# Chapter 2
# Geometry



## 2.1 Euclid

**If not Logicism, then What?**

Holes have been picked in Euclid's arguments in the last hundred years. Hilbert uncovered a number of assumptions Euclid failed to make explicit.[1] A full formulation of the axioms necessary for the rigorous development of Euclidean geometry is now available. It is much more complicated, and much less intelligible, than Euclid's presentation, and we should ask ourselves what exactly the point of axiomatization is. Euclid high-lighted certain assumptions he needed in order to prove geometrical theorems, but took others—assumptions of order and of continuity—for granted. They are not assumptions we normally question, although undoubtedly they can be questioned. If we want absolute formal rigour, as Hilbert did, then we should make all our assumptions absolutely explicit, as Hilbert did, and produce a formally valid proof-sequence. But we achieve formal rigour at a price. Whereas Euclid's presentation is intelligible and has immense intellectual appeal, Hilbert's is unintelligible, except to those who already know their geometry

[1] David Hilbert, *Foundations of Geometry*, originally published as *Grundlagen der Geometrie*, Leipzig, 1899.

backwards, and has no appeal for the wider public. Contemporary opinion among philosophers of mathematics discounts this, and regards Hilbert as having done a proper job which Euclid did only imperfectly. But this is to assume a formalist standpoint which is itself open to question. Without decrying the value of Hilbert's work from the formalist point of view, we may wonder whether this was what Euclid was, or should have been, trying to do. Outside formal logic, the axiomatic approach remains much more in the spirit of Euclid than of Hilbert. Physicists often present Newtonian mechanics, the Special Theory of Relativity, the General Theory of Relativity, or quantum mechanics, in terms of axioms, which high-light the peculiar assumptions of the theory in question, but take a lot else for granted. It is a perfectly reasonable procedure, not only for introducing the subject to schoolboys, but for identifying it concisely for professionals. For the most part, in identifying or expounding Euclidean geometry our need is to distinguish the peculiar features of that geometry from others which might reasonably be put forward instead. That each line defines an order and is continuous is not normally in question, and it only clutters up communication to anticipate questions that are not going to be asked. Brevity is not only the soul of wit, but the condition of communication. Total explicitness is not only often uncalled for, but obfuscatory. Euclid should not be criticized for his lack of rigour, but praised for his sense of relevance.

### 2.2 The Fifth Postulate

Plato's programme was premature. But geometry was axiomatized, the programme being carried out by Eudoxus and Euclid, who succeeded in deriving it all from five special axioms or postulates, in Greek *αἰτήματα* (*aitemata*), together with some general notions, *κοιναὶ ἔννοια* (*koinai ennoiai*), of a purely logical character, *e.g.* that if $a$ is equal to $b$ and $c$ is equal to $b$, then $a$ is equal to $c$.

The five postulates Euclid wanted us to grant were:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and diameter.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the

angles less than two right angles.

These were generally taken to express self-evident truths. This is somewhat surprising, in that the first three are not really propositions at all, but instructions expressed in the infinitive, and the last too complex to be self-evident—no finite man can see it to be true, because no finite man can see indefinitely far to make sure that the two lines actually do meet in every case. Many other formulations of the fifth postulate have been offered, both in the ancient and in the modern world, in the hope of their being more self-evidently true. Among them we should note:

a) From a point not on a given line one and only one line can be drawn parallel to the given line. (Playfair)
b) The sum of the angles of a triangle equals two right angles.
c) Given a figure, another figure is possible which is similar to the given figure and of any size whatever. (Wallis)
d) There exist two unequal triangles with equal angles. (Saccheri—and perhaps also Plato)
e) In a right-angled triangle, the square on the hypotenuse equals the sum of the squares on the other two sides. (Pythagoras)

It is far from obvious that these propositions are equivalent. Playfair's is the closest to Euclid's, and can be regarded as a modern version of it, explicitly mentioning parallel lines, and alone deserving the name "parallel postulate". The triangle property, that the sum of the angles of a triangle equals two right angles, is fairly easily shown to be equivalent. Much more significant are the axioms about similar triangles, put forward in a stronger form by John Wallis, an Oxford don of the Seventeenth Century, and in a weaker by Geralamo Saccheri, a Jesuit priest in the Eighteenth Century. Granted Wallis' axiom (c), we can prove that the sum of the angles of a triangle equals two right angles, as in Figure 2.2.1 (p.36).

Another argument, displayed in Figure 2.2.2, shows that Pythagoras' theorem is easily proved by means of similar triangles.

We may ask, on behalf of generations of schoolboys who have struggled with Euclid's "windmill" proof of his proposition 1.47, why Euclid preferred his much more cumbersome proof. The answer lies in the last assumption in the proof given in Figure 2.2.2, and the difficulties due to the existence of incommensurable magnitudes, itself established as a consequence of Pythagoras' theorem. The *Meno* argument shows that the diagonal of a square has length $\sqrt{2}$ times that of the sides. But $\sqrt{2}$, as Pythagoras or one of his

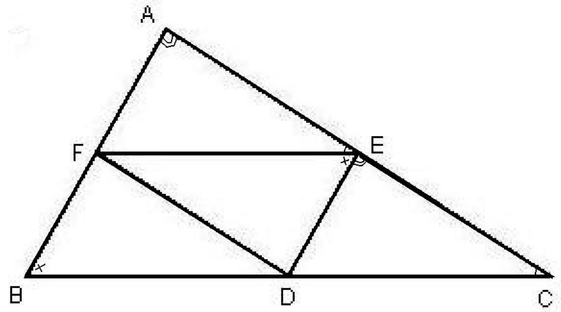



Figure 2.2.1 Proof of △ from Wallis: let $ABC$ be a triangle. Let $\triangle AFE \approx \triangle ABC$ and half the linear size. Then

$$\frac{AF}{AB} = \frac{AE}{AC} = \frac{FE}{BC} = \frac{1}{2}$$

So $AF = FB$ and $AE = EC$.
Let $BD = DC$, and hence $FE = BD = DC$.
Then (some argument left out here) $\triangle CED \approx \triangle CAB$, whence $ED = (1/2)AB = FB$.
So in $\triangle EFD$ and $\triangle BDF$ $EF = BD$, $DE = FB$, and $FD$ is common.
So $\triangle EFD \equiv \triangle BDF$, and $\angle DEF = \angle FBD$.
But $\angle BCA = \angle FEA$ and $\angle CAB = \angle CED$
[and $\angle ABC = \angle FBD$].
So $\angle ABC + \angle BCA + \angle CAB = 180°$.

followers discovered, cannot be expressed as the ratio of two whole numbers, and the similar triangles approach, which says that the ratios of the sides in similar triangles is equal is therefore suspect. Euclid later, in his theory of proportion, almost anticipated Dedekind's definition of a real number, but in his geometrical exposition preferred the technically more complicated but conceptually less suspect approach which did not invoke similar triangles at all.

A chance remark in the *Gorgias* suggests that Plato was thinking about similar triangles at about the time he was seeking to establish the foundations of geometry. In *Gorgias* 508a5-7 he distinguishes "geometrical" from "arithmetical" equality, the former

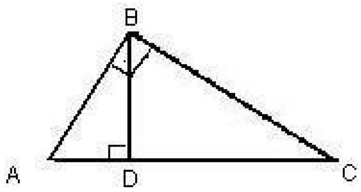



Figure 2.2.2 Proof of Pythagoras by Similar Triangles: let $\triangle ABC$ have $B$ a right angle. Drop perpendicular from $B$ to $AC$ at $D$. Then $\triangle ADB \approx \triangle ABC$ and $\triangle BDC \approx \triangle ABC$.

$$\text{So } \frac{AD}{AB} = \frac{AB}{AC}$$

$$AD.AC = AB^2$$

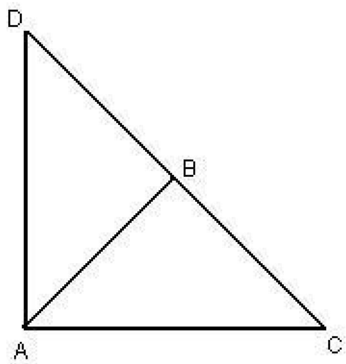
$$\text{and } \frac{DC}{BC} = \frac{BC}{AC}$$

$$DC.AC = BC^2$$

So $(AD + DC).AC = AB^2 + BC^2$. So $AC^2 = AB^2 + BC^2$. [We have assumed that angles of $\triangle$ add up to 180°, and that $AD/AB$ *etc.* are well behaved.]

being only proportionate, whereas the latter is a strict equality. Aristotle takes up the distinction in his *Nicomachean Ethics*, and again in his *Politics*, and makes it the basis of his exegesis of distributive justice. Plato and Aristotle saw that there was some universality about the concept of justice, and that justice required that we treat like alike, but wanted to avoid the implication that what was sauce for the goose was necessarily and always sauce for the gander; justice, Plato argued, required that we treat like cases alike, but also that we treat unlike cases differently. "Geometrical equality" enabled Plato and Aristotle to reconcile the underlying principle that there should be some similarity of treatment for all with there being differences of actual treatment in different circumstances. Each person should be given their fair share, said Aristotle, but their fair share was proportional to their ἀξία (*axia*), their merit, and this depended on circumstances. This distinction blunted the edge of the egalitarian arguments of fifth-century Athens, and had important consequences for political thought in the ancient world.

The proof of Pythagoras' theorem is the culmination of Euclid's first book, and we have indicated how it can be proved not merely from Euclid's own fifth postulate but from Wallis' proposition about similar triangles. It is natural to ask whether it can in turn be proved from Pythagoras' theorem taken as an axiom. In fact it can. It is easiest to show Saccheri's axiom (d), that, granted the Pythagorean proposition, there must be two triangles of the same shape but different sizes:




2.2.3 Proof of Saccheri from Pythagoras: let $\angle ABC$ be a right angle; let $BA = CB$. Extend $CB$ to $D$ so that $BD = CB$. Then $\triangle ABC \equiv \triangle ABD$; so $AD = AC$ and $\angle BDA = \angle BCA$, and, since $\triangle ABC \equiv \triangle DBA$, $\angle BAC = \angle BDA$.

$$\begin{aligned} \text{By Pythagoras} \qquad AC^2 &= BA^2 + CB^2 \\ &= 2CB^2 \\ AD^2 &= BA^2 + BD^2 \\ &= 2BD^2 = 2CB^2 \\ AC^2 + AD^2 &= 4CB^2 \\ &= CD^2. \end{aligned}$$

So $\angle CAD$ is a right angle, and $\triangle ABC \approx \triangle CAD$ QED.

It is only slightly more complicated—and left to the reader—to give a procedure for constructing a triangle of arbitrary size similar to a given triangle. The fact that the Pythagorean proposition, instead of being taken as a theorem to be proved from Euclid's axioms, could be taken as itself being the characteristic axiom of that geometry suggests that we might rename Euclidean geometry "Pythagorean geometry". Although Euclid, along with Plato and Eudoxus, was responsible for its being systematized as an axiomatic theory, we shall be led to regard the Pythagorean proposition as being from some points of view its most characteristic and fundamental feature.[2]

The alternative formulations of the fifth postulate are less cumbersome and may be more acceptable than Euclid's own version, but they are none of them so self-evident that they cannot be questioned. Certainly Pythagoras' theorem is far from being obviously true, something that should be granted without more ado. In fact, none of the alternative formulations was felt to be completely obvious, and they all seemed in need of some further justification. Both Wallis and Saccheri were seeking a better justification,[3] and Saccheri devoted years to trying to prove the fifth postulate by a *reductio ad absurdum*, assuming it to be false and trying to derive a contradiction. The attempt failed, but in the course of it he unwittingly discovered non-Euclidean geometry. The theorems of this non-Euclidean geometry were so strange that Saccheri took them to be absurd, even though he could not derive a formal inconsistency; but, though strange, they were really quite consistent, and were later recognised to be theorems of a certain sort of non-Euclidean geometry, "hyperbolic" geometry as it came to be called.

### 2.3 Non-Euclidean Geometries

Hyperbolic geometry was first discovered as such by Bolyai, a Hungarian, and independently by Lobachevsky, a Russian, at the beginning of the Nineteenth Century. Instead of Playfair's postulate that from a point not on a given line one and only one line can be drawn parallel to the given line, they postulated that from a point

[2] See further below, §2.8.

[3] John Wallis, "*De Postulato Quinto*", *Opera Mathematica*, Oxford, 1693, vol ii, pp.665-678; Gerolamo Saccheri, *Euclides ab omni naevo vindicatus*, Milan, 1733, tr. George Bruce Holland, Chicago, 1920.

not on a given line more than one, indeed infinitely many, lines could be drawn parallel to the given line. Later in the Nineteenth Century Riemann modified the parallel postulate the other way, so that not even one parallel line could be drawn; this required some other modification of the other axioms, but with such a modification produced another consistent non-Euclidean geometry, which was called "Elliptical" geometry.

Non-Euclidean geometries are strange, but seem less so now that we are familiar with them than when Saccheri first encountered them. It is easiest to visualise non-Euclidean elliptic geometry by considering the surface of some sphere, such as that of the earth or of an orange. It is easy then to see that if great circles are taken to be "lines", there are no parallel lines in elliptic geometry. Any two great circles meet; indeed, they meet not once but twice, as the meridians of longitude meet at both the North Pole and at the South Pole. (The so-called "parallels of latitude" are not parallels at all, because they are not, under this interpretation, straight lines, but, rather, circles.) If we consider an octant of an orange, or the spherical triangle on the earth's surface marked off by the meridian of Greenwich, the Equator, and longitude $90^o$ West, we see that it has a right angle at each vertex, so that the sum of its angles adds up to three right angles, $270^o$, instead of only two right angles, $180^o$. A smaller triangle will have the sum of its angles nearer to $180^o$, to which it will tend as the triangle gets smaller and smaller. Indeed, if we know how big the angles are, we can tell what the sides must be; the only spherical triangles with each of their angles $90^o$ are those whose sides are one quarter of the circumference of a great circle. This illustrates the Wallis–Saccheri thesis that there are no similar triangles of different size in this non-Euclidean geometry. It is easily seen in the case of the octant that the Pythagorean proposition is far from true, for in that case $\mathbf{h} = \mathbf{a} = \mathbf{b}$. In the same way the circumference of a circle drawn on the surface of a sphere is less than $2\pi r$. If we take the North Pole as centre and have a radius of one quarter of a great circle, we should draw the Equator, whose length is not $2\pi \times ((1/4)\times$ (a great circle)) but just (a great circle): that is to say, the ratio of the circumference to the radius of this circle is not $2\pi$ but 4. The surface of a sphere has positive curvature: if we consider two mutually orthogonal planes intersecting along a line which is itself perpendicular to the surface, each plane intersects the surface in a curve whose concave side lies in the same direction as the other's; their product, therefore, which defines the curvature of the surface, is positive, whichever way the concave sides face.

| |
|---|
| Elliptic Geometry |
| Riemann |
| No parallel (symbolically $E_0$) |
| $\triangle$ more than $180^o$ (symbolically $\Delta_>$) |
| $h^2 < a^2 + b^2$ (symbolically $P_<$) |
| circumference $< 2\pi r$ (symbolically $O_<$) |
| surface of sphere |
| positive curvature (symbolically $C_+$) |

Table 2.3.1

It is much more difficult to envisage a surface with negative curvature. The surface of a saddle, or a mountain pass, is an example. On such a surface the circumference of a circle is more than $2\pi$ times the radius, and correspondingly the square on the hypotenuse is greater than the sum of the squares on the other two sides. It is less easy to see that the sum of the angles of a triangle is less than 180°, but if we consider how a very small divergence in one's path on a mountain pass can lead to widely separated destinations, we can accept that a triangle could have its angles adding up to less than 180°. If we take this triangle feature to the limit, it follows that there is a minimum area of a triangle. This once again shows how the Wallis–Saccheri postulate fails for non-Euclidean geometry. It also draws attention to another feature of non-Euclidean geometries. Both hyperbolic and elliptic geometry have "natural units"; in hyperbolic geometry there is a minimum area a triangle can have, and in elliptic geometry there is a maximum length a line can have. (This is why for elliptic geometry we need to modify

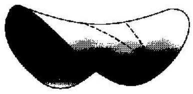


Figure 2.3.1 A saddle showing a triangle with angles all nearly zero, but still encompassing a substantial area

not only Euclid's fifth postulate but his second, which takes it for granted that a straight line can be extended indefinitely far.)

Non-Euclidean geometries remain strange. We can bring ourselves to have some understanding of them and to visualise them to some extent, but they have features that are unfamiliar and may remain unwelcome even after prolonged acquaintance, but that is not to say that they are inconsistent. And in fact non-Euclidean geometries *are* consistent.

It is easy to say that a geometry is consistent, difficult to show it. Saccheri had concluded that the system he was investigating was so absurd as to be inconsistent, and who was to prove him wrong? In the end Felix Klein did prove him wrong, by means of a "relative consistency proof" of a type that has come to be of great importance in the foundations of mathematics. Klein produced a model of hyperbolic geometry in Euclidean geometry. He considered part of the Euclidean plane—the interior of a given circle—and by redescribing the interior of that circle in a particular way showed that under the new description the axioms of hyperbolic geometry were satisfied. He then argued that if they were after all inconsistent, then there would be a corresponding inconsistency in the Euclidean plane, and Euclidean geometry would be inconsistent too. So, contrapositively, hyperbolic geometry was consistent provided Euclidean was; hyperbolic geometry was consistent relative to Euclidean geometry.

Klein's own proof is complicated. We can appreciate the thrust of his argument if we consider instead a relative consistency proof by means of the models just given. The axioms of two-dimensional elliptic geometry are satisfied on the surface of a sphere, with points in elliptic geometry being represented by points on the surface of the sphere, and lines in elliptic geometry being represented by great circles on the surface of the sphere. If the axioms of elliptic geometry were inconsistent, then we could produce a proof-sequence, as shown in Table 2.3.3, in which every line (represented on the

| Hyperbolic Geometry |
|---|
| Bolyai, Lobachevsky |
| More than one parallel (symbolically $E_+$) |
| $\triangle$ less than $180^o$ (symbolically $\Delta_<$) |
| $h^2 > a^2 + b^2$ (symbolically $P_>$) |
| circumference $> 2\pi r$ (symbolically $O_>$) |
| surface of saddle or mountain pass |
| negative curvature (symbolically $C_-$) |

Table 2.3.2

left-hand side of the table) was a well-formed formula of elliptic geometry, and either was an axiom or followed from one or more previous lines by some rule of inference, and the last line was of the form $A \wedge \neg A$. But now consider this proof-sequence not as a sequence of well-formed formulae about points and lines in elliptic geometry, but as well-formed formulae about points and great circles in a two-dimensional subspace of three-dimensional Euclidean geometry (represented on the right-hand side of the table). What were axioms under the elliptic interpretation are now true propositions and can be proved from the axioms of three-dimensional Euclidean geometry represented by additional lines at the top of the proof-sequence on the right-hand side). We could therefore fill out our proof-sequence to prove these from the axioms of three-dimensional Euclidean geometry. If the proof in elliptic geometry ended in a well-formed formula of the form $A \wedge \neg A$, the proof in three-dimensional Euclidean geometry would end in a well-formed formula of the form $A' \wedge \neg A'$, which essentially is of the form $A \wedge \neg A$ too, and three-dimensional Euclidean geometry would be inconsistent also. So granted that three-dimensional Euclidean geometry is not inconsistent, elliptic geometry is not inconsistent either; that is to say, granted that three-dimensional Euclidean geometry is consistent, then elliptic geometry is too.

| Elliptic Plane Geometry | | 3-dimensional Euclidean Geometry | |
|---|---|---|---|
| | | ······ | extra |
| | | ······ | extra |
| | | ······ | extra |
| ——— | translates into | ······ | |
| ——— | translates into | ······ | |
| ——— | translates into | ······ | |
| ——— | translates into | ······ | |
| $A \wedge \neg A$ | translates into | $A' \wedge \neg A'$ | |

Table 2.3.3

Non-Euclidean geometries are thus vindicated on the score of consistency. And in the absence of downright inconsistency mathematicians have been hard put to it to justify excluding them on any other grounds.

## 2.4 Formal and Physical Geometry

Euclidean geometry was dethroned. It was a great upset. It was thought that the foundations of truth had been entirely shaken, and that God was no longer in His heaven, because Euclidean geometry was no longer true. Established ways of thinking were unsettled in much the same way as they were by Einstein's theory of relativity at the beginning of the twentieth century. Philosophers were forced to rethink the status of geometry. They concluded that geometry could be regarded in two ways. It could be regarded formally; in that case it was simply a consistent formal system in which certain conclusions followed from given axioms; or it could be regarded substantially, as making substantial assertions about the external world of space and matter, in which case it was open to falsification in the same way as any other scientific theory, much as Protagoras had held. We shall consider these two alternatives in turn.

**If not Self-evidence, then What?**
perhaps Formalism after all

If we take geometry to be simply a formal system in which conclusions follow from premises or axioms, we are not concerned with semantics or interpretations of geometrical terms, but only the syntactical properties they are given by the formation rules, rules of inference and axioms. Thus all we can say is that *given* Euclid's axioms Pythagoras' theorem follows; or, formally,

Euclid $\vdash$ Pythagoras

This is simply a formal derivation in first-order logic, and can be equivalently expressed by the deduction theorem

$\vdash$ Euclid $\rightarrow$ Pythagoras

which would be a theorem of first-order logic in a system to which had been adjoined some additional symbols but no extra axioms. On the formal analysis there is nothing special about the axioms. We have complete freedom to choose any axioms we like. This is good news for graduates doing research: instead of being confined to just Euclidean geometry, they have a whole range of other geometries which may be investigated and written about; at the very least, we have increased the supply of topics by a factor of three, and if we start altering other Euclidean postulates, besides the fifth, the supply is increased very much more. But there is a cost. Formal freedom is purchased at the cost of substantial vacuity. As we depart further from the Euclidean prototype, the geometries become more weird and less real, and we seem more to be doodling with meaningless marks on bits of paper and less to be discovering anything significant or true. An example is given by the rules of a certain lunch club.[4]

[4] Quoted from C.W.O'Hara and D.R.Ward, *An Introduction to Projective Geometry*, Oxford, 1937, pp.17-18; and Morris R. Cohen and Ernest Nagel, *An Introduction to Logic and Scientific Method*, London, 1961, Ch.VII, 3,

(1) Periodical lunches were to be given by the club, and they were to be attended only by members of the club.

(2) Every member of the club was to meet every other member at least once, but not more than once, at one of the club's lunches.

(3) The lists of members selected by the Secretary to attend any two lunches were never to be entirely different; at least one member was to be present at both.

(4) The President, the Treasurer, and the Secretary were to be the only members present at the first lunch, and at all subsequent lunches there were to be at least three members present.

These rules are, on the face of it, perfectly straightforward, but on further examination, especially in the context of a discussion of geometry, may seem fishy. And, indeed, they are. They are the axioms of incidence of a finite projective geometry, with 'member' doing duty for a point, and 'lunch' for a line, and 'attending' for being on. Formally, if we concentrate on the syntactical structure of the system implicitly defined by these axioms and take no account of their interpretation or substantial content, there is no difference between the axioms of a finite projective geometry and the rules of a lunch club. We can, if we like, call the study of such a formal system "geometry", but we could equally well call it "lunchology", and, remembering Plato's criticisms,[5] wonder whether lunchology was a study worthy of a grown man. The purely formal characterization of geometry fails to capture what is specifically geometrical about it. Lunch clubs have nothing to do with γεωμετρεῖν (*geometrein*), measuring the earth.

**If not Formalism, then What?**
perhaps Formal Empiricism
(or Protagoreanism) after all

The standard modern view has been to distinguish the purely formal syntactic approach from a semantic one which is more in the tradition of Protagoras. We consider not just the axioms of geometry, say those of Euclid, but those axioms in conjunction with a physical interpretation. The standard physical interpretation is that under which light rays are taken to be straight lines.

pp.137-138.

[5] *Republic* VII, 533b6-c5; quoted above, in §1.9.

Instead then of postulating the axioms of Euclidean geometry, we should consider the conjunction of those axioms with a physical interpretation, and ask whether they were in point of fact really true. Riemann asked this question, and considered not the Pythagorean proposition, which we may abbreviate **Pyth**, but $\triangle$, whether the angles of a triangle added up to $180^o$, and measured the angles at the tops of three distant mountains, and found that within the limits of observational accuracy they did. In this way geometry could be regarded as an empirical science, and verified in the same way as any physical theory might be, and thus far found true. Protagoras has been re-instated.

Poincaré develops this argument to reach the opposite conclusion. Since the thesis that is subjected to empirical check is not just the set of Euclid's axioms but the conjunction of Euclid's axioms with a certain physical interpretation, we can always hold on to Euclid's axioms, provided we make suitable adjustments to the interpretation. Provided light rays are allowed not to define straight lines, then even if the measured angles of a triangle do not add up to $180^o$ the truth of Euclidean geometry is not impugned.[6]

Although Poincaré is making an important point, which needs serious consideration, his use of the word 'convention' is unfortunate. Real conventions are ones where there is absolutely nothing to choose between two courses of action—*e.g.* whether to drive on the right side or the left side of the road, or whether to regard multiplications as more or less binding, so far as omitted brackets go, than addition—and we need to have a rule in order to be able to understand one another and concert our actions. The choice of a geometry is not like this. There may be good grounds, perhaps empirical, perhaps not, for choosing between one geometry and another. Hempel claims that there are.[7] He argues that we should consider not just the simplicity of the geometry but also that of the geometry *together with* the physical interpretation. If there are two

[6] H.Poincaré, *Science and Hypothesis*, tr., pbk., New York, 1952, chs. III, IV and V; excerpted as "Non-Euclidean Geometries and the Non-Euclidean World", in H.Feigl and M.Brodbeck, *Readings in the Philosophy of Science*, New York, 1953, pp.171-180.

[7] C.G.Hempel, "Geometry and Physical Science", *American Mathematical Monthly*, **52**, 1945; reprinted in H.Feigl and Wilfrid Sellars, *Readings in Philosophical Analysis*, New York, 1949, pp.238-249.

physical interpretations, $PhysInt_1$ and $PhysInt_2$, then sometimes Euclid + $PhysInt_1$ may be more complicated than Riemann + $PhysInt_2$, and so we have rational grounds for preferring the latter, even though the former may be consistent with the observed facts. Hempel suggests that we should choose the best, that is, the simplest, combination of geometrical axioms and physical interpretation taken together. If we think of different physical interpretations $PhysInt_1$, $PhysInt_2$, $PhysInt_3$, we shall have different semantic interpretations of the axioms and theorems of geometry, which we can represent by $\models$.

So we might have $PhysInt_1 \models \mathbf{Pyth}$

where **Pyth** is the Pythagorean proposition,

but $PhysInt_2 \models \neg\mathbf{Pyth}$

and $PhysInt_3 \models \neg\mathbf{Pyth}$,

so that although there was an interpretation, say $PhysInt_1$, under which **Pyth** was true, nevertheless we were more concerned with another interpretation, say $PhysInt_2$, under which **Pyth** was false. If, for example, $PhysInt_2$ is one in which straight lines are taken to be light rays, then although Riemann's experiment confirmed the thesis $\triangle_=$, that the angles of a triangle add up to $180^o$ to within the limits of accuracy available terrestrially in his day, we now have theories, adequately confirmed by observation, according to which triangles defined by light rays have angles adding up to more than $180^o$.

$$\text{So} \quad PhysInt_2 \models \neg\triangle_=,$$

$$\text{and, correspondingly,} \quad PhysInt_2 \models \neg\mathbf{Pyth}.$$

Although we could stick to Euclidean geometry *coûte que coûte*, it would be unreasonable to do so; the combination of an elliptic geometry with Einstein's General Theory is a better buy than Euclidean geometry together with some enormously complicated physics, which interpreted lines in such a way—$PhysInt_1$—that $PhysInt_1 \models \mathbf{Pyth}$, but at the cost of a whole lot of *ad hoc* hypotheses and a number of implausible assumptions.

## 2.5 Conceptual Constraints

Hempel's exposition is beautifully clear, his arguments have much cogency, and his conclusions have been widely accepted, so much so that they constitute the current orthodoxy. Nevertheless they are open to criticism. Although he is entirely right in insisting, as against Poincaré, that we need to consider not just a geometry by itself but the combination of a geometry with an interpretation, he concentrates too exclusively on physical interpretations, and does not consider the other constraints that operate. In the first place the very distinction between geometry and physics gives rise to some conceptual pressures that ought to be recognised: and secondly, there are links between some basic concepts of geometry and other concepts outside geometry which greatly limit the range of possible interpretations.

The argument from the General Theory of Relativity is one-sided. We do not have just one unified physical theory which presupposes a particular geometry and can be put to empirical test as a whole, but a number of different theories and many untheoretical views of the natural world. Geometry is not part of one such theory of the physical world, but rather a back-cloth against which many theories, as well as untheoretical views, can operate. There is a division of labour between geometry and physics. Physics is concerned with cause and effect, and seeks to give explanations of phenomena in terms of laws of nature. It is no part of geometry's function to do any of these, but to provide a schema of reference and description which enables propositions about the world to be formulated and discussed, and to discover the relations between different propositions of this sort. Granted such a function, geometry is subject to various requirements. Thus the causal inefficacy of space and time follows from geometry's being not itself involved in giving causal explanations, and in turn imposes conditions of homogeneity and isotropy which are of geometrical consequence.

The separation between geometry and physics may, however, be denied. The programme of geometrodynamics is to bring together physics and geometry, and to reduce physical explanations to geometrical ones. In Einstein's General Theory forces are replaced by a complicated curvature of space. If the programme is successful, then there will be no distinction between geometry and physics, and the objection will fail. So much may be granted. But a programme is not a fact. And if geometrodynamics is successful, what we shall have is not geometry, but geometrodynamics. The possibility of a monistic account of the whole of nature was put forward by Spinoza, and has been vigorously pursued by contemporary physicists. If it is realised in fact, it will engender considerable alterations in our conceptual structure, and the old distinction between geometry and physics will be subverted. But as long as we are talking about geometry and not geometrodynamics, there is a contrast between geometry and physics, and this contrast imposes conceptual constraints on the sort of geometry it is reasonable to adopt.

According to the orthodox account, a point and a straight line are only implicitly defined by the axioms of geometry, and we are entirely free to choose any interpretation which conforms to the axioms. In particular, in plane projective geometry 'point' and 'line' are dual, and we can interchange them without any alteration of meaning or truth: thus in the Lunch Club it makes no difference whether we take a 'member' to be a point, and a 'lunch' to be a line, or *vice versa*. But this is not true. Points and lines are not merely implicitly defined by the axioms: other definitions have been given or attempted which show their other conceptual links. Let us survey them systematically.

We have first some mereological and categorial differences:

1. (i) A point has no parts (Pythagoras)
   (ii) A point has position but no magnitude, whereas
   (i′) A straight line has parts
   (ii′)(a) A straight line has position *and* direction
   (ii′)(b) A straight line has length but no breadth

We have secondly topological differences between a point and a line (not necessarily straight):

2. (i)(a) A point cannot have a boundary, but
   (b) can be a boundary
   (ii)(a) A line can have a point as a boundary, and
   (b) can be a boundary of a surface
   (iii)(a) A surface can have a line as a boundary, and
   (b) can be a boundary of a volume

More generally, we have a number of possible definitions of a straight line. A straight line:

3. (i) is the shortest distance between two points
   (ii) is a breadthless length
   (iii) is the path of a light ray

(iv) looks straight
(v) has no kinks
(vi) lies evenly on itself
(vii) is the axis of a 3-dimensional rotation
(viii) is the intersection of two planes
(ix) is that of which the middle covers the ends.[8]

Pythagoras' characterization of a point as something that has no parts connects the concept of a point with "mereology" (from the Greek word *μέρος* (*meros*), part), the study of the *part of* relation.[9] The *part of* relation is an ordering relation, and under that ordering points are the minimal elements. This is also why they are said to have no magnitude: which may be understood either as saying that their size is zero or as saying that the question "How much?" cannot be asked of points at all. Points are being defined in a counter-Anselmian way. Anselm defined God as *id quo maius nequeat cogitari esse*, that than which nothing can be imagined to be greater: if we define a point as that which has no parts, or as that whose magnitude is zero, we are saying in effect that it is *id quo minus nequeat cogitari esse*, that than which nothing can be imagined to be smaller. Points are thus not just entities that satisfy the axioms of some formal geometrical theory but are linked to mereology and Aristotle's category of Quantity, "How much?".

Even if we construe 'having no magnitude' in the definition of a point as meaning that the category of Quantity is inapplicable, the category of Place, "Where?" is applicable. Points have position. We can always ask of a point where it is. There is at least this categorial connexion, and it is one that differentiates points from lines, for of lines we can ask not only where they are, *ποῦ* (*pou*), but in what direction they go, *πόθεν*/*ποῖ* (*pothen/poi*), whence/whither. As regards mereology, lines are clearly distinct from points, for we can ask of a line one type of "How much?" question, namely "How long?", although we cannot ask "How wide?", or if we do we shall get only a null answer. Hence the definition of a line as a breadthless length. It is minimal in one sort of way, but not in every way, as a point is.

[8] Plato, *Parmenides*, 137e3-4.

[9] See below, §9.12, ch.10, and §11.5; but note that the *part of* relation considered there is serial, and has no minimal elements, whereas the Pythagorean definition assumes a *part of* relation that is non-serial.

The distinction implicit in the way in which a line can and cannot have parts is made more explicit in topology. Topology gives an inductive definition of dimension in terms of boundaries. The null set has dimension $-1$; points, which have no boundaries, have dimension 0; lines, whose boundaries are points, have dimension 1; surfaces, whose boundaries are lines, have dimension 2; and so on. Plato was on the track of this. In the *Meno* he defines a plane figure as the boundary of a solid: *στερεοῦ πέρας σχῆμα εἶναι* (*stereou peras schema einai*).[10] There is much proto-topology in Plato and Aristotle. Although we think of Kant as the founder of topology, and rightly, we should recognise more than we do the first efforts in that direction made by the Greeks.

Topological characterizations of lines do not imply straightness. A straight line is a particular sort of line, and one way of picking out a straight line from others is, as Hempel correctly remarks, that light travels in straight lines unless reflected or refracted. But this is not the only way; nor is it merely a contingent fact which could just as well have been otherwise. Often, admittedly we rely on the physics of the light ray, as when we look down a ruler's edge to see if it is straight. But this not an arbitrary choice on our part. We can *see* that light goes in straight lines (to within the range of observational exactitude) by looking at the beams cast by the sun through holes in a shutter in a dusty room or through the clerestory windows in a cathedral in winter. From the fact that we can see the light going in straight lines it follows that we must have some other criterion of straightness than simply that it is the path of a light ray. So it cannot be an arbitrary choice of definition on our part. And indeed, if they looked crooked, we should no more look down the edge of a ruler to judge its straightness than we actually judge the straightness of a stick by seeing how it looks when partially immersed in water.

Nor is light our only recourse for determining straightness. If light were not available, or thought to be not reliable, we could test a ruler with a taut string. A straight line is the shortest distance between two points. It is a "geodesic" characterization of straightness, and one much favoured in the General Theory of Relativity. It involves further concepts, such as that of 'distance' and 'being between' points, but they can be at least partly defined without

[10] *Meno*, 76a7.

presupposing that of straight line. A different approach again is to check two straight edges against each other by running them along each other. If they really are straight they will fit snugly together all the time: if both are kinked together, they will fit in one position but not in others. In extreme cases we can see or feel the kinks and reject the line as not being straight out of hand. Straightness excludes kinks, which are points of singularity. In this we are following Euclid's definition of a straight line as one which lies "evenly on itself". In modern parlance a straight line has perfect translational symmetry along itself. It also has perfect rotational symmetry around itself. An axis of rotation is a straight line, and I could test a thin rod for straightness by turning it slowly on a lathe.

It is worth also noting the sophisticated way in which an "optical plane" is made. Optical planes need to be very flat indeed, and lens grinders would first grind two planes together, then each against a third, and then against each other, and so on. Clearly, when two planes are ground against each other we cannot be sure that they are flat: one might be slightly convex and the other correspondingly concave. If the second is then ground against a third, any high point of the second will score a corresponding low point on the third. But when this is ground against the first, low point will be opposite low point, and high opposite high, and so the highs will grind each other down. More mathematically, if any point of the first is $x$ above the plane, the corresponding point on the second will be $x$ below, *i.e.* $-x$ above, and on the third will be $x$ above, which will tend to grind the original high down to $-x$. Each grinding operation tends to convert $x$ into $-x$, *i.e.* multiply by $-1$. So with three planes being ground against one another, the total effect is to multiply $x$ by $(-1)^3$. The only stable situation is that in which $x = (-1)^3 x$, *i.e.* $x = -x$, which is possible only when $x = 0$. Given a way of producing an optical plane, we can then produce a very straight line as the intersection of two optical planes.

We thus have many different ways of producing, or checking, straight lines. In our normal experience these different ways cohere, although it is possible to conceive worlds—for example with continually varying refractive media—in which they do not all cohere. This is not to say that they are all independent. As with some other basic concepts which are linked with several others, some of the linkages are secured by deeper connexions. Those between straight lines and light rays seem relatively contingent. That between straight lines and the geodesic property of being the shortest distance between two points is not a *physically* contingent matter, but we could perhaps conceive things being arranged differently—it much depends on our concept of distance, and the curvature of space. The symmetry properties of straight lines seem more fundamental, together with what we have taken for granted thus far, its continuity and one-dimensionality, which characterize not only straight lines, but lines and curves of any sort. It seems reasonable, therefore to make no mention of light rays or geodesic properties in defining a straight line, but to characterize it conceptually as a one-dimensional continuous curve unbounded and infinite in extent which is symmetric under translation along itself and rotation around itself.

### 2.6 Which Geometry?

**If not Formal Empiricism, then What?**

If purely formalist and empiricist accounts of geometry are rejected, we are faced once again with the question "Which geometry should we choose?". We cannot say, as the Formalists do, that they are all on a par, so long as they are all consistent, and there is nothing for it but to make a purely arbitrary choice—although we should concede to the Formalists that we are free to make an arbitrary choice if we so please, and that much may be learned from the study of a geometry considered solely as a formal system: and we cannot, as the Empiricists do, leave it entirely to sense-experience to decide between different formal systems—although the verdict sense-experience gives to a geometry **plus** an interpretation is weighty, and we cannot go on maintaining, without some modification or gloss, a geometry together with an interpretation which flies in the face of the empirical evidence. We therefore have to ask ourselves how we should choose a geometry, granted that there are many consistent ones to choose from.

Let us tabulate them with their special features in order to make an informed choice between them as in a consumers' magazine:

The *Which?* Guide to Geometries

| Hyperbolic | Euclidean | Elliptic |
|---|---|---|
| Bolyai, Lobachevsky | Pythagoras, Euclid | Riemann |
| More than one parallel<br>(symbolically $E_+$ | Exactly one parallel<br>$E_1$ | No parallel<br>$E_0$) |
| $\triangle$ less than 180°<br>(symbolically $\Delta_<$ | $\triangle = 180°$<br>$\Delta_=$ | $\triangle$ more than 180°<br>$\Delta_>$) |
| $h^2 > a^2 + b^2$<br>(symbolically $P_>$ | $h^2 = a^2 + b^2$<br>$P_=$ | $h^2 < a^2 + b^2$<br>$P_<$) |
| circumference $> 2\pi r$<br>(symbolically $O_>$ | $c = 2\pi r$<br>$O_=$ | $c < 2\pi r$<br>$O_<$) |
| surface of saddle<br>or mountain pass | flat plane surface | surface of sphere |
| negative curvature<br>(symbolically $C_-$ | zero curvature<br>$C_0$ | positive curvature<br>$C_+$) |
| minimum area | no natural unit | maximum length<br>maximum area |

Table 2.6.1

None of these geometries is conclusively ruled out: none of them is inconsistent. We cannot say of any of them Not Recommended, and must allow that any one of them may be the most suitable for some particular purpose. Nevertheless, we can give rational guidance for the general user on the strength of some of the features listed in the table, and conclude that Euclidean Geometry is the Best Buy and this for several reasons.

In the first place Euclidean geometry is both more specific and more flexible than either of its competitors. It is more specific in respect of the first six of the features listed in the table after the makers' name, for instance when it specifies an equality instead of merely a *greater than* or *less than*. On each of these counts, it is given a star. It is also given further points for its greater flexibility, manifested in the bottom row. With an elliptical geometry we have to ask what its unit of length is—the length of its great circles, so to speak. Such a geometry would not be available for lengths greater than the maximum one. In choosing it we are foreclosing our subsequent freedom of conceiving. Although if the length is very large, we are unlikely to come up against empirical evidence against it, we may always want to consider, if only hypothetically, some length that is greater, and it is a restriction on our freedom of thought to rule it out in advance. Similarly, though less embarrassingly, it is a pity to be cumbered with either a maximum area of the whole space—the area of the surface of the earth—in elliptic geometry, or a minimum area any triangle can have in hyperbolic geometry. It is better to be able to triangulate our space as closely as we please without any geometrical restriction.

The flexibility of Euclidean geometry is shown more clearly in the Saccheri–Wallis formulation, which in effect assigns to Euclidean geometry alone the possibility of having two figures the same shape but different sizes. In elliptic geometry, as we saw with the octant of a sphere, the shape determines the size, and the same holds good of hyperbolic geometry. There is no possibility in those geometries of a scale model, and instead of being able to characterize objects and other figures by reference to their shape and their size independently, we should have only one, linked way of characterizing them. Euclidean geometry has more degrees of freedom, and is therefore better suited to its function of being a descriptive back-cloth against which physical phenomena can be described and physical theories formulated and tested. Whereas for a scientific theory, such as physics, flexibility may be a fault, and prevent the theory being put to the test and made liable to falsification, for geometry, with its different aims, flexibility is not a weakness but a strength. It increases the descriptive potentiality of geometry, which is what we want, and the fact that it is at the cost of falsifiability is no criticism, since it is not the function of geometry to offer falsifiable predictions or explanations.

It may seem paradoxical that we claim on behalf of Euclidean geometry both that it is more specific and that it is more flexible and generally available. But there is no paradox. In saying that a geometry is Euclidean, we are saying all we need to say in order to characterize it completely: in saying that it is hyperbolic or elliptic, we are not characterizing it completely: we need to say further what its curvature is—it has to be some particular negative or positive number, not just negative or positive; we need likewise to say what the minimum area of a triangle or the maximum length of a straight line is, by how much the angles of a triangle fall short of or exceed two right angles, by how much the circumference of a circle exceeds or falls short of $\pi$ times its diameter. There is just one Euclidean geometry, whereas there are whole families of hyperbolic and elliptic geometries, each different from the others, and each having its own peculiarities, of curvature, of the sum of the angles of a triangle, and of the ratio of the circumference of a circle to its diameter, which preclude its easy application to some conceivable cases. Euclidean geometry, by contrast, is exact: in every Euclidean geometry triangles add up to the same, the ratio of circumference to diameter is always the same, the curvature is always the same—*viz.* 0; but the one and only Euclidean geometry, once specified, is available in a wide variety of cases, and is thus more multi-purpose.

The same considerations apply with the number of parallels. Euclidean geometry, having exactly one, is more specific than hyperbolic geometry, which has infinitely many, though in this case not more specific than elliptic, which has none. But the latter is a defect when it comes to establishing a system of reference. On the surface of the globe, lines of longitude intersect at the poles: $10^o$E and $90^o$N is the same as $10^o$W and $90^o$N. This is a defect in a system of reference. We want there to be a one–one correspondence between points in the space and sets of coordinates. If this is to be so, we need "topological parallelism", that is that lines (not necessarily straight) defined by all the coordinate(s) except one being constant should always exist and never intersect. These do not have to be, so far as this argument goes, straight lines—we can have curvilinear coordinates—and do not demand geometrical parallelism. But the use of straight lines in elliptic geometry is ruled out, and the use of Euclidean parallel straight lines is strongly suggested.

## 2.7 The Theory of Groups

Plato had argued against all forms of operationalism and constructivism in mathematics, because mathematics ought to turn the mind towards the abstract contemplation of timeless reality. He admitted that the actual linguistic practice of geometers would suggest the opposite, but he, and Aristotle after him, saw it as a weakness, not a clue to an understanding of what was really going on: "they talk ridiculously *τε καὶ ἀναγκαίον* (*te kai anankaion*),

<for want of better terms> or <yet this is how they have to>[11] as though they were doing things and using the language of action—'squaring', 'applying', 'adding'—whereas the point of study is knowledge, not action".[12]

It was unfortunate that Plato's and Aristotle's influence was so great that although the language of operations remained part of the standard vocabulary of geometers—as we noted in §2.2, Euclid's first three postulates are instructions, couched in the infinitive, rather than primitive propositions couched in the indicative—it was not taken seriously until Felix Klein propounded his *Erlangen Program*, in which he suggested that geometry be approached not axiomatically but through the groups of operations which left geometrical features invariant.[13] Topology was to be seen as the study of what was left unaltered under the group of all continuous transformations. Hyperbolic geometry, and with somewhat more difficulty elliptic geometry, could also be given a group-theoretical characterization. Euclidean geometry turned out to be the geometry which was unaffected by the group of translations, rotations and reflections, which was therefore called the Euclidean group. The Euclidean group leads us to Euclidean geometry in much the same was as, on one presentation, the Lorentz group leads us to the Special Theory of Relativity.[14]

A sceptic about the pre-eminence of Euclidean geometry might allow the propriety of the group-theoretical approach, but query the importance of the Euclidean group, and ask:

**What is so good about the Euclidean Group?**

[11] The former version is that of F.M.Cornford *The Republic of Plato*, Oxford, 1941, p.238; the latter is my tendentious rendering to bring out the nuance of Plato's recognition that operational terms were, however much he disliked them, a necessary part of the mathematician's vocabulary.

[12] *Republic* VII 527a6-b1; *Metaphysics* K, 1064a30, cf. A, 989b32.

[13] Felix Klein, "*Vergleichende Betrachtungen über neuere geometrische Forschungen*", Erlangen, 1872; revised version in *Mathematische Annalen*, **43**, 1893, pp. 63-100. See also H.Helmholtz, "The origin and meaning of geometrical axioms", *Mind*, **1**, 1876, pp.301-321.

[14] See P.E.Hodgson and J.R.Lucas, *Spacetime and Electromagnetism*, Oxford, 1990.

One partial answer is the purely abstract one that the group generated by the operation of reflection is the simplest non-trivial group, while the group of rotations is the paradigm continuous cyclic group and the group of translations is the paradigm serial continuous group. The Euclidean group is thus a group of peculiar simplicity, and corresponding importance.

Another partial answer is due to Helmholtz.[15] The Euclidean group preserves rigid motions, and rigid motions are presupposed by our philosophy of measurement,[16] and are obviously of importance if we are to manipulate material objects in the world around us. This is a practical consideration. There is also a "communication argument". Granted that you and I are different corporeal beings, differently located in space, but talking about the same things, what we shall be best able to talk to each other about will be those aspects that are the same from either of our points of view. That is, the pressures of communication between observers who necessarily, when communicating, are doing so at approximately the same time, and therefore while occupying different positions in space, will lead them to pick out those features that are invariant under translation and rotation—which together form the "proper Euclidean group", (thus differing from the full Euclidean group in not containing reflection)—and so the geometry defined by that group naturally assumes importance in the eyes of limited communicators who cannot be in the same place at the same time.[17]

This argument has been countered by T.G. McGonigle, who points out that rigid motions are possible in any space of constant curvature.[18] At first sight there seems to be an inconsistency between the claim that Euclidean geometry is characterized by the Euclidean group and the claim that rigid motions are possible in spaces in which geometrical features are not invariant under the

[15] "The Origin and Meaning of Geometrical Axioms (ii)", *Mind* **3**, 1878, pp. 212-225.

[16] See below, §2.8 and §12.3; and J.R.Lucas, *Space, Time and Causality*, Oxford, 1985, pp. 85-86.

[17] See further, J.R.Lucas, "*Euclides ab omni naevo vindicatus", British Journal for the Philosophy of Science*, **20**, 1969, pp. 185-191; and J.R.Lucas, *Space, Time and Causality*, Oxford, 1985, pp.111-113.

[18] T.G.McGonigle, *British Journal for the Philosophy of Science*, **21**, 1970, pp.185-191.

Euclidean group. If we consider the surface of an orange, it is evident that spherical triangles and other shapes can be slid around on the surface without distortion, and it would seem, therefore, that they are being translated and rotated. But when we consider it more deeply, we see that the apparent translations are not real translations, because when iterated enough they come back to where they began. They are in fact not translations, but rotations around a somewhat distant centre of rotation. Instead of having a group of simple rotations together with translations, we have a group of more complicated rotations with different radii of rotation. Such a group would indeed preserve rigid motions, and in the limit would be indistinguishable from the Euclidean group. But it is in a sense more complicated. Although there is only one sort of operator—rotation around some centre of rotation—it is one with a variable parameter—the radius of rotation—whereas the Euclidean group, though having two sorts of continuous operator, has no further parameters to specify. There is a trade-off between one type of simplicity and another, but we can argue both abstractly that the Euclidean group is the simplest group that preserves rigid motions, and as a matter of practice that it is in terms of translations and simple rotations that we construe the motions of material objects around us. We could be wrong. It could be that what we regard as translations are really rotations around a very distant centre of rotation. But we regard them, naturally enough, as translations, and once we distinguish translations from rotations, we are committed to the Euclidean group, and so to Euclidean geometry.

## 2.8 Pythagorean Geometry Has a Better Metric

The Pythagorean proposition, we saw,[19] can be taken as an axiom instead of a theorem to be proved, and in many ways is a more characteristic feature of the resulting geometry than Euclid's complicated parallel postulate. And we can argue for Pythagorean geometry on the score of its characteristic proposition, $P_=$, being better than the characteristic propositions, $P_<$ and $P_>$, of hyperbolic and elliptic geometries respectively. For $P_=$ **is the simplest sensible rule for assigning an overall measure to separations spanning more than one dimension.**

[19] In §2.2.

We need first to explain why geometry should be concerned with assigning measures, and secondly to justify the claim that the Pythagorean rule is the simplest sensible one to adopt. As regards the first question, we may be content to make the etymological point that geo*metry* should be concerned with assigning a *metric*. Although geometry has developed since the Egyptians used it to measure the plots of earth along the Nile, and has been more concerned with shapes than with absolute size, it is only in Pythagorean geometry that the two are distinct, and in them relative size remains important. Some geometries—projective geometry, for example—take no account of size, but most do; and in so far as geometry is to be applied, as in laying out a tennis court, it is important that numbers should be assignable to segments of straight lines and curves.

Mathematicians often define a metric on a space as a function $d$ from the cartesian product of the space with itself into the non-negative real numbers.

$$S \times S \overset{d}{\longmapsto} \Re^{0,+},$$

subject to four conditions:

(i) $d(x, y) = 0 \quad \text{iff} \quad x = y$
(ii) $d(x, y) > 0$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, z) \geq d(x, y) + d(y, z)$.

We may naturally ask why these conditions should be imposed on any metric function. (i) expresses the thought that a point has no length (magnitude)[20] and that any two distinct points define a straight line and a segment of it having length. (ii) expresses the thought that a point is a minimal limit of a line.[21] (iii) expresses the thought that length is isotropic. (iv) is the so-called triangle inequality, and lays down that the distance between any two points cannot be more than the sum of the distances between each and some third point, but may be less. It establishes an upper bound for composite magnitudes, and expresses the ancient principle that the whole may not be more than the sum of its parts.[22]

[20] See above, §2.5, 1.(i) and (ii).

[21] See above, §2.5, 2.(ii)a) and below, §12.5.

[22] See below, §12.4.

If we accept these conditions as partly constituting what it is to be a measure of distance, we are led to look to certain functions as plausible measure functions. (i) suggests that $d(x, y)$ should be a function of $x - y$; (ii) and (iii) suggest that it should be a function of $(x - y)^2$ or $(x - y)^4$, or $(x - y)^6$, ...*etc.*; and if we are to take the $n$-dimensional case, then the most natural additive, always non-negative, symmetrical function is

$$d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}.$$

Of the three geometries on offer, we should therefore choose the one which has $P_=$, that is to say Euclidean geometry, in which alone Pythagoras' theorem holds true, and yields an exact equality between the square on the hypotenuse and the sum of the squares on the other two sides of a right-angled triangle.

The right angle is not just a particular angle, but expresses the independence of the different dimensions. The Greek for 'rectangular' is ὀρθογώνιος (*orthogonios*), from which the word 'orthogonal' comes. Orthogonality often expresses independence, especially in Fourier analysis, where different periodic components of a function are represented by different dimensions, and Parseval's theorem expresses in Hilbert space a close analogue of Pythagoras' theorem. In quantum mechanics, as we move from Schrödinger's wave mechanics to Heisenberg's matrix representation, we set great store by diagonal matrices in which the product of two vectors comes out as a sum of squares, thus again paying tribute to the pre-eminence of the Pythagorean rule. (But we should note that the analogy is not exact. In quantum mechanics we deal with Hermitian matrices, operating on complex vectors and their conjugates; nevertheless the analogy is close enough to be suggestive.)

In a slightly different way, Euclidean geometry facilitates the use of the parallelogram rule for compounding displacements in different directions, and hence velocities, accelerations and forces also. If, however, in elliptic geometry I start as it were at the North Pole, go $1/4\times$(a great circle) South, turn 90° and go $1/4\times$(a great circle) East, turn 90° and go $1/4\times$(a great circle) North, turn 90° and go $1/4\times$(a great circle) South, I shall not end up where I started. The order in which I carry out the displacements is not commutative; so that the parallelogram rule is no longer natural for compounding displacements, and not available at all for compounding velocities, accelerations, or forces.

It is clear that these arguments, like those of the two previous sections, are not deductive. There is no inconsistency in supposing that the distance function is given by

$$d(x,y) = \Sigma|x_i - y_i|$$

or by

$$d(x,y) = \quad \Sigma(x_i - y_i)^4 \quad ^{1/4}$$

or . . . , nor in having shape dependent on size, nor in having no topological parallels. It is clear also that these arguments are not inductive. Equally, they are not just promulgating an arbitrary convention. They are rational arguments, though neither deductive not inductive ones. It is rational to seek greater simplicity, greater generality and greater unification, and these arguments appeal to those considerations.

## 2.9 Desargues

We have found a number of different ways in which the axioms of a geometry may be justified. There is one further one which cannot be called in aid of Euclid's axioms, but is available for plane projective geometry, and is of importance elsewhere in mathematics. In plane, that is two-dimensional, projective geometry "Desargues' Theorem" is *not* a theorem but has to be postulated as an extra axiom. Desargues' theorem states that if two triangles are centrally perspective, they are axially perspective, that is, if $AA'$, $BB'$, $CC'$ are concurrent (at $O$), then if $D$ is on $BC$ and $B'C'$, and $E$ is on $CA$ and $C'A'$, and $F$ is on $BA$ and $B'A'$, $D$, $E$ and $F$ are collinear.

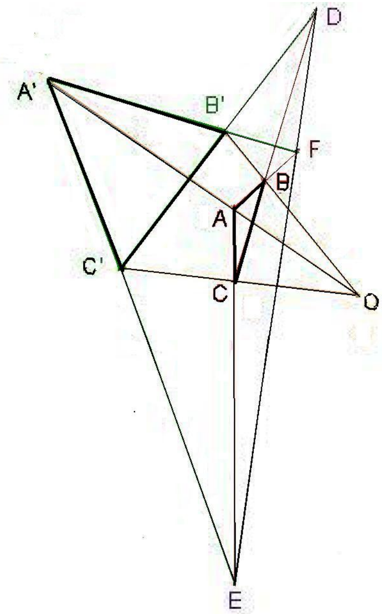
Desargues' theorem is *not* a theorem in two dimensions. There are non-Desarguian two-dimensional geometries, although they are fairly unlovely. In standard two-dimensional projective geometry it is necessary to postulate Desargues' theorem as an axiom. But in three, or more, dimensions it is a theorem, and can be proved quite easily. For we can prove that $DEF$ is a straight line in three-dimensional geometry if it is the intersection of two planes. And that is easily proved by considering the various different planes the relevant lines must be in.

Desargues' theorem thus offers a further criterion of mathematical truth. If we suppose that the other axioms of plane projective geometry are $P_1$, $P_2$, $P_3$, $P_4$, and Desargues' theorem is $D_5$,

$$\text{then} \quad P_1, P_2, P_3, P_4 \neg\vdash D_5$$

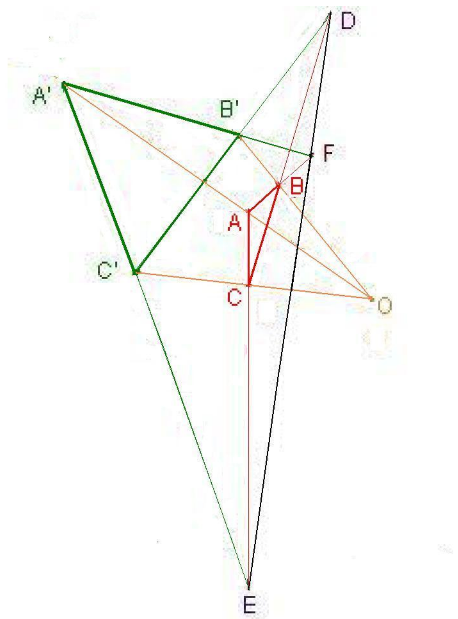
$$\text{but} \quad P_1, P_2, P_3, P_4, P_6, P_7, P_8 \vdash D_5$$

where $P_1$, $P_2$, $P_3$, $P_4$, $P_6$, $P_7$, $P_8$ is a natural generalisation of $P_1$, $P_2$, $P_3$, $P_4$.




Desargues' "Theorem": if $AA', BB', CC'$ are concurrent (at O), then if $D$ is on $BC$ and $B'C'$, and $E$ is on $CA$ and $C'A'$, and $F$ is on $BA$ and $B'A'$, $D, E$ and $F$ are collinear.

Not all generalisations are natural generalisations, but obviously 3-dimensional projective geometry is a natural generalisation of 2-dimensional projective geometry, and the fact that Desargues' theorem *is* a theorem in a two-dimensional subspace of a three-dimensional projective geometry is a weighty reason for holding it to be true even if it cannot be deduced from the other axioms of two-dimensional projective geometry alone.




Desargues' Theorem in three dimensions turns entirely on the intersections of the various planes: since $BB'$ and $CC'$ intersect at $O$, the lines $BB'O$ and $CC'O$ are co-planar, and so the points $B, B', O, C, C'$ are all co-planar, and so $BB'$ and $CC'$ must intersect, at $D$. Then $D$ is co-planar with $B, B', C, C'$, and so is in every plane that includes $BC$, and so in $ABC$, and is in every plane that includes $B'C'$, and so in $A'B'C'$. By exactly similar reasoning, since $CC'$ and $AA'$ intersect at $O$, the lines $CC'O$ and $AA'O$ are co-planar, and so the points $C, C', O, A, A'$ are all co-planar, and so $CC'$ and $AA'$ must intersect, at $E$, say. Then $E$ is co-planar with $C, C', A, A'$ and so is in every plane that includes $CA$, and so in $ABC$, and is in every plane that includes $C'A'$, and so in $A'B'C'$. Again, by exactly similar reasoning, since $AA'$ and $BB'$ intersect at $O$, the lines $AA'O$ and $BB'O$ are co-planar, and so the points $A, A', O, B, B'$ are all co-planar, and so $AA'$ and $BB'$ must intersect, at $F$, say. Then $F$ is co-planar with $A, A', B, B'$ and so is in every plane that includes $BA$, and so in $ABC$, and is in every plane that includes $A'B'$, and so in $A'B'C'$. So $D, E$ and $F$ are all in both $ABC$ and $A'B'C'$ and thus in the line common to these two planes.

### 2.10 Conclusions

Our *Which?* survey of geometry yields a more complex picture than either the Formalist or the Empiricist had supposed. Our choice of geometries and of interpretations of geometrical terms is not arbitrary, but is guided by six different sorts of consideration:

1. There are conceptual links between geometrical and other concepts which limit the application of terms such as 'point', 'line', or 'plane', and lead us to adopt some propositions as true and to reject others as false.

2. As between one geometry and another, it is rational to choose the one which is more specific. Euclidean geometry is more specific than either hyperbolic or elliptical geometry because

(a) its curvature is exactly zero, whereas theirs is either any constant negative number (hyperbolic) or any constant positive number (elliptical);

(b) the angles of a triangle in Euclidean geometry add up to exactly $180^o$ whereas in hyperbolic geometry it is merely less than, and in elliptical geometry it is merely more than, $180^o$;

(c) the square on the hypotenuse is exactly equal to the sum of the squares on the other two sides of a right-angled triangle, whereas in the other geometries, again, there is only an inequality, not an equality;

(d) the ratio of the circumference of a circle is exactly $2\pi$ in Euclidean geometry, but only $> 2\pi$ in hyperbolic, and $< 2\pi$ in elliptic, geometry.

3. Conversely, Euclidean geometry is more flexible than hyperbolic and elliptic geometry, in that the size of figures is not determined by their shape, and there is no fixed unit of length or of area in Euclidean geometry. Euclidean geometry is scale-invariant and metrically amorphous. In choosing Euclidean geometry, we are not committing ourselves to other choices, which are therefore left open. Euclidean geometry thus provides a suitable back-cloth to physics or any other explanatory scheme, in that it does not pre-empt the answers they may give to further questions. If we want geometry to have a low profile, being able to accommodate a variety of other explanatory schemata, then Euclidean geometry is the least obtrusive one we can have. But to accept this option is to choose a particular role for geometry, and not that proposed in geometrodynamics, where a single integrated explanatory scheme

is sought.[23]

4. We can look at geometry not axiomatically but group-theoretically; in that case the Euclidean group is singled out as the most basic, in as much as it is generated by the simplest non-trivial discrete group, the simplest continuous cyclic group and the simplest continuous serial group. There are good epistemological and practical reasons why communicating agents who occupy different locations in space at any one time should attach significance to features that are invariant under the Euclidean group of transformations.

5. A metric needs to satisfy some conditions if it is to be coherent. If there is more than one dimension, some symmetric function is needed to add distances in independent directions. The Pythagorean rule is the simplest one that satisfies the conditions.

6. A geometry, or indeed any mathematical theory, may be embedded in another theory which is a natural generalisation of it, and the axioms of the smaller theory may then emerge as theorems of the larger. In particular, an $n$-dimensional geometry may be embedded in an $(n+1)$-dimensional one, or an $n$-dimensional one in one with an infinite number of dimensions (a Hilbert space, for example). The more general theory, just because it is a more general theory, justifies the axioms of the more specific theory, which are seen as being just a special case of a more fundamental truth.

These considerations are not, of course, deductive proofs. They are, rather, what Mill described as "considerations... capable of determining the intellect either to give or withhold its consent".[24] They could be denied without inconsistency. But often the consequences of denying them would be awkward, and would need some explanation themselves, which we might find hard to give. There is a *prima facie*, though not conclusive, argument in their favour.

They apply not only to geometry, but to other axiomatic systems too, such as set theory and Peano Arithmetic.[25] Set theory is not just a formal system in which we can choose any axioms we

[23] How do considerations 3 and 4 relate? 3 says that if something is relevant its bearing should be spelled out precisely: 4 says that often it is good to have one feature irrelevant to another. So either deem one feature totally irrelevant, or else say exactly what its relevance is.

[24] J.S.Mill, *Utilitarianism,* ch.1, p.4 in Everyman ed.

[25] See below, §6.5.

like, but is, also, constrained by conceptual links with other ranges of discourse. $\in$ could not be a reflexive relation; the axiom of foundation asserts an anti-Anselmian doctrine of ultimate *ur*-elements that are utterly minimal. A set theory with the Generalised Continuum Hypothesis is more exact than one without, in as much as it specifies that there is no cardinal between any transfinite cardinal and the cardinal of its power set, whereas without such an axiom there is an indeterminate range of possible situations. It is better to come down definitely in favour of discreteness (or, for that matter, density) than to leave the question open. In a rather different way the Axiom of Choice may be defended as a natural generalisation of a principle of finite choice that is unquestionably true. The group-theoretical approach to geometry may be paralleled by an entirely fresh approach to set theory from the stand-point of category theory or the theory of games, which may yield entirely new insights into it, and may make some axioms seem obviously true (or obviously false).