and a charm of goldfinches.

§4*.2 The Uses of Numbers

The moral to be drawn is that we use numerals in many different ways. We use them as answers to the question 'How many?', as a means of expressing an order, and as a repetitively enlargeable stock of basically similar names.

The difference between the first and second uses is illustrated by the song "Green grow the rushes, Oh!".

I'll sing you seven, Oh!
What are your seven, Oh?
Seven are the seven stars in the sky...

The first two uses of the word 'seven' are as **counting numbers**: I'll sing you one, I'll sing you two, I'll sing you three, I'll sing you four, I'll sing you five, I'll sing you six, . . *etc.* In its third use, where we say 'Seven are the seven stars in the sky', we are using seven as a cardinal number, answering the question 'How many stars are there in the sky?'. If I am asked 'How many stars are there in the sky?', I can answer 'seven', and this is a different use of the word 'seven' from when I am just counting.

The chorus makes a further important point about the natural numbers:

Seven are the seven stars in the sky,
Six are the six Proud Walkers,
Five are the symbols at your door,
Four are the Gospel Makers,

[2] I am indebted to the Crown Prince of Japan for authoritative confirmation of this fact.

Three, three are the rivals,
Two, two are the lilly-white boys,
One is one and all alone and ever more shall be so.

And at that point the chorus stops. It cannot go on. We have run out of counting numbers. For, whatever may be the case with ordinary counting upwards, count-downs are always finite. Even with transfinite numbers, so long as I count backwards, I shall always in the end get to an end.[3] The case is very different if I count forwards. Then there is no end to how far I can go. As soon as I have finished 'I'll sing you seven, Oh!' with its chorus, off I go on 'I'll sing you eight, Oh!'. This is a further lesson to be learned from the song. There is no mathematically determined end to the progression of natural numbers, as there is to any regression: although they have a least number, they do not have a greatest number. And yet in fact the song does not go on for ever, but stops when we have reached twelve. So far as mathematics goes, we need not stop there: we could go on to thirteen; we could go on to fourteen; we could go on to one hundred and forty four: but the mind boggles. After twelve, there is a certain repetitiveness about our numbers. It is rather clever to be able to count up to twelve, and shows a certain mastery of the English language, but when we go beyond twelve, it becomes somewhat mechanical. Twelve is, as we claimed in the previous section, the last number with an individuality all of its own.

Besides the cardinal and ordinal uses of numbers, we use them as names when we want to play down the differences between the entities we are naming—criminals are often referred to only as a number, soldiers have a number, as did civilians during the War. We find it offensive, because it suggests that they are qualitatively identical and only numerically distinct, and implies that people are merely units without any individuality of their own. It is not similarly offensive to use numbers to name things. Car registration numbers and telephone numbers use numerals and letters to constitute an identifying name or address.

Mathematicians are similarly varied in their use of numerals, although more inclined to the third type of use than the first or

[3] This has important consequences for the proof-sequences of §2.3 and §3.4; given any purported proof, I can always work back to the beginning, and thus check every step in the sure knowledge that I shall be able to check them all. So proofs are two-way decidable.

the second. Sometimes numerals are used simply as labels, as when we index a set of entities $a_1, a_2, a_3, \ldots$ *etc*. Sometimes they are addresses, as when we use coordinates to label the points of a space. Often they are taken to refer to abstract entities, which may be studied in their own right, or may be used as models, a set of distinct entities which can be shown to satisfy some set of postulates. Sometimes they are clearly ordinal, as when we talk of a second-order differential equation, or solutions of quintic equations. And sometimes they are cardinal, as when we say a quadratic equation has two roots. Often the uses overlap, as when we use real numbers both as coordinates and to indicate the distance from the relevant axes. But the uses do not always overlap, and it is easy to obtain a distorted view of number by concentrating on only one use.

The three primary senses of the words used for natural numbers will be discussed in this and the following two chapters. Numerals can be used to refer to **Cardinal Numbers**, answers to the question 'How many?'. Frege and Russell took cardinal numbers as basic, and hoped by elucidating them to elucidate the nature of all other numbers as well. They can also be used to refer to **Ordinal Numbers**. Ordinal numbers, as we saw in the previous section, can be expressed in two different ways. They can be expressed by the ordinal terms, 'first', 'second', 'third', *etc.*, or they can be expressed simply by counting, 'one', 'two', 'three', *etc.* In the latter case, we may call them 'Counting Numbers' rather than ordinals, but the important thing about them is just the order they have; and they should be regarded as being logically much the same as the ordinals. Dedekind, Cantor, Peano and Kronecker, all took ordinal numbers as basic, and tried to elucidate them as a step to elucidating numbers generally. Numerals can, thirdly, be used **Symbolically**, where we have them formed repetitively, according to some definite rule, rather dully; we have a finite stock of digits, together with some sort of rule, so that we can always go on, forming new expressions, as required. It is the sort of thing computers can be programmed to do, but, so far as finite numbers are concerned, presents none but technical problems of programming expertise. Peano, again, can be seen as protagonist of this approach, although he was dealing not with decimal, nor even with binary, digits, but with unary ones.

Numbers

1. Cardinal	Answers to question 'How many?' natural numbers Nought, One, Two, Three,...	Frege, Russell
2. Ordinal	Place in a list ordinals counting numbers First, Second, Third,... One, Two, Three,...	Dedekind, Cantor, Kronecker (Peano)
3. Abstract	Symbolic $1, 2, 3, ..., 10, 11, ..., 100, ... 1024, ...$ $1,000,000,000,000,000$ 10^{10} $10^{10^{10}}$ 35K	Peano

§4*.3 How Many?

Frege maintained that we should locate natural numbers on the conceptual map by considering cardinal numbers. Cardinal numbers are, he argued, answers to the question 'How many?', in Latin *Quot?*, in Greek πόσοι; (*posoi?*). He further pointed out that in order to ask the question 'How many?', we need to specify *two* variables: that is, if I ask the question 'How many?' I have got to fill it out in two ways, and ask "How many *whats* there are in *what*?".[4] We get different answers if we ask how many books there are in the *Iliad*, or how many lines; how many packs there are in a pile of cards, or how many honour cards in the game of skat. Berkeley, and before him Plato, had noticed the same point, but had wrongly seen it as an argument for subjectivism.[5] For although it is up to me what question I ask, it is not up to me what the correct answer is. Having asked 'How many books are there in the

[4] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §22, §46,§49,§52, pp.28-29,59-60,62,64.

[5] G.Berkeley, *New Theory of Vision*, §109; *Principles of Human Knowledge*, §xii; *Republic* VII, 522c-526a; and *Hippias Major*, 300e-302e.

Iliad?', I cannot get as a correct answer 'Twenty three' (although if I want to get the answer 'Twenty three', I can elicit it by asking a different question, *e.g.* 'How many un-boring books are there in the *Iliad*?').

The logical status of the two variables is different. The first is an individual variable, ranging over books, un-boring books, lines, packs of cards, cards, honour cards at skat. The second variable is a predicate variable. It indicates what the individuals have in common that prompts us to ask how many there are. It is books or lines in the *Iliad*, not in the *Odyssey* nor the *Aenead*. It is cards in the pile in my friend's hands, not in the cupboard or in a shop. Logicians make the distinction between individual and predicate variables in formal logic by using lower case letters for the former and capital letters for the latter. The convention is to write the capital letter for the predicate first, and the lower-case letter(s) for the individual variable(s) second. We might use b for books, l for lines, I for in the *Iliad*, O for in the *Odyssey*, and A for in the *Aeneid*. We could then ask 'How many?', or in Latin *Quot?*, in these combinations:

Quot I(b)?	*Quot O(b)*?	*Quot A(b)*?
Quot I(l)?	*Quot O(l)*?	*Quot A(l)*?

and obtain the answers

24	24	12
15,688	12,225	9,896

In the standard predicate calculus as it has been developed by logicians in this century, all the emphasis has been on the predicates, with the individual variables being reduced to the status of mere dummies, rather like the dx in an integration. If we want to express 'All humans are mortal' we do not write it as[6]

$$(\mathrm{A}h)Mh$$

but as

$$(\mathrm{A}x)(Hx \to Mx).$$

where $(\mathrm{A}h)$ and $(\mathrm{A}x)$ are "universal quantifiers" (or "quotifiers" as we shall be led to call them) binding h and x respectively.Instead of talking about the noun, 'humans', we use the adjective, 'human', and talk about all beings, not only human ones, saying

[6] For (A) and (V) (normally written (∀) and (∃)) see note on logical symbols on p.xi.

of each of them that if it is Human, it is Mortal.[7] In a similar fashion we rephrase sentences of the form 'Some humans are Wise' which might naturally be expressed by means of the "existential quantifier", $(\mathrm{V}h)Wh$, as 'Some beings are Human and Wise' $(\mathrm{V}x)(Hx \wedge Wx)$. Whatever their merits for ordinary formal logic, such rephrasings are not helpful for an analysis of number, where we do not want to slur over the difference between individual variables and predicate variables. Instead of the standard predicate calculus, we need to use the "Many-Sorted" predicate calculus developed by Professor Smiley,[8] in which there are many sorts of individual variables, with appropriate rules correlating each individual variable, such as h, for humans (as a noun), with the corresponding predicate variable, such as H, for Human (as an adjective). Thus Smiley has a typical axiom

$$(\mathrm{A}h)Hh$$

i.e. All humans are Human. From this, and other highly acceptable axioms, he is able to develop a predicate calculus which is much more natural and closer to our normal ways of thinking and expressing ourselves than the standard predicate calculus studied by logicians. For our purposes, however, we do not need to follow Smiley's development of the Many-Sorted Predicate Calculus. All we need is the general form of question

$$(Quot\ h)Wh?$$

symbolizing the question 'How many humans are Wise?'; or

$$(Quot\ s)Ps?$$

for the question 'How many honour cards at skat are there in this pile?'; or

$$(Quot\ m)Jm?$$

[7] In this passage, for ease of reading, I follow a counter-Germanic convention, giving adjectives a capital initial letter and nouns only a lower-case one.

[8] T.J. Smiley, "Syllogism and Quantification", *Journal of Symbolic Logic*, **27**, 1962, pp.58-72; see also Hao Wang, *Journal of Symbolic Logic*, **17**, 1952, pp.105-116; and Alonzo Church, *Introduction to Mathematical Logic*, Princeton, 1956, pp.339-341.

for the question 'How many moons of Jupiter are there?'; or

$$(Quot\ b)Ab?$$

for the question 'How many books are there in the *Aeneid*?' *etc.* and possible answers to these questions, which we shall symbolize in the general form

$$(\mathcal{Q}\ h)Wh, \quad (\mathcal{Q}\ s)Ps, \quad (\mathcal{Q}\ m)Jm, \quad (\mathcal{Q}\ b)Ab, \ etc.$$

$\mathcal{Q}$ is a sort of quantifier, that is a term of the same logical type as 'some', 'all', 'none', 'no' $(\mathrm{V}x), (\mathrm{A}x)$, ...*etc.* The name 'quantifier' is unfortunate, as the question to which there are possible answers is not *Quantum?* (How much?) but *Quot?* (How many?). On this score, contrary to the general tendency, Latin proves itself a better philosophical tool than Greek. In Greek the word for 'how many?', *πόσοι;* (*posoi*?) and for 'how much?', *πόσον;*, (*poson?*) are very similar, whereas in Latin *quot?* and *quantum?* are obviously different.[9] The Greeks knew that there was a distinction to be drawn, and and were careful in distinguishing the sort of thing, which they called *πλῆθος* (*plethos*) a multitude, about which the question *πόσοι;* (*posoi?*), *quot?*, 'how many?' could be asked, from the sort of thing, which they called *μέγεθος*, (*megethos*) a magnitude, about which the question *ποσόν;* (*poson*?), *quantum?*, 'how much?' could be asked. But although they draw this distinction, and although the Schoolmen were very clear about its importance, it has been largely obscured in modern thought. In modern logic we talk about "Quantification Theory", in which we use quantifiers. But it is "Quotification Theory" we should be studying in logic, with quotifiers answering our questions 'How many men are mortal?', 'How many pigs can fly?', 'How many undergraduates are clever?', and 'How many professors are absent-minded?', by the answers 'All', 'None', 'Some', and 'Not all', respectively. Quantification is something that a quantity surveyor does, and when we come to consider the application of real numbers to magnitudes, and are properly concerned with quantifying them and assigning to them a suitable quantitative measure, we shall need to keep the

[9] But Greek does have another word, *πηλίκος* (*pelikos*), which is specifically concerned with bulk rather than multitude.

distinction very clear.[10] And so, although it is too late to hope to reform the speech-habits of logicians, I shall be pedantic, and speak of quotification, quotifiers, and quotities throughout.

Let us, then, invent the word 'quotifier', and say that $(\mathcal{Q}\ h)$ is a quotifier. Since there are different possible answers to the question 'How many humans are Wise?', we leave at present a blank after the $\mathcal{Q}$, which can be filled by the appropriate numeral when we know the answer to the question. Thus $(\mathcal{Q}_{24}\ b)Ib$ is our answer to the question: *Quot I(b)?*, 'How many books are there in the *Iliad*?'. Strictly speaking the individual variables, h and b in the examples above, are redundant, being adequately specified by the h in the Wh and the b in the Ib. But we shall have occasion to use more complex formulae in which there are other bound individual variables, where it will help to indicate which was the relevant individual variable in question; and in any case it is desirable to stress the analogy between

$(\mathcal{Q}\ h)$ and $(\mathcal{Q}_{24}\ b)Ib$ on the one hand,
and $(\mathrm{A}x)$ and $(\mathrm{V}x)$ on the other.

Besides the quotifiers studied in formal logic, there are some others, such as 'many', 'most', 'few', and 'a few', which can also be given as answers to the question 'how many?', in addition to the numerical answers which seem most often to be called for. All these should be classed as quotifiers.[11] Although less specific than numerical answers, they are analogous in being of the same logical shape as numbers; or, more illuminatingly, numbers are analogous to them.

Frege calls the complex expression, Wh in our example, which can be read as 'humans who are Wise', or in his examples 'moons of Jupiter', 'horses who draw the king's carriage', 'leaves on a tree', a "concept", but he is at pains to argue that they are not merely creations or attributes of our minds. Concepts are objective (as also are numbers, which are, in Frege's terminology "assigned" (*legen*)

[10] In ch.11.

[11] It is worth quoting the sentence: "I am the one who gave his all in the fight of the few against the many", cited by Paul Benacerraf in his "What Numbers Could Not Be", *Philosophical Review*, **74**, 1965; reprinted in Paul Benacerraf and Hilary Putnam, eds., *Philosophy of Mathematics*, 2nd ed., Cambridge, 1983, p.284.

to concepts, or "belong" (*kommen*) to concepts[12] but are not material objects or spatially located ones, nor attributable only to collections of material objects, as he accuses Mill of supposing. He quotes Locke and Leibniz, who point out that we can ask and answer numerical questions about non-material objects just as well as about material ones[13]—How many roots does a quadratic equation have? How many groups are those of order eight? How many persons in the Trinity? How many figures of the Aristotelian Syllogism? Frege also points out the difference between numbers and ordinary predicates which might be said to be assigned to, or to belong to, objects. We speak of a tree's having 1000 leaves in a quite different sense from its having green leaves.[14] From the latter it follows that each leaf is green, but it does not follow from the former that each leaf is 1000. Arguing the other way, Frege points out that whereas from 'Solon was wise' and 'Thales was wise' there follows 'Solon and Thales were wise' from 'Solon was one' and 'Thales was one' there does not follow 'Solon and Thales were one'.[15] A similar ambiguity occurs in English childhood experience. When the hostess at a children's party says 'Now we are six', she may mean it individually, that each one of us is six years old, and go on to say 'we are too old to cry', or she may mean it collectively, and go on to say 'we can form a ring, and play ring o' ring of roses'. *Being six years old* is logically speaking, an adjective: our together being six in number is not. The logical grammar of numerals is thus clearly different from that of ordinary adjectives. Even in languages where the numeral declines and agrees with the noun, the entailment patterns are like those of the word 'all', *omnes*, πάντες (*pantes*), so that numerals in these uses should be classed as quotifiers rather than ordinary adjectives or nouns.

[12] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §46, p.59; §55, p.67; and elsewhere.

[13] *The Foundations of Arithmetic*, §24, p.31; citing John Locke, *Essay on Human Understanding,* II, 16, 1, and G.W. Leibniz, (Erdman ed., pp.8 and 162).

[14] *The Foundations of Arithmetic*, §22, p.28.

[15] *The Foundations of Arithmetic*, §29, pp.40-41.

§4*.4 Nought

That Frege's arguments lead to the conclusion that numerals are a sort of quotifier is shown most clearly by his account of the number Nought. To say that the number of moons of Venus is nought is simply to say that there are no moons of Venus, which is to say that there does not exist any moon of Venus, which can be expressed symbolically:

$$\neg(\mathrm{V}m)Vm) \qquad (\text{or} \quad (\mathrm{A}m)\neg Vm)$$

where m stands for moon, and V for orbiting round Venus (to be distinguished from (V), the existential quotifier). The case for explaining 'nought' in terms of 'naught' or 'no' or 'none' is overwhelming. Further support, if any were needed, comes from Frege's explicit statement that existence is analogous to number, and his reference to the standard critique of the Ontological Argument.[16] The core of the criticism is that the Ontological Argument assumes that existence is a property whereas it is really something expressed by a quotifier. But if existence is something expressed by a quotifier and is analogous to number, number must be something expressed by a quotifier too.

Nought is the cardinal number to begin with: if asked 'How many moons of Venus are there?' or 'How many stars in the Pleiades can you distinguish?', I may say 'Nought' but I may well say 'None'. 'Nought' and 'None' are both possible answers to the cardinal question 'How many?', and no account of cardinal numbers that excludes them will be satisfactory. Nought thus plays a crucial role in the cardinal approach. It clinches the argument for holding that numerals are quotifiers, and it anchors them by identifying one numeral as one of the quotifiers we are already familiar with. If we can give a recursive definition of each numeral in terms of its predecessor, then we shall have achieved a general schema of definition for every one. We can, as it were, sing our way through the chorus of "Green Grow the Rushes, Oh!", defining each in terms of its predecessor, until our count-down reaches nought, and then we can define nought in entirely non-numerical terms. Every other numeral is defined generically as a quotifier,

[16] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §53, p.65.

but specifically in numerical terms as the successor of an quotifier. But nought is not the successor of any number; instead, it is equivalent to None, and None is definable in terms of either All or Some, either, that is, as $(\mathrm{A}x)\neg$ or as $\neg(\mathrm{V}x)$, so that we can give a completely explicit definition of the nought quotifier, and say:

$(\mathcal{Q}_0)Fx$ is defined as $(\mathrm{A}x)\neg Fx$

'Nought' is not the only word for nought. We talk of absolute zero, and use the symbol '0' to express a cypher. There are, correspondingly, other conceptual connexions. Absolute zero, 0K, is that temperature lower than which it is impossible to go. Zero is the measure of a minimum magnitude, and when we come to measure theory in Chapter Twelve, minimality is a more important link than none-ness. We can also characterize 0 as the natural number that is not the successor of any natural number, and as the Identity element under the operation of addition.[17] Thus if, as in Chapter Three, we introduce negative numbers as equivalence classes of ordered pairs of positive numbers, (x, y), subject to the equivalence relation $(x, y) \approx (z, w)$ iff[18] $x + w = y + z$, we can introduce 0 as (x, x), or equivalently (y, y), or (z, z), *etc.* We then have $(z, w) + (x, x) = (z, w)$, for all (z, w). Another characterization, less obvious, but of some theoretic interest, is that 0 is the "universal element" under multiplication: that is, $0 \times x = 0$ whatever the x. In this it resembles ∞ under addition, the speed of light in the Special Theory of Relativity, $\top$ under $\vee$ and $\perp$ under $\wedge$ in propositional calculus, where $p \vee \top$ is $\top$ for all p, and $p \wedge \perp$ is always $\perp$.[19]

§4*.5 Quotifiers and Quotities

If we can give an adequate account of nought, we can reasonably hope to do the same for other numerical quotifiers. Frege sketches a recursive definition, and shows that it satisfies requirements essentially similar to those of Peano's postulates,[20] but the modern

[17] See below, §6.2, §12.7.

[18] Short for 'if and only if'.

[19] See further, §11.7.

[20] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §76-§79, pp.89-92.

exposition of David Bostock is simpler.[21] We need to make a distinction between the assertion that there are *exactly* n things of type x that are F and the assertion that there are *at least* n things of type x that are F. The distinction is brought out by the limerick

There was an old man of Lyme
Who married three wives at a time;
When asked "Why the third?"
He replied "One's absurd,
And bigamy, Sir, is a crime!"

It is fairly easy to give recursive definitions of 'There are at least n things of type x that are F' and 'There are at most n things of type x that are F', which are the concepts we need for defining bigamy and monogamy. We could then give Lyme definitions, and define 'There are exactly n things of type x that are F' as the conjunction of 'There are at least n things of type x that are F' and 'There are at most n things of type x that are F', but we can more economically define 'There are exactly n things of type x that are F' by the conjunction of 'There are at least n things of type x that are F and the denial of 'There are at least n' things of type x that are F', where n' is the next number after n. If there are at least six Proud Walkers and it is not the case that there are at least seven Proud Walkers, then there are exactly six Proud Walkers.

So let us start defining 'There are at least.....things of type x that are F'. The usual existential quotifier $(\mathrm{V}x)Fx$ expresses that there is at least 1 thing of type x that is F. We could write it explicitly as $(\mathrm{V}_1x)Fx$. Let us set out this definition formally:

$(\mathrm{V}_1x)Fx$ is defined as $(\mathrm{V}x)Fx$

It then is easy to define $(\mathrm{V}_2x)Fx$ in terms of $(\mathrm{V}_1x)Fx$

$(\mathrm{V}_2x)Fx$ is defined as $(\mathrm{V}x)(Fx \wedge (\mathrm{V}_1y)(Fy \wedge y \neq x))$

We should note in this definition that $(\mathrm{V}x)$ is used on one occasion and (V_1y) on the other, even though we have defined the latter as being exactly the same as the former. The reason for this becomes clear when we generalise and make similar definitions for $(\mathrm{V}_2x)Fx$, $(\mathrm{V}_3x)Fx$, and generally $(\mathrm{V}_nx)Fx$, thus:

$(\mathrm{V}_3x)Fx$ is defined as $(\mathrm{V}x)(Fx \wedge (\mathrm{V}_2y)(Fy \wedge y \neq x))$

Similarly

$(\mathrm{V}_4x)Fx$ is defined as $(\mathrm{V}x)(Fx \wedge (\mathrm{V}_3y)(Fy \wedge y \neq x))$

and, generally,

[21] David Bostock, *Logic and Arithmetic*, I, Oxford, 1979, ch.1, §2, pp.9-25.

$(\mathrm{V}_{n^0}x)Fx$ is defined as $(\mathrm{V}x)(Fx \wedge (\mathrm{V}_n y)(Fy \wedge y \neq x))$

This last definition can be rephrased in terms of predecessors rather than successors, so as to make it obvious that we are embarking on a "count-down" definition which must come to a conclusion. Let us write n^* for the predecessor of n. Then

$(\mathrm{V}_n x)Fx$ is defined as $(\mathrm{V}x)(Fx \wedge (\mathrm{V}_{n}\ y)(Fy \wedge y \neq x))$

Having now given a general recursive formula for defining 'There are at least n things of type x that are F', we define our numerical quotifiers in terms of there being at least n things of type x that are F, but not n' things of type x that are F. Formally, we first define $(\mathcal{Q}_0 x)Fx$:

$(\mathcal{Q}_0 x)Fx$ is defined as $\neg(\mathrm{V}_1 x)Fx$

(which is the same as $\neg(\mathrm{V}x)Fx$, which in turn is the same as $(\mathrm{A}x)\neg Fx$, as used in our first definition); we then go on to define $(\mathcal{Q}_1 x)$ in terms of $(\mathrm{V}_1 x)Fx$ and $(\mathrm{V}_2 x)Fx$:

$(\mathcal{Q}_1 x)$ is defined as $(\mathrm{V}_1 x)Fx \wedge \neg(\mathrm{V}_2 x)Fx$

Similarly $(\mathcal{Q}_2 x)$ is defined as $(\mathrm{V}_2 x)Fx \wedge \neg(\mathrm{V}_3 x)Fx$

and generally: $(\mathcal{Q}_n x)$ is defined as $(\mathrm{V}_n x)Fx \wedge \neg(\mathrm{V}_{n^0}x)Fx$

This completes the formal definition of the numerical quotifiers. The definition is a bit more cumbersome than we might have hoped, but it has none the less achieved our aim of giving an account of the cardinal use of numerals and grounding them in other, non-numerical concepts. To the question "What are numerals?" we answer: "Numerals are quotifiers, that is to say, answers to the question 'How many?', and of the same logical type as 'All', 'Some', 'None', and 'Not all'; and indeed, the numeral 'nought' is just another word for 'none'." But if we were further asked what *numbers* are, as distinct from numerals, we could not answer similarly. As Hunter felicitously puts it:[22]

> Any numerical adjective, but no numerical noun, can be unambiguously defined in the language Q <*i.e.* first-order logic>, interpreted only with respect to the quantifiers, the connectives, and a symbol for identity.

Some further step is needed. Plato, similarly dissatisfied with accounts of how words such as 'just' or 'virtuous' were used in particular contexts, postulated some abstract entity that lay behind their various uses, and eventually, somewhat apologetically, coined

[22] Geoffrey Hunter, *Metalogic*, London, 1971, §49, [49.6], p.207.

the word πoιότης (*poiotes*), 'qual-ity' from πoῖoς (*poios*), the equivalent of the Latin *qualis*, to mean what he had in mind.[23] We might likewise say that numbers are "quotities", where quotities stand to quotifiers as qualities do to adjectives and as substances do to substantives.

In first-order logic we use two quotifiers, (Ax) and (Vx), and they are classed along with the truth functors, $\neg$, $\leftarrow$, &, $\vee$, $\leftrightarrow$, $\top$ and $\perp$ (otherwise known as sentential functors, sentential connectives, or logical constants). We can say that there are *sixteen* different binary truth functions, which can *all* be expressed in terms of negation and disjunction. If truth functions can be counted, and quotities are like truth functions, then quotities look the right logical shape to be counted too. If we can, so as to be able to say that there are three prime numbers between 10 and 20, then we are in effect quotifying over quotities, and treating them as variables, in which case Quine's *dictum* "To be is to be the value of a variable" applies, and we are ascribing to quotities high ontological status.[24] It is a test that qualities pass: "He has three good qualities", we say in a reference, "loyalty, adaptability and industry"; and we use the universal quotifier of qualities in Russell's example of the man who had all the qualities of Napoleon. Quotities and qualities are in the same case. We can quotify over qualities, and so should be able to quotify over quotities themselves, being likewise countable; and in the absence of cogent metaphysical arguments against their being counted or otherwise quotified over, we should be happy to accept them as respectable abstract entities with a reasonably clear conscience.

If we have a whole series of quotifiers:

$$(\mathcal{Q}_0\ ), (\mathcal{Q}_1\ ), (\mathcal{Q}_2\ ), (\mathcal{Q}_3\ ), \ldots, (\mathcal{Q}_{24}\ ), \ldots,$$

we can discern a pattern, and pick out the subscripts:

$$0,\ 1,\ 2,\ 3, \ldots,\ 24, \ldots,$$

in much the same way as mathematicians do when they discern the same group structure in a number of different groups, or the

23 *Theaetetus* 182a.

24 W.V.O.Quine, *From a Logical Point of View*, Cambridge, Mass., 1953, chs. 1 and 6; *Methods of Logic*, §38.

similarity between a finite projective geometry and the rules of a lunch club.[25] It is not so copper-bottomed a procedure as a formal logician would like: he may complain that it is merely hand-waving. But the edge of that criticism is turned, by the fact that hand-waving—or at least informal argument—cannot be altogether dispensed with. If the procedure is generally intelligible, and mathematicians seem to be able to be talking about the same things, when they talk about numbers, we need not be greatly put out by hard-line Logicists bewailing our sloppy methods, once we realise that they are themselves, in their desire for absolute rigour, crying for the moon.

But Frege did cry for the moon.

§4*.6 Frege's Extensions and Sets

Frege drew back from identifying cardinal numbers with quotities, and defined them instead in terms of extensions—*Umfänge*—of concepts. It was a disastrous move, and led to the wreck of his logicist programme. He was led to it because the definition in terms of quotifiers was merely a contextual definition. It enabled us to explain what cardinal numbers were when they turned up in certain contexts, but not to say what they were ἁπλῶς, (*haplos*), simply. We do not use numbers just to answer the question "How many?" but in many other contexts too. We talk about them, and in particular we count them. We need, therefore, to be able to tell them apart from other things and apart from one another.

Frege was concerned with how to exclude interlopers, such as Julius Caesar, from being counted as numbers.[26] It is tempting to take a brusque line. We do not need a definition of number to be able to know that Julius Caesar is not a number: a simple acquaintance with Roman history is enough. Julius Caesar is clearly not a quotity, just as he is clearly not a quality. We do not need an exact definition of a quality in order to know that some things are not qualities, and similarly we do not need to define a quotity—certainly we do not need to replace quotities by extensions or sets—in order to assure ourselves that Julius Caesar is not one of them. Later, however, we shall have to soften our response. There is a real problem of how to exclude interlopers

[25] See above, §2.4.

[26] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §56, p.68.

from the realm of numbers, and as we shall see, there is no answer to that problem within the confines of first-order logic.[27] But the problem is not a categorial one. It is not because we cannot define the category of quotity—answers to the question *Quot?*—that we have problems in delimiting exactly what the numbers are. And so this, though a real problem, is not an argument for taking numbers to be extensions rather than quotities, whatever quotities are.

Frege was much more concerned with the problem of there being different numerical expressions referring to the same number, *e.g.* 'twelve' and 'dozen', or 'hundred' and 'century', in much the same way as 'The Morning Star' and 'The Evening Star' refer to the same planet. Numbers must be, he held, *selbständiger Gegenstände*, self-subsistent objects, and hence capable of re-identification, and so re-identifiable (*wiederkennbare*) on other occasions and under other descriptions.[28] Venus is a self-subsistent object because it can be referred to under different descriptions; it does not matter whether we talk of the Morning Star or of the Evening Star; in both cases we are talking of the one and the same object which exists independently of us. In the same way Frege wants to be able to be sure that however we refer to the number 12, we are referring to the same thing, and so seeks a criterion of identity to enable us to tell when different quotifier expressions such as 'twelve', 'dozen', 'the number of the Apostles', 'the number of eggs in this box', 'the number of old-time pennies in a shilling', 'the number of calendar months in a year', which evidently have a different *Sinn*, connotation, have nonetheless the same *Bedeutung*, denotation. He does this by talking of the *Umfang*, extension, of a concept, and considering the equivalence class of all those extensions of concepts that are *gleichzahlig* (translated by Austin as 'similar', and by others as 'equinumerous') with one another,[29] quoting Hume,[30] to the effect that the numbers assigned to two collections are equal when there is a one–one correlation between the members of each collection;

[27] See below, §6.4.

[28] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §56 *ad fin.*, p.68.

[29] See further below, §5.2.

[30] G.Frege, *The Foundations of Arithmetic*, §56 *ad fin.*, p.73, quoting David Hume, *Treatise on Human Nature*, Bk.I, Part iii, Sect.1, in Selby-Bigge's ed., Oxford, 1888, p.71.

he defines number in terms of extensions that are equinumerous to one another. He is able to define equinumerosity in terms of a one–one mapping, which can in turn be defined in first-order logic. So he can reasonably reckon that he has succeeded in building the whole of arithmetic on the sure foundation of logic:

> ... the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one ...[31]

But Frege's extensions were unable to bear the weight he wanted to place on them. His own account of them was not sufficiently clear or acceptable to be absorbed into the mainstream of philosophical thinking, and they were construed as classes by Russell, and more generally as sets. Although it is not a historically accurate representation of Frege's thought, it is reasonable for us to construe the approach of Frege and Russell in set-theoretical terms, and evaluate the logicist definition of numbers as sets of **equinumerous** sets. The use of equivalence classes to introduce new mathematical concepts is standard,[32] and has the merit of carrying over to transfinite numbers.

Naive set theory, however, was shown by Russell's paradox to be inconsistent.[33] Axiomatized set theory avoids the paradoxes, and is, so far as we know, consistent, but axiomatic set theory is messy, and much more mathematical than logical. Books on axiomatic set theory turn out to be about transfinite arithmetic. It is not much catch to define the natural numbers in terms of set theory, and then find oneself diving into transfinite cardinals. It may be a useful technical exercise, to confirm, what we already believe, that almost all mathematics can be expressed in set theory, but if our aim is philosophical illumination, we feel that we have succeeded only in explaining the obscure in terms of the even more obscure.

The set-theoretical exegesis is open to other objections, some of which can be countered, but which cumulatively draw us back from Frege's way out, and lead us once again, after we have developed the concepts of isomorphism and factoring,[34] to ask what quotifiers refer to, and if they are said to refer to quotities, what quotities really are.

[31] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §87, p.99.

[32] See above, §3.7, for the introduction of rational numbers.

[33] See below, §12.2.

[34] in §9.2-§9.4 and §9.8.

§4*.7 Paradigm Sets

Frege's criterion of equinumerosity enables us to recognise the same cardinal number exemplified in different collections, but in practice we seldom talk of 'the number of the Apostles', and nearly always talk of 'twelve' and other numerals. Frege explicates them by means of a sequence of paradigm sets:[35]

$$\{0\}, \{0,1\}, \{0,1,2\}, \{0,1,2,3\}, \{0,1,2,3,4\}, \{0,1,2,3,4,5\}, \ldots$$

which have, respectively $1,2,3,4,5,6,\ldots$ members, and in effect suggests that we characterize first 0 as the set of all sets equinumerous with the null set, Λ, and then
1 as the set of all sets equinumerous with $\{0\}$,
2 as the set of all sets equinumerous with $\{0,1\}$,
3 as the set of all sets equinumerous with $\{0,1,2\}$,
4 as the set of all sets equinumerous with $\{0,1,2,3\}$,
5 as the set of all sets equinumerous with $\{0,1,2,3,4\}$,
6 as the set of all sets equinumerous with $\{0,1,2,3,4,5\}$, *etc.*
This is a very satisfactory characterization of the successor relation in set-theoretical terms, and one which Cantor generalised to cover transfinite numbers too. If we now express $0,1,2$, *etc.* explicitly in terms of Λ, we obtain the following explicit definitions:
0 is the set of all sets equinumerous with the null set, Λ
1 is the set of all sets equinumerous with $\{\Lambda\}$,
2 is the set of all sets equinumerous with $\{\Lambda, \{\Lambda\}\}$,
3 is the set of all sets equinumerous with $\{\Lambda, \{\Lambda\}, \{\Lambda, \{\Lambda\}\}\}$,
etc.

This seems very satisfactory, but it is not the sole satisfactory way of characterizing numbers in set-theoretical terms. Instead of

[35] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §79, p.92. T.J.Smiley, "Frege's 'series of natural numbers'", *Mind*, **97**, 1988, pp.583-584 points out that Austin wrongly uses 'the series of natural numbers' in §76 and thereafter. The correct rendering is 'the natural series of numbers'. It is natural because it is determined by the successor relation. The numbers are the cardinals, infinite as well as finite. The contrast between the natural series of numbers and the natural series of numbers beginning with 0 is that the former consists of various islands, each infinite number being unrelated to anything but itself, while the latter picks out just the island to which 0 belongs, that is finite numbers.

using equinumerosity, we might accept type theory, and identify $0, 1, 2, 3$, *etc.* by $\Lambda, \{\Lambda\}, \{\{\Lambda\}\}, \{\{\{\Lambda\}\}\}$, *etc.* Or, again, we might identify $0, 1, 2, 3$, *etc.* by $\Lambda, \{\Lambda\}, \{\Lambda, \{\Lambda\}\}, \{\Lambda, \{\Lambda, \{\Lambda\}\}\}$, *etc.* And this ambiguity has been held by Benacerraf to tell against the explication of natural numbers in terms of sets.[36]

When C.P. Snow's novel, *The Masters*, first came out, it was rumoured that the head of a certain Cambridge college was dissuaded only by the most earnest entreaties of his friends from suing Snow for libel, on the grounds that Snow's character Jago was clearly meant to be a likeness of him. If the case had come to court, it would have posed an interesting problem. For there was another head of another Cambridge college who also was convinced that Jago was a portrait of him, and who also was dissuaded only by the earnest entreaties of his friends from suing Snow. Two cases of libel, each claiming that a fictional character is a recognisable portrait of two different people are self-defeating. In the same way Benacerraf argues that since there are more than one set-theoretical representations of the natural numbers, none of them can be *the* representation, and all must be inadequate. He tells the story of two boys, Ernie and Johnny, each brought up by militant Logicists to believe that numbers really were sets, but as it transpired different sets. Ernie is brought up to accept Frege's exegesis, Johnny to accept the first alternative given above. And then they have a schoolboy quarrel about which is right, a question to which there is, in the nature of the case, no definitive answer. If numbers really were sets, there would be a unique set that really was the number in question to the exclusion of all other impostors.

Benacerraf's argument is not cogent against the Frege–Russell account, but only against those set-theorists who identify cardinal numbers *as*, rather than *by means of*, particular paradigm sets. After all, what Frege was doing was to introduce an equivalence class. He identified numbers with an equivalence class of extensions. And the Logicists have a perfectly good equivalence class, even though different Logicists choose different paradigms of it. It is the same as with colour. Ernie and Johnny were both taught to use the word 'yellow', but Ernie was taught that it was the colour of daffodils,

[36] Paul Benacerraf, "What Numbers Could not Be", *The Philosophical Review*, **74**, 1965, pp.47-73; reprinted in Paul Benacerraf and Hilary Putnam, eds., *The Philosophy of Mathematics*, 2nd ed., Cambridge, 1983, pp.272-294.

while Johnny was taught that it was the colour of primroses. They both could use the word in the same way, but if they were of a disputatious disposition, they could go on quarrelling about what the word really meant, until they realised that there was no real point at issue. In a more closely analogous case, if Benacerraf's argument were valid, we could have two militant rationalists, one of whom used fractions and the other used decimals. Or again one mathematician might always use binary notation, and use 100 to express the number four, and express fractions in "perquartages", as we sometimes do in percentages, insisting that a half was really 10%. That would be perfectly possible. It does not show that rational numbers are not to be explained in terms of ordered pairs of natural numbers; it is just that what they are is an equivalence class of all those that cross-multiply. So, too, with Benacerraf's boys: their disagreement is not a substantial one; it is not about the nature of natural numbers, which are equivalence classes of all those sets that are equinumerous with one another, but rather about numerals, and is no less surprising than if one of them used Arabic, and the other Roman, numerals; or if one of them was speaking in German, and saying *drei*, and the other in Greek, and saying τρεῖς (*treis*). Although they indubitably disagree, there is an underlying agreement. Benacerraf's argument would wean us from maintaining that a particular natural number just *was* a particular set, in the same way as we could be shown that the rational number one-half was not *identical with* the ordered pair $\{1, 2\}$. But a realist could still identify a natural number with a quotity, or indeed, with an equivalence class of equinumerous sets, just as he identifies rational numbers with an equivalence class of ordered pairs granted a suitable equivalence relation.

In Benacerraf's example, the suitable equivalence relation will not be that of equinumerosity, which Frege and Russell used, but a more complicated one mapping the numerosity of Ernie's sets onto the type of Johnny's. Let us, for the sake of completeness, list the sets of a third boy, Tommy, whose sets are the third sequence suggested above, of the same type as Johnny's, but not as "thin".

Different Paradigm Sets for Natural Numbers

	Ernie	Johnny	Tommy
0	Λ	Λ	Λ
1	$\{\Lambda\}$	$\{\Lambda\}$	$\{\Lambda\}$
2	$\{\Lambda, \{\Lambda\}\}$	$\{\{\Lambda\}\}$	$\{\Lambda, \{\Lambda\}\}$
3	$\{\Lambda, \{\Lambda\}, \{\Lambda, \{\Lambda\}\}\}$	$\{\{\{\Lambda\}\}\}$	$\{\Lambda, \{\Lambda, \{\Lambda\}\}\}$
...	...	...	...

Table 4.7.1

Although Benacerraf's argument does not refute Logicism, it none the less points to a weakness. We can best see this if we erase all the symbols except the right-hand brackets, thus:

Erased Paradigm Sets for Natural Numbers

	Ernie	Johnny	Tommy
0			
1	$\}$	$\}$	$\}$
2	$\}\}$	$\}\}$	$\}\}$
3	$\}\}\}$	$\}\}\}$	$\}\}\}$
...	...	...	...

Table 4.7.2

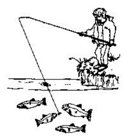
It is evident then that all we have really done is to replace the old symbols, $0, 1, 2, 3$, by a string of right-hand brackets; we are using unary digits, only writing '}' instead of '1', or '/', *etc.* and open to the same criticisms as Frege made against his predecessors.[37] The Logicists, like the Formalists, make out that they are giving an analysis of number in purely logical terms, but are having to assume that they have numbers available in their meta-logic.[38]

Benacerraf's argument is right, moreover, in two further respects. It stresses the uniqueness requirement implicit in realism,

[37] G.Frege, *The Foundations of Arithmetic*, tr. J.L.Austin, Oxford, 1950, §34-§39, pp.44-51.

[38] See above, §3.6, and below, §14.9.

and it raises the importance of structure. Reality is not plural.[39] And the cardinal approach fails to register the importance of the numbers as an order. The picture we obtain from Frege and Russell is of the equinumerosity criterion being used to pick out large bundles of classes from a huge pool of classes, but each equivalence class constituting a cardinal number being entirely separate from every other cardinal number, not taking into account that if twelve are the twelve Apostles, and one falls out, then eleven will be the number of those who went to heaven. Numbers are by nature not only cardinals saying how many there are in a given collection of items, but ordinals too, whose essence in each case depends on its relations with others, that is, its position in a structure.



Small boy fishing for numbers in Frege's pool of equinumerous extensions

[39] See below, §15.2.