Chapter 7
The Infinite

7.1 In Defence of Doubt

Once we allow some principle of recursive reasoning, we are landed with the infinite. There are an infinite number of natural numbers: for 0 is a natural number, and if anything is a natural number, its successor is a natural number. So there are natural numbers, and for any finite number, n, it is demonstrably false that there are exactly n natural numbers. We feel impelled to allow the question 'How many natural numbers are there?', and the only possible answer seems to be 'an infinite number'. But we have qualms. Infinity seems out of this earth. It smacks of Platonism, mysticism and theology. The word 'infinite' is a negative concept, contrasting not only with 'finite', in a strict mathematical sense, but with 'definite' and with 'comprehensible'. Often, especially in theology and ancient philosophy, the Infinite is the Whole, *τὸ πᾶν* (*to pan*), the Universe, the Absolute, whose logic is difficult and fraught with inconsistencies. We are wise to be wary.

Although Parmenides, Plato, Augustine, William of Alnwick, Leibniz, Cantor, Dedekind and most modern mathematicians are fairly happy with infinity, other philosophers have had doubts. Aristotle allowed the existence of the potential infinite but denied the actual infinite, and was followed by Aquinas and most of the Schoolmen. It is fair to place Weyl and the modern Intuitionists in that tradition. Locke had considerable difficulty in articulating a coherent and satisfactory account of infinity. Berkeley was deeply critical of infinitesimals and mathematical infinity generally, and modern Finitists likewise reject every sort of infinity, and try to confine themsleves to finite numbers alone. But infinitistic arguments keep seeping in, and strict Finitists distinguish themselves from their laxer brethren, and still are not stringent enough in their scepticism to escape the strictures of the ultra strict.[1]

It is easy to stifle doubts, and accept the infinite as part of the mathematical exercise, justified by the general success of mathematics. Geometry runs more smoothly if we allow points at infinity: then we can say that two straight lines always intersect, parallel lines meeting at a point at infinity. If we add the symbol ∞ to our other numerical symbols, we are able to discuss series, sums and integrals much more incisively. Once we allow the question 'How many natural numbers are there?' it is illuminating to answer $\aleph_0$. It seems silly to let philosophical scruples deter us from entering Cantor's paradise.

But credulity conduces to inconsistency. Easy acceptance is not on. It is important, though difficult, to articulate the objections to infinity that we are inclined to feel. What tends to happen is that the objections are only half-formulated, and never adequately considered; instead, the mathematician gets used to operating with infinity, and, as it were, represses the doubts he once felt without ever thinking them through properly and either seeing that they are valid and accepting them or seeing that they are not valid and why. As in psychoanalysis, repression causes trouble later on. Many Intuitionists show symptoms of not having come to terms with infinity in their youth, and suffering in consequence from psychological lesions in middle age. The only prophylactic is to give full rein to doubts at the first encounter. Doubt now, doubt bravely: *dubita fortiter* to parody Luther.

[1] Terminology varies. Hao Wang (*From Mathematics to Philosophy*, London, 1974, p.52; "Eighty Years of Foundational Studies", *Dialectica*, **12**, 1958, pp.466-497.) distinguished, besides orthodox objectivistic mathematics, three sceptical positions—Predicative set theory, Intuitionism, Finitism—and subsequently distinguished within Finitism, a further, even more sceptical, position, Ultrafinitism. In this chapter the positions distinguished and discussed will be Intuitionism, Lax Finitism and Ultrafinitism.

Doubt bravely. Most sceptics are too timid. They lack the courage of their lack of conviction, and doubt some things, while cravenly believing others, which, according to their lights, are equally open to question. And correspondingly, one cogent counter to scepticism is to disallow selectivity, and deny to the sceptic on grounds of a more complete scepticism the tenets he requires in order to formulate and communicate his doubts.

The cardinal and ordinal approaches to infinity will first be developed, and their difficulties exposed. Intuitionism will then be considered in its own right, and then increasingly sceptical forms of Finitism. Once these are shown not to offer a satisfactory alternative to full-blooded infinitistic accounts, further attention will be given to the difficulties of infinity and ways of remedying them.

7.2 Cardinality

We have seen in Chapter Four that natural numbers are grounded in a certain sort of quotifier which enables us to answer the question 'How many *a*s are F?'. These numerical quotifiers can be defined one after another, and form a natural sequence, or progression. The natural numbers are thus adequately characterized. We can, therefore, ask how many there are as we can ask 'How many three-figure numbers are there?'—the question is definite, though the answer is not. But whereas the answer to the latter, 900, is straightforward, the answer to the former must take us into some transfinite realm. For, as Peter of Spain argued, feeling towards the principle of recursion, there is more than one finite natural number, there is more than two finite natural numbers, there is more than three finite natural numbers, and thus by a "ladderlike" inference from these, *et sic de aliis* for any finite natural number there are more finite natural numbers than that.[2] So if we ask the question 'How many natural numbers are there?', no numerical quotifier thus far introduced will give the right answer; it must, therefore, be a new sort of quotifier. We resort to the same tactic as when faced with the questions 'How many times does 5 go into 12?', 'What is left when we take 12 from 7?', 'What is the square root of 2?', and

[2] Peter of Spain, *Tractatus called afterwards Summule Logicales*, ed. L.M. De Rijk, Van Gorcum, 1972, p.231, cited by Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., *Cambridge History of Later Medieval Philosophy*, p.568, n.11.

'What is the square root of -1?'. In each of these cases we extended our concepts so that the question, which previously could not always be given an answer, now can. In the same way, we want to extend our concept of numerical quotifier, so that to the question 'How many natural numbers are there?' we give an answer, and say 'There are $\aleph_0$ natural numbers' (pronounced aleph nought, from the first letter of the Hebrew alphabet).

A key step in the cardinal approach to number was to establish a criterion for the same cardinal number to be applicable in different cases. If I say that the number of natural numbers is $\aleph_0$, how do I tell whether the number of negative integers, of positive rationals, of real numbers, is $\aleph_0$ too? Frege's answer is that the number of negative integers is the same as the number of natural numbers iff they are equinumerous (*gleichzahlig*) with each other, that is to say iff there is a one-one mapping between the negative integers and the natural numbers, so that to each negative integer there corresponds one and only one natural number. But then there is a problem: whereas a finite multitude is never equinumerous with a proper subset of itself, infinite multitudes characteristically can be. If we pair off each natural number with its double, thus:

0	1	2	3	...	...
0	2	4	6	...	...

we establish a one-one mapping between the natural numbers and the even numbers. So we have to say that there are $\aleph_0$ even numbers as well as $\aleph_0$ natural numbers, even though the even numbers are a proper subset of the natural numbers. Similarly the pairing

0	1	2	3	...	...
1	2	3	4	...	...

shows that there are $\aleph_0$ counting numbers even though the counting numbers are a proper subset of the natural numbers.

Even more paradoxical is Cantor's proof that there are **only** as many rational numbers as there are integers. Although, as we shall shortly see, we cannot pair off the positive rationals in their natural order of size with the positive whole numbers (the counting numbers), we can do so if we rearrange them, expressed as fractions, in the form m/n, where m and n are natural numbers in an array:

1/1	2/1	3/1	4/1	...	...
1/2	2/2	3/2	4/2	...	...
1/3	2/3	3/3	4/3	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...

This array will more than include all the positive rational numbers (since 1/2, 2/4, 3/6, which all represent the same rational number, will occur separately and be counted separately). Although we still could not establish a one–one pairing if we considered each row, one after another, nevertheless if we count along successive diagonals, taking first 1/1, and then 2/1 and 1/2, and then 3/1, 2/2, 1/3, where in each diagonal the total of the numerator and the denominator is the same, we shall reach every member of the array in due course, and can thus set up a one–one correspondence between the members of the array and the natural numbers. Hence the cardinal number of the positive rationals is the same as that of the natural numbers; and a further extension of the argument shows that so is the cardinal number of all the rationals, positive and negative, together with nought. Such a conclusion seems highly counter-intuitive. We are denying the truism that the whole is greater than the part: by accepting Frege's, Dedekind's and Cantor's definition of equinumerosity, we are led to the paradoxical conclusion that the whole is also equal to the part. But a thing cannot be both greater than and equal to another: if infinity leads to this conclusion, it is absurd, and the whole concept must be rejected.

Many philosophers have accepted that conclusion. John Philoponus used it against the Aristotelian thesis that the world had always existed, and so did many of the Schoolmen; essentially the same argument is used by Kant and has been put forward again in recent years.[3] Others, however, have sought to evade the apparent

[3] See Richard Sorabji, *Time, Creation and the Continuum*, London, 1983, chs. 13, 14 and 15 for ancient and mediaeval discussions of infinity. See G.J.Whitrow, "The Age of the Universe", *British Journal for the Philosophy of Science*, **5**, 1954-5, pp.215-225, and **29**, 1978, pp.39-45, esp. p.41. See also K.R.Popper, *British Journal for the Philosophy of Science*, **29**, 1978, pp.47-48 and J.Bell, *British Journal for the Philosophy of Science*, **30** 1979, pp.161-165.

paradox. Nicole Oresme and Albert of Saxony anticipated Galileo and Newton, and simply denied the applicability of the relations *less than* and *equal to* to infinities. Such a response is not open to us. We are already committed to some sense of *equal to* in the relation of equinumerosity we have used to define the equivalence classes from which we abstract cardinal numbers; and we could hardly call them *numbers*, if the relation *less than* were altogether inapplicable. Henry of Harclay allowed the applicability of these relations, but questioned what exactly the part-whole relation was. William of Alnwick began to distinguish different ways in which one infinity could be greater than another, and Gregory of Rimini did distinguish the *greater than* of proper set inclusion from the *greater than* of having a higher cardinality.[4] Gregory was right. Ordering relations, as we shall see in Chapters Nine and Ten, constitute a whole family, as do equivalence relations, so that we need to distinguish different orderings and different equivalences, and although proper set inclusion is incompatible with the set-equality defined as $A = B$ iff $A \subseteq B \wedge B \subseteq A$, it does not follow that it is incompatible with set-equality defined as equinumerosity. Once it is recognised that *equinumerous with* and *less than* belong to different families, the paradox is resolved.[5] Indeed, instead of being a paradox, it can be used as an alternative definition of infinity. Dedekind defined an infinite set as one that was equinumerous with a proper subset of itself.[6] Infinity thus defined comes to the same thing as the recursive approach adopted in this chapter, though it is not quite trivial (and requires the Axiom of Choice) to establish the equivalence.

We might suppose that since infinite sets can be equinumerous with proper subsets of themselves, all infinite sets were equinumerous with one another. But this is not so: there is more than one infinite cardinal. Cantor showed that it was impossible to list all the *real* numbers between 0 and 1, one after another in an infinite list. Of course, they can be ordered. Given any two distinct real numbers we can say that one of them is greater than the other. But that does not give us a list, because there is always another between them—the case is similar to that of the rationals which

4 See Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., *Cambridge History of Later Medieval Philosophy*, pp. 567-573.

5 See below, §9.5.

6 See above, §5.2.

cannot be listed while in their natural order. Nevertheless there might be another ordering which was a list—*i.e.* there was a first, a second, a third, ... *etc.* with nothing between the first and the second, or the second and the third. But Cantor showed it to be impossible. Every real number can be expressed as an infinite decimal, and *vice versa*. Some—those ending in 0 recurring or 9 recurring can be expressed in two ways; thus .9999... is the same as 1.0000... In all such cases he fixes on the 0 recurring form. Suppose then we did have a list of all the real numbers between 0 and 1. It would have the general form:

$$\begin{array}{llllll}
.a_{11} & .a_{12} & .a_{13} & \dots & .a_{12} & \dots \\
.a_{21} & .a_{22} & .a_{23} & \dots & .a_{22} & \dots \\
.a_{31} & .a_{32} & .a_{33} & \dots & .a_{32} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
.a_{n1} & .a_{n2} & .a_{n3} & \dots & .a_{n2} & \dots
\end{array}$$

where a_{11}, *etc.* are digits, either $1, 2, 3, 4, 5, 6, 7, 8, 9$, or 0. By his method of diagonalization he then constructs a new number $.b_1b_2b_3 \dots b_n \dots$ which he showed was not in the list. The rule was to define b_n in terms of a_{nn} but so as to be different from it. If a_{nn} was $0, 1, 2, 3, 4, 5, 6, 7$ or 8, b_n was $a_{nn} + 1$: if a_{nn} was 9, b_n was 0. So $b_n \neq a_{nn}$. And in that case the number $.b_1b_2b_3 \dots b_n \dots$ cannot occur in the proffered list. It cannot be the first one, because $b_1 \neq a_{11}$: it cannot be the second one, because $b_2 \neq a_{22}$: it cannot be the third one, because $b_3 \neq a_{33}$: and so on down the list, it always fails to be the n^{th} one on the list because it differs from it in its n^{th} digit. Thus there is no way of correlating the real numbers between 0 and 1 with the counting numbers $1, 2, 3, 4, \dots$ *etc.* Since there are evidently at least as many real numbers as counting numbers, and they are not equinumerous, it follows that there must be more real numbers than there are counting numbers, that is, the number of real numbers is greater than $\aleph_0$. We say that the number of real numbers, the cardinal number of the continuum as it is sometimes called, is **non-denumerable**. Further, if every decimal of the form $.c_1c_2c_3 \dots c_n \dots$ represents a real number between 0 and 1, there must be $10^{\aleph_0}$ real numbers between 0 and 1. For there are 10 possible choices for the first digit after the decimal point, 10 for the second, and so on, *ad infinitum*. Equally, we could have used the binary notation, expressing each real number between 0 and

1 as $.d_1d_2d_3\ldots d_n\ldots$ where d_i could take only the values 1 and 0. So, on this reckoning there are $2^{\aleph_0}$ real numbers between 0 and 1, with what would have seemed previously the paradoxical conclusion that $10^{\aleph_0} = 2^{\aleph_0}$.

That there should be these two infinities is not altogether surprising. The Greeks had distinguished two, the infinite by addition and the infinite by division. They called the former the infinitely big, and the latter the infinitely small. It is ironic that Cantor has led us to the opposite conclusion, with the continuum, which we reach by infinite division, having a larger cardinality than the natural numbers, which were the paradigm of infinite bigness.

One might wonder whether these were the only two infinities there were, or whether there were others besides; more specifically, whether there are any between the natural numbers and the continuum, and whether there are any beyond the continuum. Cantor believed that the answer to the first question was No, and spent much of his life trying to prove "the Continuum Hypothesis", as it was called, to wit that there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$. We now know that the Continuum Hypothesis can neither be proved nor be disproved within standard axiomatizations of set theory. Cantor proved that the answer to the second question was Yes. He considered the "power set", that is to say, the set of all the subsets of a given set, and showed that it could not be equinumerous with the original set: the assumption that there was a one–one mapping from the original set to the power set led to an inconsistency. He thus established that the cardinal number of the power set was always greater than that of the original set, or in general terms that $2^{\aleph} > \aleph$. Some care is needed in defining *greater than* and *less than* between cardinals. We cannot simply use $\subset$, proper set inclusion, as we can with finite cardinals. Instead, we need **first** to pick out the equivalence classes of equinumerous sets, and **then**, among these, the quotient set of sets under the equivalence relation of equinumerosity,[7] use $\subset$, proper set inclusion, to order these equivalence classes. In effect, we are defining a new relation on sets, and saying that X has a lower cardinality than Y, iff X is properly included in Y **and** X is **not** equinumerous with Y; or formally (and clauses in correct order)

[7] See below, §9.5.

$$X \prec Y \qquad \text{iff} \qquad (X \subset Y) \wedge (X \not\equiv Y).$$

It is clear that this definition works both for finite cardinals and for $\aleph_0$. It is natural to extend it from sets having a cardinality to cardinal numbers themselves, and reckon that one cardinal is less than another iff there is a set having the first cardinality which is not equinumerous with a set having the second cardinality, but is properly included in it.

There is thus an infinite succession of infinities. Whether there are any others, between adjacent members of this succession, that is, whether there is an $\aleph$ between $\aleph_\alpha$ and $2^{\aleph_\alpha}$ is again an open question. The hypothesis that the answer is No is the claim of the General Continuum Hypothesis. We are naturally led also to wonder how many cardinal numbers—how many $\aleph$s—there are. But to this question no answer can be given. The cardinal number of all cardinal numbers is an incoherent concept. In which case, as Dummett observes, we are hardly entitled to press the question 'How many natural numbers are there?' or to posit $\aleph_0$ as an acceptable answer. Even on the simple cardinal approach, the critics have a strong case, and we cannot take infinity for granted without considering and meeting their objections.

7.3 The Mostest

Although most mathematicians first come across infinity as a useful symbol, ∞, or as the cardinal number of the natural numbers, the approach in terms of some ordering relation is more profound, since infinity, coming from the Latin *finis*, an end or a boundary, is, like the other concepts used in discussions of infinity, such as limit, bound, or extreme, susceptible of analysis in terms of some ordering relation. In the Middle Ages the Schoolmen defined the infinite in terms of *maius*, the Latin comparative of *magnum*, meaning 'greater than'.[8] And Cantor was led to develop transfinite arithmetic in order to make sense of successive operations iterated without end, but nevertheless tending towards some sort of limit. He considered the topological operation of taking the derived set of a given set, which could be indexed by a single dash or a subscript $_1$,

[8] Thomas Bradwardine, *Tractatus de Continuo*, cited by Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., *Cambridge History of Later Medieval Philosophy*, p.567, n.8.

and then of taking the derived set of the derived set, which could be indexed by a double dash or a subscript $_2$, and so on, *ad infinitum*. It seemed reasonable to consider the set that would result from *all* these operations having been done, and to assign to it the index ω so that ω could be said to come after all the other indices. It makes sense. Moreover we can assimilate the way in which such a limit number comes after all its predecessors to the way in which an ordinary number comes after its predecessor. We can define 4, the successor of 3, in terms of the set $\{0, 1, 2, 3\}$:[9] and we can define ω, the successor of the whole numbers, as the ordered set $\{0, 1, 2, 3, 4, \ldots\}$.

The difficulty arises as soon as we start talking of the limit as something definite. The limit of 'more' is 'most'. If it is more than all (other) members, it is the most. But could not there then be something more than it? If ω is admitted as a definite ordinal number, a "limit ordinal" coming after $0, 1, 2, 3, 4, \ldots$ can we not have $\omega + 1$ coming after ω? It seems that there is an ineradicable inconsistency in attempting to comprehend that which is necessarily not bounded in one completed whole. Whatever is said cannot be the last word on the topic, and must in the nature of the case be overtaken by the further development of the ever developing incomplete process.

This objection was first formulated by Aristotle, and was much considered in the Middle Ages.[10] Aristotle argued against the "actual" infinite and was willing to allow only the "potential"

[9] See above, §4.7 and §5.6.

[10] Many of Aristotle's arguments are physical, and seek to show that no infinite substance actually exists in nature. The theologians, particularly St Augustine (*De Civitate Dei*, XII, xviii, 19.), were concerned with God's unbounded power and knowledge, quite independently of the contingent restraints of physical reality. The only limitations on the power of an omnipotent God are conceptual limitations. The Schoolmen discussed whether there was any logical or conceptual impossibility in the actual infinite as opposed to physical arguments against it (see Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., Cambridge *History of Later Medieval Philosophy*, Cambridge, 1982, pp.566-569.) In a later age still, Georg Cantor viewed the actual infinite, and the whole of transfinite arithmetic, from a theological standpoint (see Michael Hallett, *Cantorian Set Theory and Limitation of Size*, Oxford, 1984, pp.9-11, 14, 21-24, 35-37).

infinite.[11] His conceptual argument rested upon a distinction in definition. The infinite is not, as is customarily thought, that which has nothing outside it, but that which always has something outside it: *οὐ γὰρ οὗ μηδὲν ἔξω* (*ou gar hou meden exo*), but *ἄπειρον μὲν οὖν ἐστιν οὗ κατὰ τὸ ποσὸν λαμβάνουσιν αἰεί τι λαμβάνειν ἔστιν ἔξω* (*apeiron men oun estin hou kata to poson lambanousin aiei ti lambanein estin exo*).[12] Bradwardine and many of the Schoolmen rendered his thesis as saying that the infinite was not *tantum quod non maius* but *non tantum quin maius*[13] in preference to the literal rendering *non enim cuius nihil est extra, sed cuius semper aliquid est extra*, and some, making the crucial distinction between multitudes and magnitudes, said that an infinite multitude was not *tot quod non plura* but *non tot quin plura*.[14]

We can express Aristotle's thesis in symbols, if we take *ἔξω*, (*exo*), outside, to be representable by $>$: it is not

$$(\mathrm{V}x)\neg(\mathrm{V}y)(y > x)$$

$$\text{but} \quad (\mathrm{A}x)(\mathrm{V}y)(y > x)$$

The former can be rewritten

$$(\mathrm{V}x)(\mathrm{A}y)\neg(y > x)$$

which, granted a linear ordering for $>$, is equivalent to

$$(\mathrm{V}x)(\mathrm{A}y)(x \geq y).$$

The former thesis thus claims that there is a superlative, a $>$est, a "mostest", an all-embracing all; whereas Aristotle denies this, and

[11] *Physics* III, §4-§8, $202^b30 - 208$, esp. $207^b27 - 34$; *Metaphysics*, K, 10, $1066^a35 - 1067^a37$, *De Caelo*, I, §5-§7, $271^b - 275^b$.

[12] *Physics*, III, 6, $207^a7 - 9$.

[13] Thomas Bradwardine, *Tractatus de Continuo*, cited by Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., *Cambridge History of Later Medieval Philosophy*, p.567, n.8.

[14] Anneliese Maier, "Diskussionem über das aktuell Unendliche in der ersten Hälft des 14 Jahr-hunderts", in Maier, *Ausgehendes Mittellalter*, Vol I, 1964-7, pp.41-44; cited by Norman Kretzmann, Anthony Kenny, Jan Pinborg, eds., *Cambridge History of Later Medieval Philosophy*, p.567n7.

says that however much there is, there is always more. Aristotle is arguing for a comparative rather than superlative account, a more-than rather than a mostest doctrine of infinity. It is clear that the order of quotifiers is crucial. As he sees it, there are two different concepts, on the one hand a metaphysical and theological one, and on the other a conceptual and mathematical one. When we are talking about the Whole, *τὸ πᾶν* the Universe, or God, it is appropriate to say that it is not included in, enclosed in (we should note Aristotle's spatial word *ἔξω* (*exo*), outside) or surpassed by, anything else, and so is not comprehended, not bounded. To say that God is infinite is to say that He is unbounded and incomprehensible. To say that the Universe is infinite, although it may now be construed as saying that the measure of some aspect of it—its volume, its duration, its mass—is infinite in the mathematical sense, often means rather that it is not a closed system, that it is not subject to external constraints or external boundary conditions, because, by definition, it includes all such, and every condition is already taken into account. The logic of the word 'universe' is a simple exercise in the superlative mode. There is some ordering relation—usually the part–whole relation—and the universe is the maximum. In theology the relation is ontological: there are greater and less degrees of reality, giving rise to a great chain of being, with God conceived as the most real, the ultimate reality, as in St Anselm's definition, *id quo maius nequeat cogitari*, that than which a greater cannot be thought of. The contrast between the Anselmian absolute, as we may call it, and Aristotle's comparative concept is expressed by the reversal of the order of the quotifiers and the use of the strict asymmetrical ordering relation $>$ instead of the antisymmetrical $\geq$. The two are clearly incompatible. If for every x there is a y that is greater than it,

$$(\mathrm{A}x)(\mathrm{V}y)(y > x),$$

then any candidate for being the Anselmian absolute will be surpassed by some other entity, and so will forfeit its claim to maximality.

$$\text{So} \qquad (\mathrm{A}x)(\mathrm{V}y)(y > x) \vdash \neg(\mathrm{V}x)(\mathrm{A}y)(x \geq y).$$

We thus have a straight inconsistency between Aristotle's doctrine of the potential infinite and the Anselmian account of the actual infinite, and it might seem that the former was to be preferred. Intuitionism offers a coherent philosophy of mathematics allowing only the potential infinite.

7.4 Characterization of Intuitionism

Intuitionism was first propounded by L.E.J. Brouwer,[15] who was himself greatly influenced by the philosophy of Kant. It was considerably developed, and modified, by Heyting and Beth. In our own time it has been practised by Bishop[16] and Troelstra,[17] and its chief protagonist has been Michael Dummett.[18] Dummett has been influenced more by Wittgenstein than by Kant, and modern discussions of Intuitionism have been carried on in terms of a theory of meaning, which, it is argued, reveals the invalidity of certain classical principles of logic when infinite sets are involved, notably the Principle of Bivalence, or, equivalently, the Law of the Excluded Middle, and, concomitantly, the Law of Double Negation.[19]

Intuitionism can be seen as a version of conceptualism. Mathematics exists in minds, and is manifested in the activity of mathematicians. Like musical composition, it is invention, not discovery. The emphasis is on proving—the analogue to making music—rather than on what is proved—the score. Musicians form a community of those who love making and listening to music, and mathematicians also constitute a community, of those who love noticing proofs and sharing them with others.

Metaphysically, Intuitionism is cheap, like conceptualism, without being crassly reductionist, as Formalism often seems to be. Intuitionists see themselves as offering an alternative and better

[15] L.E.J.Brouwer, *Collected Works*, Volume I: *Philosophy and Foundations of Mathematics*, ed. A.Heyting, Amsterdam, 1975.

[16] E.Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[17] A.S.Troelstra, "Aspects of Constructive Mathematics", in J.Barwise, ed., *Handbook of Mathematical Logic*, North-Holland, 1977, pp.973-1052.

[18] M.A.E.Dummett, *Elements of Intuitionism*, Oxford, 1977; and "The Philosophical Basis of Intuitionistic Logic" in M.A.E.Dummett, *Truth and Other Enigmas*, Duckworth, 1978.

[19] In addition to the works already mentioned, the reader may find it convenient to consult P.Benacerraf and H.Putnam, *Philosophy of Mathematics: Selected Readings*, 2nd ed., Cambridge, 1983, pp.52-61, 77-129.

answer than the Formalists to the question "If not Platonic Realism, then What?"; and answer not in terms of what is drawn on paper or blackboard, but what goes on in mathematicians' minds, and the collective experience of the community of mathematicians.

If not Platonic Realism, then What?
Intuitionism?

The Intuitionists are not committed to some platonist realm of entities over and above those of the familiar world. They do not have to posit the existence of Numbers, or Sets, or any other abstract objects, only people, who think and talk. And they do not have to make out that mathematics is merely the study of marks on paper, or inscriptions on a blackboard: the marks are merely the means of communicating what is in the mathematician's mind. The one claim that the Intuitionist must make is that there are valid proofs, and that these can be seen to be such by other competent mathematicians.

The emphasis the Intuitionists lay on the activity of the mathematical community is a welcome antidote to Plato's ban on all operational terms,[20] and the solipsistic tendencies of most classical accounts. The emphasis on sharing proofs focuses attention on one of the distinguishing characteristics of mathematics. These are real merits. But Intuitionists are unwilling to accept the logic of these emphases. For all their talk of the mathematical community, they are unwilling to accept the general view of the mathematical community as regards their own contentions. Most mathematicians accept classical arguments as valid. Intuitionists beg to differ. It makes sense to disagree with everybody else, if there is some objective standard of truth, independent of what any, or every, one thinks. On that basis, it is perfectly legitimate to hold the great majority of mathematicians mistaken in their acceptance of the Law of the Excluded Middle. But on that basis, too, mathematics is not simply constituted by the communal activity of mathematicians, but aspires to some truth independent of what mathematicians actually do, and in the light of which what they actually do can be corrected. Intuitionists are faced with a dilemma. Either

[20] See above, §2.7.

mathematical practice is corrigible or it is not: if it is corrigible, then what is claimed in a putative proof can be different from the proof itself; if it is not corrigible, then the mathematics mathematicians actually do is classical, not intuitionist, and Intuitionists can no more argue that classical mathematicians are wrong to argue as they do than a jazz pianist can argue that Bach is wrong to compose as he did.

Activity is important in mathematics, but is importantly different from musical activity. In music, if it seems right, it is right: in mathematics, the constraints are more rigid, and the object of the operation is external to the operation itself. If by extending a line, I can construct a proof, or by simplifying an equation, factorise and extract its roots, well and good. But always there is an end-result the mathematician is trying to achieve; and it is natural to characterize this as the discovery of truth rather than the creation of beauty.

Proving is not the only mathematical activity. Mathematicians also calculate and solve problems. And although proofs are often shared, Intuitionists too easily assume that everyone can recognise a mathematical proof when he sees one. But most people cannot, and many otherwise intelligent men, like Hobbes, have been gravely mistaken in their judgement that some proof was valid when it was not. Although the feeling of compellingness of mathematical argument is an important feature, an exegesis of mathematical theorems exclusively in terms of their proof, is misconceived. It is easy to understand Pythagoras' theorem, and use a 3-4-5 triangle for marking out tennis courts, without being able to prove it. There are said to be forty-seven proofs of Pythagoras' theorem, but they are all proofs of one and the same theorem, not proofs of forty-seven different propositions. For many years in the last century it was believed that the Four Colour Theorem had been proved by Kempe, until Heawood pointed out a flaw in the putative proof, but mathematicians were able to explain to school boys what it meant, and still can, even though very few people understand the modern proof, if indeed it is a valid one. Similarly, many people understand Fermat's Last Theorem, though only a few can follow Andrew Wiles' proof of it.[21] The logic of 'p is true' is different from that of 'I have a proof of p'. If p is true, it always was true; if Fer-

[21] But see further below, §14.3.

mat had not discovered a valid proof of his last theorem, it would none the less still be true, even though he could not have written truthfully that he had a proof of it. Unless a mathematician could understand the proposition to be proved independently of its actually being proved, he could not conduct an intelligent search for a proof. Mathematicians are not interested in proofs alone, but are primarily interested in them on account of the propositions they establish as true. The final argument for insisting on a fundamental distinction between a proof and what a proof actually proves, must be left to the next chapter, where Gödel's theorem establishes beyond doubt that truth outruns provability.

7.5 Proofs and Dialogues

The emphasis laid by Intuitionists on proofs rather than on what is proved explains the most controversial and characteristic tenet of Intuitionism, namely the denial, where infinite sets are involved, of the Principle of Bivalence (p is either true or false) and the Law of the Excluded Middle (It is necessarily the case that either p or not-p). Since in ordinary propositional calculus the Law of the Excluded Middle is inter-derivable with the Law of Double Negation, $\neg\neg p \vdash p$, the Intuitionist also denies that. If we are talking about proofs, then the fact that I have not proved a proposition goes no way towards proving that I have disproved it. To maintain otherwise would be like the Irishman who was accused of stealing a pig, and produced seven witnesses who had not seen him do it. We have

'I have not proved p' does not entail 'I have proved (not-p)'

It is tempting to follow a suggestion originally made by Gödel to reinterpret Intuitionist statements as being really about provability. If we use a "dictionary", due to McKinsey and Tarski,[22] every thesis of intuitionistic propositional calculus becomes a thesis of $S4$; so if we construed the Intuitionist mathematician as talking about

[22] J.C.C.McKinsey and A.Tarski, "Some theorems about the sentential calculi of Lewis and Heyting", *The Journal of Symbolic Logic*, **13**, pp.1-15; discussed by A.S.Troelstra, in Kurt Gödel, *Collected Works*, vol. I, New York, 1986, pp.296-299. Gödel's original paper is printed there, pp.300-303; and also in J.Hintikka, ed., *The Philosophy of Mathematics*, Oxford, 1969, pp.128-129. See also A.N.Prior, *Past, Present and Future*, Oxford, 1967, pp.25-27.

proofs rather than propositions proved, it would make good sense. When he asserted p for a simple propositional variable, he would be saying that he had a proof of p, that is, that p is provable. When he asserted $\neg A$, for any well-formed formula, he would be saying that he had proof of $\neg A$, that is that $\neg A$ is provable. When he asserted $A \vee B$, for any well-formed formulae, he would be saying that he either had a proof of A or had a proof of B. When he asserted $A \wedge B$, for any well-formed formulae, he would be saying that he both had proof of A and had a proof of B. When he asserted $A \rightarrow B$ he would be saying that if he had a proof of A, he would *eo ipso* have a proof of B. If we construe the box of modal logic as expressing the 'is provable' of the Intuitionist, Gödel's translation seems natural.

But though classical mathematicians can accommodate within their own terminology what the Intuitionist seems to them to be trying to say, the Intuitionist will have none of it. Although admittedly concerned with proofs, he claims to be talking about what can be proved, and is propounding a radical reconstruction of its logic; and though classical mathematicians regard it as perverse to deny these generally accepted laws of thought, they cannot convict him of inconsistency. It is possible to formulate a version of propositional calculus in which the Law of the Excluded Middle is not a theorem, or equivalently, in which it is not permissible to use the Law of Double Negation to infer p from $\neg\neg p$. A consistency proof for the intutionistic propositional calculus is available—proof by erasure will clearly work—and we can similarly formulate a provably consistent intuitionistic predicate calculus. More generally, intuitionistic mathematics can be developed without any apparent inconsistency, and can be considered as a piece of formal mathematics in its own right without regard to the substantive claims of Intuitionism. Equally, the Intuitionist is unable to convict the classical mathematician of inconsistency either. Gödel and Gentzen have each shown independently that the consistency of intuitionist arithmetic implies the consistency of classical Peano Arithmetic.[23] It is a stand-off. Neither side can show the other to be completely wrong.

[23] Paul Bernays, "Sur la Platonisme dans les mathématiques", *L'Enseignment Mathématique*, **34**, 1935, pp.52-69; tr. in Paul Benacerraf and Hilary Putnam, *Philosophy of Mathematics: Selected Readings*, 2nd ed., p.271.

It should be noted that the Intuitionist is not alone in calling at least some of these theses in question. Aristotle raised the problem of "Future Contingents" in his discussion of tomorrow's sea battle, and in the controversy that has raged ever since about the correct resolution of the problem, it has been often suggested that perhaps the Principle of Bivalence does not always apply, and in particular not to contingent statements about the future.[24] Though to deny the Principle of Bivalence is not manifestly absurd, it still seems strange in a mathematician. Benenson put forward an argument by dilemma to show that an irrational number raised to the power of an irrational number may be itself rational.[25] For consider $\sqrt{2}$ raised to the power of $\sqrt{2}$,

$$\sqrt{2}^{\sqrt{2}}$$

Call it x. Then x is either rational or irrational: if it is rational, then the case is proved; if it is irrational, consider x raised to the power of $\sqrt{2}$,

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

By the usual rules for exponents, it is equal to $\sqrt{2}$ raised to the power of $(\sqrt{2} \times \sqrt{2})$, namely 2. So once again we have an irrational number raised to the power of an irrational number yielding a number that is rational.

The Intuitionist denies the force of this argument. The only time a mathematician should say

$$\sqrt{2}^{\sqrt{2}} \text{ is rational}$$

is when he has a proof of it. And the only time a mathematician should say

$$\sqrt{2}^{\sqrt{2}} \text{ is irrational}$$

[24] Aristotle, *De Interpretatione*, ch.9, tr. J.L.Ackrill, *Aristotle's Categories and De Interpretatione*, Oxford, 1966, pp.50-53, 132-142; N. Rescher, "Truth and Necessity in Temporal Perspective", in R.M.Gale, ed., *The Philosophy of Time*, London, 1968; Gilbert Ryle, *Dilemmas*, Cambridge, 1954, ch.2, "It Was To Be"; J.R.Lucas, *The Future*, Oxford, 1989.

[25] See M.A.E.Dummett, *Elements of Intuitionism*, Oxford, 1977, pp.9-10.

is when he has a proof of that. And the only time a mathematician should say p or q is when he either has a proof of p or has a proof of q. We therefore have no warrant for saying either

$$\sqrt{2}^{\sqrt{2}} \text{ is rational}$$

or

$$\sqrt{2}^{\sqrt{2}} \text{ is irrational}$$

and, according to the Intuitionist, the proof fails.

The Law of the Excluded Middle, which underlies Arguments by Dilemma, is inter-derivable with the Law of Double Negation. which better illustrates the issues between Intuitionist and Classical logic. Classical logic is dialectical in its underlying style, and views argument as a cooperative pursuit of truth. I make a claim and issue a challenge. I invite you, *for the sake of argument*, to counterclaim. 'Do you deny it?', I ask. If you do, I shall show you that you are wrong. If not, I claim to have the undisputed mastery of the field. But of course, you might question without denying. You might just say 'Prove it' without picking up my challenge. But if you accept first that my claim is intelligible apart from having been proved, and secondly that I am a serious operator who would not put forward frivolous claims, you will be prepared to cooperate, and either put up a counter-thesis for consideration or shut up, and accept my word for it. And if having tried to gainsay me you find you cannot on pain of inconsistency, then you must let my claim pass without more ado.

Of course you could refuse to cooperate, and I could not convict you of inconsistency if you did. But I should not want to pursue such a policy for myself, and I doubt if you would. If Euclid has an argument by *reductio ad absurdum*, I want to follow it out and discover new truths about geometry. Although by refusing to suppose the opposite I could foil his winning the truth-game, I should do so at the cost of myself forgoing some knowledge of the truth. Rather than do that, I am prepared to stick my neck out, and see what happens. Maybe I shall provide the premise for a *reductio ad absurdum*, and see that your claim cannot be controverted on pain of inconsistency, and acknowledge that if I cannot gainsay it, you must be allowed to assert it.

The Law of Double Negation arises from the concept of contradiction, which goes beyond the minimal concept of negation that stems from the concept of inconsistency that is a necessary condition of a formal system if it is not to be a trivial one, in which to be a theorem is nothing more than to be a well-formed formula.[26] Contradiction is a prerequisite not of a formal logical system, which is essentially a monologue, but of ordinary language and any dialogue which is carried on between two persons. If we have any communication between two autonomous agents, it is essential that either should be able to say 'No' to the other, and contradict what he has been saying. For there is no point in talking, if there is no possibility of disagreement. Communication is only necessary between different, finite, fallible centres of knowledge and ratiocination, where each may know some things and have thought out some things right, but each may be ignorant of some things, and is liable to have got some things wrong. Under such conditions it is always possible that one communicator will wish to dissent from what another communicator has said, and will need to use a word like 'No' by means of which he can contradict him. But those who contradict are also finite, fallible centres of knowledge and ratiocination, who may be ignorant of some things, and are liable to have got some things wrong. They may need to be contradicted in their turn, so that the record may be set straight again, and their foolish negation annulled. If a person contradicts me, and I am not persuaded to change my mind, I shall want to contradict his contradiction, and re-affirm my original assertion. Thus, although in a logistic calculus which is being used by only one person the requirement of consistency does not yield the Law of Double Negation, in a natural language, which is used to listen to as well as to talk with, it is essential that there should be a principle of cancelling, and of cancelling cancellations. It is a dialectical necessity between two persons symmetrically placed, and enjoying parity of epistemological esteem.

Since mathematics, too, is something that is argued about by more than one person, mathematical statements can be assessed by hearers as well as asserters, and I can return your service by a denial, and you can play back a counter-denial, and we can keep up a volley of denials and counter-denials. Of course, a player might refuse to volley. He might hit the ball down, instead of returning it across the net. He might reject his opponent's non-acceptance of his

[26] See above, §3.4.

claim, without reinstating it positively. But then the truth-game would peter out. Rather than force a draw, each player extends his claim, exposing himself to the risk of being proved wrong in the hope that either his or his opponent's claim will be found to be right. Truth on this view is not merely a matter of assertibility conditions but also of aspirations, and the Principle of Bivalence represents not an unwarranted assumption that we know what we do not know but an entirely respectable hope that we may know it, coupled with a willingness to play the truth-game in that spirit which will best enable finite beings of limited abilities but unlimited aspirations to discover as much truth as possible.[27]

The Intuitionist's rejection of the Law of Double Negation thus reveals him as occupying a special epistemological position of disparity, in which he does not accept the normal rules of dialectical discourse. In simple cases, like Benenson's argument for there being some irrational number which when raised to the power of an irrational number results in a rational one, where only some singular proposition is involved, his position seems implausible, and almost perverse. In most cases, however, it is where infinite sets are involved that the Intuitionist is unwilling to countenance the Law of Double Negation or Principle of Bivalence; and here his argument is stronger. We do not at present know whether it is in fact true or whether it is in fact false, but it must be one or the other. The Intuitionist demurs. Classical mathematicians, he says, are working with a fundamentally wrong model in mind, supposing that they can inspect each one of the infinite number of even numbers, and see if any are not the sum of two primes, in which case Goldbach's conjecture is false, or that none are not the sum of two primes, in which case all are, and Goldbach's conjecture is true. But there is no surveyable set of the even numbers, and hence there is a difference between genuine existence proofs, which are constructive, and those purported ones that depend on *reductio ad absurdum*. If we came across an even number which was not the sum of two primes, we should have falsified Goldbach's conjecture in a very definite and indisputable way. We would have established that there was a number, and could also say what it was. If, however, we merely devised a proof that showed that Goldbach's conjecture led to an

[27] The last three paragraphs are taken from J.R.Lucas, "Mathematical Tennis", *Proceedings of the Aristotelian Society*, 1984, pp.68-69.

inconsistency, we should have established less; it would be more difficult to persuade the determined sceptic, who kept on saying "Show me". If we were writing up our proofs in Latin, we should use the words mentioned in the previous chapter, and express the first result by the words *quidam numerus*, 'a certain number is even and not the sum of two primes', a number we could specify if we wanted, but are not so doing in order to cut out irrelevant detail. In the second case all we could say would be *nonnullus numerus*, 'it is not the case that no number is even and not the sum of two primes'; if asked, we should have to come clean and admit that we did not know which the number was, and we might forestall the question by using the Latin *nescioquis*, 'some number, I know not which, is even and not the sum of two primes'. There is a real difference in the force of the existential quotifier which is obscured in standard classical treatments, but can be accommodated in a dialectical account of the quotifiers: *quidam* invites the question 'Which one?'; *nescioquis* forestalls it; *nonnullus* is non-committal.

The difference between the different Latin renderings of the existential quotifier becomes important in mathematics when the quotification is over an infinite domain. If the domain is only finite, we could in principle go over each case one by one, and either find that it was not a case in point and pass on, or find that it was, in which case we should know not only that there was such a case, but which it was. We cannot do this in the infinite case, and in the eyes of the Intuitionist that fact constitutes a profound difference.

> $(\mathrm{V}x)F(x)$ is not the same as
> $F(a_1) \vee F(a_2) \vee F(a_3) \vee F(a_4) \vee \ldots \vee F(a_n) \vee \ldots$
> Intuitionists think this is **very important**

For them the infinite is always only a potential infinite, and never an actual one. Not only can we never actually survey it, but we cannot conceive of it as surveyable even by God. Classical mathematicians are fond of taking the God's-eye view, and suppose that God can survey an infinite totality as easily as we can finite ones. But this, according to the Intuitionist, is a profound mistake. God can no more survey an infinite totality than He can alter the past. Infinite totalities are not there for Him to survey. They do not exist. They cannot exist. They are a contradiction in terms.

It is difficult to argue theology with an Intuitionist, and our grip on ontology is still weak. But existence is not only a profession of ontological commitment. It is also a counter in argumentative discourse. In mathematics its most important function is to license talk. If I start talking about the greatest prime number, it is the same as if I start talking about the present king of France, or Ryle's youngest son. My discourse, though grammatical and consistent, fails to carry information because it fails to refer to anything. Rather than hear me out as I go rabbiting on, you interrupt, and say "But there is no greatest prime number". That is a conversation stopper. Once it has been alleged that there is no object that my purported referring term refers to, I must either meet the allegation and give it the lie, or else shut up. I cannot go on talking about nothing at all.

In ordinary conversational discourse it is easy to interrupt, and we need take no special pains to guard against failure of reference because should reference fail we shall soon be interrupted and brought to heel. Mathematics, however, is much more monologous. If you are reading my book, you cannot interrupt. Since I cannot argue from your silence that my referring attempts are being successful, I need to anticipate possible interruptions and deal with them in advance, so that if you were minded to interrupt, you would find the point already dealt with. I forestall your "But it does not exist", or, more colloquially, "But there isn't one", by an existence proof. Once I have proved that the nine-point circle exists, I can go on to talk about it and its centre and say that the latter is on the Euler line with complete confidence that I cannot be faulted on the score of talking about nothing at all. Equally if I am disposed to talk about Brouwer's fixed point, I can cite his Fixed Point theorem to ward off the counter that it does not exist, even though it is a non-constructive existence proof. If someone were to prove that Goldbach's conjecture was true, we should thereafter give short shrift to the computer buff who wanted us to give him a grant while he searched for a counter-example: and *per contra* if it were proved that Goldbach's conjecture leads to an inconsistency, then it would be a reasonable proposal to program a computer to find a counter-example.

What is at issue betwen the Intuitionist and the classical mathematician is the validity of inference patterns. The Intuitionist allows the *quidam* inference

$$F(a) \vdash (\mathrm{V}x)F(x),$$

which is the Existential Introduction rule of some systems, but not the *nonnullus* inference

$$\neg(\mathrm{A}x)\neg F(x) \vdash (\mathrm{V}x)F(x),$$

which classical logic also allows, as, so to speak, a pre-emptive strike, whereby it is shown that the objection, "There is no such instance", cannot be maintained. Whereas in the first, the *quidam*, dialogue, I claim that there is an instance, and, if you challenge me, I can say which it is, in the second, the *nonnullus*, dialogue, I counter-challenge you: "Do you maintain that there is no such instance?", I ask. "If you do, I shall show that you are wrong: if you do not, because you cannot, you are no longer in a position to question my original assertion."

The argument, like that with the *Sorites* Arithmetician in Chapter Six, is an argument by challenge, in the context of a cooperative search for truth. In §6.7 the cooperative arguer was prepared to hazard a counter-*example*, and thence be shown that that led to inconsistency, and thus to be convinced that no counter-example was sustainable, and the universal claim must, therefore, be allowed: here the cooperative arguer needs to hazard a counter-*thesis*, in order to be convinced that the counter-thesis is untenable, and thus that the original claim in undeniable. The only way of saying that I cannot say *quidam* is to say *nullus*, and if you say this, I can show you that you are wrong.

The fine-structure of the existential quotifier can thus be displayed by the different dialogues that warrant its use. Best of all is the straightforward *quidam* rule:

$$F(a) \vdash (\mathrm{V}x)F(x).$$

If I could find an even number that was not the sum of two primes, say $10^{11}+2$, then I should have disproved Goldbach's conjecture in the best way possible, so that even the Intuitionists were satisfied. When I said that there was an even number that was not the sum of two primes, and they asked "Which one?", I could answer them by saying which it was. If, however, I could merely show that the assumption that there was an even number that was not the sum of two primes led to inconsistency, I should lose the first round, having failed to answer their question. But if we were keen to know the truth, we should go on to a second round, which would give you a chance of winning the game, by showing that *if* I had tried to take up your challenge, I should have been bound to fail; that is, you attempt to show that *any* example I care to choose (from your point of view *quivis*: from mine *quivolo*) will prove fatal to my original contention. If you win this round, you win the game. I can no longer claim *nonnullus*, if *quivolo* has been refuted. If, however, you do not win the second round, the game continues to a third round, with *nonnullus* still in the field. In this third round I force you to take the initiative again—this time, since examples have not got either of us anywhere, in terms of a universal proposition as counter-thesis. Although I had not, as it happened, produced an instance of an even number that was not the sum of two primes, your counter-thesis that I could not, in principle, do so, is shown to be itself untenable. It would be unreasonable of you to try and shut down my research programme of looking at even numbers to see if they were the sum of two primes. Although, as of now, I have not succeeded in identifying any such number, or even ascribing any properties to it, I have answered the question *an sit*, and shown that it exists and is talkable about, even though I have not, as yet, answered the question *qualis sit*, or said anything significant about it.

7.6 Verificationist Arguments for Intuitionism

The emotional pressure towards Intuitionism is a general queasiness about actual infinities and platonism generally, but it is argued for, in its modern version, on the basis of a verificationist theory of meaning. Words in general, it is claimed, have meaning only in virtue of the way they are used, and the sentential connectives of propositional calculus can only acquire their meaning through the rules laid down for asserting the complex propositions they are used to build up. If assertability conditions alone are constitutive of meaning, then it is the conditions under which mathematical propositions can be legitimately asserted—that is, when they are proved—which alone determine their sense, and in particular the sense of the sentential connectives.

The key thesis is that assertability conditions alone are constitutive of meaning. It is supported by a rhetorical question:

> How are we supposed to be able to *form* any understanding of what it is for a particular statement to be true if the kind of state of affairs which it would take to make it true is con-

> ceived, *ex hypothesi*, as something beyond our experience, something which we cannot confirm and which is insulated from any distinctive impact on our experience?[28]

There is a sense that the classical mathematician is claiming some sort of acquaintance with entities he has no knowledge of, and that only by confining discourse to what we have a firm grasp of, can we avoid talking nonsense.

Another argument is the Manifestation Argument, which depends on Wittgenstein's emphasis on the identity of meaning and use. Wright understands this as establishing that to "understand an expression is to know how to use it properly, and the proof of such knowledge is that one does actually so use it".[29] Wright has an extremely stringent standard of proper use. He concedes that someone might manifest many signs of being able to use some expression—

> No doubt someone who understands such a statement can be expected to have many relevant practical abilities. He will be able to appraise evidence for or against it, should any be available, or to recognize that no information in his possession bears on it. He will be able to recognize at least some of its logical consequences, and to identify beliefs from which commitment to it would follow. ... In short: in these and perhaps other important respects, he will show himself competent to use the statement. But the headings under which his practical abilities fall so far involve no mention of evidence-transcending truth-conditions.[30]

Only if the classical mathematician can specify exactly the conditions under which he would say that Goldbach's Conjecture had been proved or refuted, will he have manifested a proper understanding of what that conjecture states.

Wright distinguishes a third argument, the argument from normativity. "Meaning is normative. To know the meaning of an expression is to know, perhaps unreflectively, how to appraise uses of it; it is to know a set of constraints to which correct uses must

[28] Crispin Wright, *Realism, Meaning and Truth*, Blackwell, Oxford, 2nd ed., 1992, p.13.

[29] p.16.

[30] p.17.

conform. Accordingly, to give the meaning of a statement is to describe such constraints; nothing has a claim to be regarded as an account of a statement's meaning which does not succeed in doing so." And, Wright argues, the realist's truth-conditional conception has indeed no such claim.[31] For, according to the realist, what the statement means is the same, whether or not it can be actually verified, and therefore is not constrained by whatever it might be that would verify it. It is as if the meaning were represented by one of two small boxes, identical in appearances, but one containing a beetle made of something that vaporises instantly if the box is opened. The realist's claim to mean something by an expression, independently of whether or not it can be verified, is like the claim to be thinking of the box with the beetle when in fact there is no discernible difference between the two boxes. And in such a situation we have Wittgenstein's authority for saying that that is a meaningless claim.

All these arguments are based on theories of meaning. But arguments from meaning should always be treated with caution. It is very easy to devise a theory of meaning, thinking of a few paradigm examples, and then, finding that other locutions cannot be accommodated within it, to dismiss them as meaningless. Many theories of meaning are vitiated by apparently taking for granted a correspondence theory of truth. 'Snow is white' is true, they say, if and only if snow is white, and from this they infer that mathematical statements are made true by their corresponding to the appropriate mathematical states of affairs. "How are we supposed to be able to *form* any understanding of what it is for a particular statement to be true", asks Wright, "if the kind of state of affairs which it would take to make it true is conceived, *ex hypothesi*, as something beyond our experience, something which we cannot confirm and which is insulated from any distinctive impact on our experience?"[32] But there is no *state of affairs* that makes mathematical statements true, in the way that there, arguably, is in the case of snow's being white, or the cat's being on the mat. Although sometimes, even in mathematics, we can extend Tarski's account of truth to enable us to construct models of theories, the model is being devised to bring out the desired formulae of the theory as true: the real direction

[31] p.24.

[32] p.13.

of truth-conferring is from the theory to the model, rather than *vice versa*. We cannot argue from our sometimes making use of semantic approaches to a general correspondence theory of truth. For many kinds of discourse—moral discourse, literary criticism, some parts of physics, most of philosophy and metaphysics—the correspondence theory of truth is inapplicable.

The Acquisition Argument carries little weight. Empiricist philosophers often have difficulty in accounting for our formation of concepts. That has always been a difficulty for empiricism. There is also room for a psychological enquiry into the way we learn mathematical concepts. But whatever its outcome, the fact remains that we are able to use mathematical terms in a way that seems to make sense to most mathematicians. We do not need direct acquaintance with some mathematical state of affairs to enable us to learn by ostensive definition what some mathematical proposition means. Rather, having learned on other, perhaps simple, occasions what the meaning of the constituent terms is, we can understand what the new complex must mean also. I come to understand Goldbach's conjecture not by knowing some state of affairs, but by knowing, as I think, the meaning of *every, even, number, sum, two, prime number*. Of course, I may deceive myself. The Intuitionist may persuade me that I do not really know the meaning of *every* in this context. But he has to show it. *Prima facie*, the word *every* is well understood, and the suggestion that we need to make out afresh on each new occasion of using it a justification for its use in the new context is not a suggestion we should take seriously.

The Manifestation Argument is likewise unpersuasive. Wittgenstein identified meaning with use. It is not clear that he was right to do so, but certainly someone cannot be said to understand an expression unless he knows how to use it properly. The anti-realist argues that since, *ex hypothesi*, there are no circumstances in which there is conclusive evidence for an "evidence-transcendent state of affairs", there can be no practical occasion for the assertion of the expression, and hence no grounds for believing that it is really understood. But our understanding is not so limited that we can only understand what we sometimes know. We engage in many speculations about the beginning and end of time, the possibility of spaces disconnected from our own, about the nature of God, about life after death, without ever being able to make warranted assertions about them. Wright himself concedes as much; but, according to Wright, it is not enough, for "the headings under which his practical abilities fall so far involve no mention of evidence-transcending truth-conditions". It is difficult to see what lack is being bemoaned. Goldbach's conjecture is a conjecture. I know what would falsify it—the instancing of some even number with a list of all the primes less than it, and the difference between the even number and each member of the list, with some factor of that difference. This, of course, is not an exact specification—if a particular even number were so specified, then the conjecture would be no longer a conjecture, but false. Much more so, I cannot specify what a proof would look like. I might not be able to understand it even if it were instanced, any more than I can understand the proof of Fermat's Last Theorem. But I have an unspecific idea of what a proof might be like. In putting forward Goldbach's conjecture I am putting forward a *hypothetical*: under certain, not exactly specified, conditions I should withdraw it, recognising it to be false, and under other, even less exactly specified, conditions, I should accept it as definitely true. I do not need to be able to specify these conditions exactly in order to manifest my ability to use the expression meaningfully. It would be absurd to make such a demand. In other realms of discourse—science, history, morals—we often make assertions and entertain hypotheses which may be subsequently confirmed or refuted by considerations not then known to us. Thirty years ago cosmologists were arguing for continuous creation without having then any idea of the echoes of the Big Bang which have subsequently refuted their hypothesis. If being able to use an expression properly is construed as being able to specify exactly the conditions under which it would be conclusively verified or falsified, then hardly any expressions are ever used properly, or are even meaningful.

There is much confusion over hypotheticals and possibilities. The Intuitionist discounts hypotheticals, and seeks to restrict possibilities to the actual. But this is not how we normally use words. When I make an assertion, I stick my neck out. However well warranted it may be, I am taking on the risk of being wrong. It may well be the case that thus far, under a wide range of conditions, whenever there has been an event of type A, there has subsequently been an event of type B, so that I am amply justified in making the causal claim that A causes B. I still may be wrong. New evidence, or a deeper understanding of science, may falsify my assertion. Its meaning cannot be constituted by its assertability conditions, but must take into account also the conditions under which I *might* subsequently *have to* withdraw it, or, alternatively, be entitled to claim that I had been vindicated further. These conditions are fuzzy-edged. They are not tightly specified in advance. We often do not know what we should say if such and such an eventuality should arise, though sometimes, if it does arise, we are fairly clear what the right response is, and do not think it is simply a matter of arbitrary choice on our part how we should respond.

Once it is recognised that meanings are not fully determinate, and that we do not have to cross all bridges in advance of setting out on a journey, other Intuitionist arguments lose their appeal. A mathematical proposition may be not effectively decidable, without being absolutely and always so. We distinguish between the difficult-to-discover beetle in the box, and the logically undiscoverable one. In the latter case I am rationally suspicious, and do not see that there is any difference between the two boxes which could justify my choosing one rather than the other; but if the difficulty is merely a technical one, I can imagine its being overcome—a sufficiently sensitive determination of the moments of inertia would in fact differentiate between a box with a beetle in it and a box with the same material as part of the lining—and then it is reasonable to mean one box rather than the other.

We can allow that the meaning of a word is given by its use, in a sufficiently wide sense of 'use', which includes not only actual use in clearly defined situations, but hypothetical use in conjectural ones. In particular, it includes not only Introduction Rules but Elimination Rules, and the use of sentences in argument and conjecture as well as in warranted assertions, and the relations between sentences and the words they are composed of. The meaning of the word 'or' is not given just by the introduction rules for $\vee$, but at the very least by the vel-elimination rule as well, which would license the argument by dilemma that Intuitionists refuse to accept. So too, the meaning of the word 'every' in Goldbach's conjecture is given not only by the proof of the conjecture, should there be one, but by it consequences. If I believe Goldbach's conjecture to be true, I shall not spend time programming a computer to search the even numbers for one that is not the sum of two primes, whereas if I had a non-constructive proof of its negation, I might well.

7.7 Selective Scepticism

Many of the arguments in favour of Intuitionism fail. Often underlying them there seems to be an exercise in selective scepticism. The Intuitionist professes himself unable to understand what the classical mathematician is saying, or doubtful about the validity of classical arguments, often insinuating that the classical mathematician is working with a wrong picture of what he is doing. It is difficult to defend oneself against such a charge. G.E. Moore was able to reduce his opponents to speechless confusion by asking them exactly what they meant when they used some common word. It is something we are characteristically unable to do. Satisfactory definitions are hard to articulate, and often cannot be given at all. It does not follow that we do not know the meaning of the words we are using. It is one thing to know how to use a word, quite another to know, and to be able to say, how that word is used. *Knowledge how* does not depend on *knowledge that*, but on the contrary, *knowledge that* often is only true in as much as it encapsulates an unarticulated *knowledge how*. Nothing follows from the fact that the classical mathematician is typically unable to give any account of the meaning of the basic logical connectives he uses. I may well be reduced to spluttering, as I attempt to explain the meaning of the word 'not' to someone who professes not to understand, but all that shows is that he is a better gamesman than I, not that I do not know the meaning of the word 'not'.

The best tactic is counter-attack. The classical mathematician cannot make the sceptic understand something he does not want to understand, nor acknowledge the force of an argument he does not want to acknowledge, and it is a lost labour to try and answer the sceptic's questions to the satisfaction of the sceptic. It is much more effective to turn the sceptic's own doubts against himself, and show that if we are to be really scrupulous in what we claim to understand and what arguments we are prepared to regard as acceptable, we must abandon not only the classical arguments the sceptic does not like, but the arguments he cherishes as well.

Intuitionists are vulnerable to the more extreme forms of unbelief practised by stricter finitists, who deny even the potential infinite, and accept not what could in principle be done but only what is in actual fact done. Alexander George calls them the

"Actualists".[33] Actualists do not know what Intuitionists mean by a proof, and are not willing to extrapolate from actual procedures to in principle effective procedures. Like the classical mathematician, the Intuitionist reckons that

$$(10^{(10^{(10+1)}+1)} + 1)$$

is either a prime number or else has factors. The Intuitionist thinks that this is a different case from Goldbach's conjecture, because although nobody has ever worked the sum out, it is clear that in principle someone could: we have an "effective procedure" for doing so. The classical mathematician is quite willing to concede this, and more also, but the Actualist is not. The number is not one that has ever been entered on a computer, and until it has been actually done, he is unwilling to concede that the calculation could be done.

In ordinary computer logic or recursive function theory we are able to characterize an effective procedure as one which will terminate in a finite number of steps. But what do we mean by 'finite'? The classical mathematician knows, the Intuitionist knows, but the Actualist cannot easily be told. The concept of finitude cannot be expressed in first-order logic, which is the logic computers can understand. It is only in second-order logic that we can express what it is to be finite, and hence what it is to be an effective procedure;[34] but second-order logic is not itself completely axiomatizable, and hence does not itself have an effective procedure for

[33] Alexander George, "The Conveyability of Intuitionism", *Journal of Philosophical Logic*, **17**, 1988, pp.133-156. For the "Actualist" position, see P. Bernays, "On Platonism in Mathematics", in P.Benacerraf and H.Putnam, *Philosophy of Mathematics: Selected Readings*, 2nd ed., Cambridge, 1983, pp.258-271; Crispin Wright, "Strict Finitism", *Synthese*, **51**, 1982, pp. 204, 208-210; and R.Gandy, "Limitations to Mathematical Knowledge", in D. van Dalen, D.Lascar, J.Smiley, eds., *Logic Colloquium '80*, North Holland, Amsterdam, 1982, pp.129-146.

[34] This is somewhat surprising, since in §5.2 we were able to express in first-order logic that there are infinitely many individuals possessing a given property. In order to show that a set is (Dedekind-)infinite, it is enough to specify *one* one–one function mapping the set into a proper subset of itself. But it is much more difficult either to tell or even to formulate the thesis that a set is *not* infinite: for we need to be sure that there does not exist

determining which of its well-formed formulae are true. Though the classical mathematician can be reasonably happy with second-order logic, the Intuitionist ought not to be—at least, if he is happy with second-order logic, he has little reason for cavilling at the classical mathematician's notion of truth as something that goes beyond our ability to give an effective procedure for recognising it. And if, *per contra*, he insists on disbelieving the classical mathematician, he is equally vulnerable himself to even more radical scepticism. He has started down a slippery slope that has Ultrafinitism at the bottom.

7.8 Ultrafinitism

We may ask what is wrong with the strongest version of Actualism, Ultrafinitism as we have called it. The answer is "Nothing—only it is not mathematics"; indeed, with the advent of computers it has become a highly relevant discipline. We are concerned to know not what might in principle could be done, but what our own actual computer does, and if we are wise, we are chary of hand-waving assurances about the potentialities of the hard- and soft-ware the salesman has on offer, and do not believe that anything can be done unless it actually has been done. This is good computational sense: but it is not mathematics. Although a computer engineer might refuse the question whether

$$(10^{(10^{(10+1)}+1)} + 1)$$

is composite or prime, on the grounds that it was too big to be entered into his computer, or the prime-number program was too long and required more memory than he had available, it would be difficult for him to refuse to answer the question whether it is even or odd. Unmanageable though

$$(10^{(10^{(10+1)}+1)} + 1)$$

may be in some respects, it is perfectly manageable in others. We do not have to pair off a sequence of strokes, and see whether at the

any such one–one function mapping it into a proper subset of itself, and that may require an extensive search to carry it through properly, and must require our being able to speak of *all* one–one functions even to formulate it. See further below, §13.5.

end we are left with one over, as the only means of telling whether a number is odd: we can tell at once from the decimal representation of it.[35] And just as we can tell at once from the way it is presented to us that

$$(10^{(10^{(10+1)}+1)} + 1)$$

is odd, we cannot be sure that we shall never have some simple algorithm that will determine in a few lines whether it is prime or not. Whether a number is manageable or not depends not just on the number but the question we are asking about it, and the means at our disposal for answering that question; that is to say, manageability is not something definite, given by the capacity of computer hardware, but depends also on our questions and purposes.

Ultrafinitism is a legitimate discipline, but not the only one. It is perfectly proper, as a Cartesian exercise, to avoid making any leaps, to be very sticky about allowing that something could be done in principle, and to insist that it actually be done in practice, before conceding that it can be done at all. Only the actual proves the possible beyond all shadow of doubt. *Ab esse valet consequentia posse*, and no other proof of possibility is as ungainsayable as actuality. The extreme actualist position is the most cautious possible, bar that of refusing ever to allow anything. He does admit that some things can be done, but only when they have actually been done.

But there is a price to be paid. By restricting the possible to the actual, the Actualist leaves out the characteristic feature of the possible, namely that more things are possible than actually happen. The realm of the possible is for him no other than the realm of the actual. He ceases to be able to understand the mathematician who is trying to tell him about what can be done, and is confined to the more boring bits of autobiography—just those moves which he, with the aid of his computer has, as it happens, actually accomplished.

Besides the restriction in scope, there is a restriction is knowledge. Knowledge is inherently risky. When I claim to know something, I am staking my credit on its being true, and could turn out to be wrong and to have to eat my words. Similarly, when I

[35] This point is convincingly made by Mark Steiner, *Mathematical Knowledge*, Cornell University Press, 1975, pp.16-19 and ch.1, §II, pp.41-70. See further below, §14.3.

try and find something out, I could get it wrong, and be misled in my enquiries into believing something false. The only way I can be sure of not having to eat my words is to say nothing, and the only way I can avoid being misled is not to try to find out. Goldbach made a conjecture. He could have been wrong: possibly being wrong is the price of possibly being right and propounding something that is true. In general mathematicians want to know the truth—so much so, that they are willing to run the risk of being wrong in their search for it. Of course, if they are wrong, that is bad, and they withdraw the claim they had made, and eat their words with red faces. But they are not prepared, in their efforts to avoid error, to avoid it by saying nothing at all, or by adopting the safest possible rules of inference. They are in the knowledge business, and knowledge characteristically outruns brass-bottomed certitude. In general—outside mathematics—when I lay claim to know, what I claim to know is logically more than the grounds I have for claiming it, as when on the evidence of your statement today, I say I know that you will be at the meeting tomorrow. In mathematics our rules of inference are of a different, non-empirical and very stringent, kind, but they too are intended to lead to truth and not mere warranted assertibility, and so are not necessarily the safest and most restrictive rules possible.

Ultrafinitism is not mathematics, because it is more concerned with safety than with truth. It remains a perfectly respectable exercise. We are not always seeking after truth, and may on occasion sacrifice knowledge for greater security. In the law courts the jury is told not to seek after truth but to decide whether the accused has been proved guilty beyond reasonable doubt. Engineers are required to build in large margins of safety into their calculations. As regards purely numerical computations, the Actualists are the accountants, who double-check every calculation, and make sure that the figures add up correctly, and never embark on speculations whether

$$(10^{(10^{(10+1)}+1)} + 1)$$

is prime or not. Mathematicians think accountancy is boring. Perhaps that is the reason why it is much better paid.[36]

[36] See further D.Dantzig, "Is 10*(10*10) a Finite Number?", *Dialectica*, **9**, 1956, pp.273-277; and Crispin Wright, "Strict Finitism", *Synthese*, **51**, 1982, pp.272-282.

7.9 Lax Finitism

The boundary drawn by Wang between manageable and unmanageable numbers is unsatisfactory from a theoretical point of view. For one thing, it depends on technology, and shifts as better hardware is developed. For another, and more importantly, it is vulnerable to new formalisms and new methods of proof, which may make what was previously unmanageable and unsurveyable amenable and perspicuous. Once we have the decimal notation, we have available certain methods of proof which make previously horrendous problems easy to solve. We can tell at once that

$$10^{(10^{(10+1)}+1)}$$

is even, and that

$$(10^{(10^{(10+1)}+1)} + 1)$$

is odd, as is also

$$(10^{(10^{(10+1)}+1)} - 1),$$

and that the first and last of these numbers are composite. Other notations are feasible, other methods of proof possible. We are therefore always being tempted to go beyond the limits set by our specification of acceptable methods, and help ourselves to new insights, which show us new truths we can see to be true though they cannot be established by the acceptable methods specified thus far. And thus the Lax Finitist, who confines himself to computable quotifier-free methods, is led to acknowledge a potential infinity of cases.

The temptation can be resisted. But it is difficult to give a satisfactory rationale for resistance. For the crucial concepts are only available to those who are not restricted to the specified methods. *Infinita sunt finita*: there are an infinite number of finite numbers, and the concept of a finite number can be defined only in second-order logic. If I understand finitude, I can resolve to use only finitary methods; but only if you and I both understand finitude, will you be able to take in why I restrict myself to some methods of proof, and refuse to allow others. But since to understand finitude involves being able to understand infinitude too, and to define the concept of a finite number requires the generous resources of second-order logic, it is implausible to give as a reason for restricting ourselves to finitary methods that we cannot understand any others. Either we restrict ourselves by a series of arbitrary *fiat*s, without giving any general characterization at all, or, in explaining what we are doing, we show that the reason why we are doing it cannot be that we could not understand anything more. We cannot say what the limits are, unless we are already able in our thinking to go beyond them.

7.10 Actualising Potentiality

There are, indeed, difficulties in the concept of infinity. But if we are to be consistent sceptics, we must abandon mathematics altogether for the different discipline of computer studies. We need, therefore, to return to infinity, and try and resolve the difficulties. Aristotle's account of the two incompatible concepts of infinity reveals three areas of ambiguity: the concept of a **limit**, which in turn involves the range of the **quotifiers**; and, thirdly, the **modalities** involved.

Plato and Aristotle found the infinitely small more difficult to doubt than the infinitely large,[37] and we can get a better grip on infinity if we think of lower limits and infinitesimals. However slowly I am going, I can always go slower, for so long as I am not standing still—in which case I am not *going* at all—I am moving with a speed greater than zero, and so there is another speed, also greater than zero but less then mine, at which I could be going; and, more generally, for any extension with a given magnitude there is another extension with a smaller, and so there is no extension of smallest magnitude. Yet a point, although not itself an extension, is a lower bound to all extensions.[38] It seems natural to extend the domain of the relation $<$ so that a point as well as an extension can be said to have an extension which is less than that of the given extension, and to extend the range of the quotifier (V) so that it ranges over points as well as extensions. If we signal these changes by writing $(\mathrm{V}s)$ whose universe of discourse is S, a proper superset of X, the universe of discourse of $(\mathrm{A}x)$, we can affirm both the converse of Aristotle's comparative thesis

$$(\mathrm{A}x)(\mathrm{V}y)(y < x)$$

and a counter-Anselmian superlative thesis

$$(\mathrm{V}s)(\mathrm{A}y)(s \leq y).$$

[37] *Physics*, III, 4, 203^a15, III, 6, 206^b27 and III, 7, 207^a25.

[38] See above, §2.5. See further below, §12.5.

To alter the universe of discourse over which a variable ranges, is often to alter the ontology. A state of rest is different from a state of motion: a point is different from an extension. We may have qualms about the ontological status of states of rest and points, but the two accounts of infinity cited by Aristotle are no longer *inconsistent* with each other. Once we allow the existence of point-like lower bounds which are themselves of a different status from the extensions they are bounding, we cannot in principle disallow the existence of upper bounds, likewise of a different status from the entities they are bounding. Instead of speeds, we might consider their reciprocal, slownesses, as we might call them, measured in hours per mile, or hours per inch. For each finite speed, there would be a corresponding finite slowness, and corresponding to the state of rest as the limit of minimum speed, there would be a state of infinite slowness as the limit of maximum slowness. A number greater than all the finite numbers could similarly be spoken of without inconsistency, so long as it was not itself a finite number.

Thus far we have shown that the Anselmian view of the infinite is not ruled out as being simply inconsistent with Aristotle's account, but we have not given any argument to justify Cantor's claim that any potential infinity presupposes a corresponding actual infinity.[39]

Aristotle elucidated modality in temporal terms. Although he is sometimes careful to distinguish logical from temporal priority, he often construes successor, *ἐφεξῆς* (*ephexes*), or, occasionally, *ἐφεξῆς ὕστερον* (*ephexes husteron*), in temporal terms.[40] In particular the Greek word *τέλειον* suggests a temporal succession of tasks accomplished. He draws attention to the close relation between *τὸ ὅλον* (*to holon*), the whole, and *τὸ τέλειον* (*to teleion*), the complete, and *τὸ τέλος* (*to telos*), the end; and the latter has even stronger temporal overtones in Greek than in English—Herodotus regularly uses *τελυτέω*, (*teleuteo*) to mean 'die'. There is a suggestion that an infinity is always becoming, "an ever increasing sequence" and therefore never complete, always potential, and therefore never actual. But that suggestion depends on the temporal metaphor, and otherwise is unconvincing. Although 'before' and 'after' do define

39 Michael Hallett, *Cantorian Set Theory and Limitation of Size*, Oxford, 1984, §1.1, p.7.

40 *Physics* V, 2, esp. 227^a4; *Metaphysics*, K, 12, esp. 1068^b35

an order, it is not the only order: there are many other orderings defined by other ordering relations, which do not need to be, and sometimes cannot be, explained in temporal terms, but which may be unbounded. Nor need modality be construed temporally. Although there are temporal modalities, there are non-temporal modalities too. We should, if we can, avoid importing temporal elements into our exegesis of infinity. Although a purely temporal account is inappropriate, an operational one, *pace* Plato,[41] is not. As with quotifiers, an exegesis in dialogue terms is illuminating, and shows how Aristotle's purely potential infinite leads on to an actual infinite.

The development of the infinitesimal calculus required an exact exegesis of the infinitely small. Although non-standard analysis has vindicated the infinitesimals that Newton and Leibniz used, it was the epsilon-delta account, in terms of a universal and existential quotifier, that really explained infinitesimals and made them respectable in the eyes of rigorist mathematicians. Nor was it enough that points should be the limits of smaller and smaller extensions; a tangent might touch a curve in two coincident points, but two coincident points would not determine the slope of a line. It was necessary, rather, to consider the slope of chords, and determine their limit as their end-points approached each other. In saying that the slope of, say, $y = x^2$ at the point $(1, 1)$ was 2, I am claiming that however close to 2 you require me to get, I can find values of $\mathrm{d}x$ and $\mathrm{d}y$ such that the chord between the points $(1, 1)$ and $(1 + \mathrm{d}x, 1 + \mathrm{d}y)$ has a slope within the limits you have specified. More generally, infinite sequences can be assigned limits, which can be explained in terms of quotifiers, which in turn can be explained in terms of a dialogue, where an existential quotifier represents a choice I am allowed to make, the best instance I can put forward to make my case, and a universal quotifier represents a choice my adversary is allowed to make, the worst instance he can cite against my case. But we share rationality. Just as in *Sorites* Arithmetic, it would be unreasonable to argue every case, so with limits of infinite sequences, the underlying rationale can be understood by both parties, and the general point will be conceded by the adversary, who is more concerned to acknowledge truth than to escape dialectical defeat by forcing an agnostic draw.

[41] See above, §2.7.

The transition from the potential to the actual is a feature of rationality, if modal operators are elucidated in terms of a game, played between two rational truth-seekers, who, being rational, stand back, and consider the meta-game. If meta-gamely the object of the game is the discovery of truth, then certain strategies and certain concessions will be rational for both parties. The Aristotelian potential infinity is actualised by being acknowledged by both sides as being always and necessarily available. Once my adversary realises that whatever natural number he chooses, I always can choose its successor, it is no longer just something I *can* do, but something he *must* accept. Seen from my point of view, the stress is on the 'can': seen from his point of view, the stress is on the 'always'. A perpetual possibility on my part is for everyone else an actual constraint. Whereas in the physical world there is an important difference between what can exist and what actually does exist, in discussing concepts and games and mathematical manoeuvres we are already discussing possibilities, and so what necessarily can be is. Once we have shown that we must be able to go one step further, we are in a position to envisage all those further steps actually taken.

7.11 All

Tensions still remain, arising from Cantor's two definitions of successor. Cantor modified the successor relation so that ω could be said to come next after all the finite numbers.[42] Thus by modifying the simple successor relation, where each successor has a unique predecessor, to $\succ$ which allows ω to be the successor of all its predecessors, we can affirm both

$$(\mathrm{A}x)(\mathrm{V}y)(y > x)$$

and

$$(\mathrm{V}w)(\mathrm{A}y)(w \succ y),$$

where w is a different sort of number, and $\succ$ is a slightly expanded version of $>$. But the two-pronged definition of $\succ$ raises difficulties. With derived sets it was possible not only to consider the set that would result from all the operations having been done, but then

[42] See above, §7.3.

to consider the derived set of that set. We cannot similarly consider slower and slower speeds, or successively smaller extensions. If the limit of an infinite sequence of nested extensions is a point, the point is a *terminus ne plus ultra*. We approach it as $n \to \infty$, but attach no further sense to $\infty + 1$. If pressed, we should simply say that $\infty + 1 = \infty$. You cannot move more slowly if you are at rest, and correspondingly your slowness cannot exceed infinity. For Cantor, however, although $1 + \omega = \omega$, $\omega + 1$ does not; ω has a successor, $\omega + 1$, which is turn is succeeded by $\omega + 2$, $\omega + 3$, ... *etc.* ... and then $\omega + \omega$, *ad infinitum.* Every ordinal has a successor, either the next after its predecessor, or the ordered set of all its predecessors. Granted the two-pronged nature of ordinal ordering, it is not surprising that the ordinal of all the ordinals is an inconsistent concept. The trouble lies in the word 'all'. There is a parallel between universality, expressed by the universal quotifier $(\mathrm{A}x)$, and necessity, $\Box$, and between necessity, $\Box$, and inference, $\vdash$.[43] If, as will be argued in the next chapter, mathematical inference cannot be completely formalised, but is inherently extensible and fuzzy-edged, it follows that the concept of necessity, some uses of 'all' and some applications of the universal quotifier are inherently extensible and fuzzy-edged too. In Cantor's development of transfinite arithmetic there is a tension between the word 'all', which has a superlative import, and captures the sense of the infinite being the "mostest", and the word 'next', which has a sense of "more-than-ness", implicit in Aristotle's exegesis of infinity, which seems to deny the possibility of the superlative. If both of these are combined in the definition of ordinal succession, there is an obvious and ineradicable inconsistency in attempting to comprehend that which is necessarily not bounded in one completed whole, the ordinal of all ordinals, or the cardinal number of all cardinal numbers. Whatever it is said to be cannot be the last word on the topic, and must in the nature of the case be overtaken by the further definition of further successors. We may modify the concept of successor to reach into the transfinite, and sometimes it may provide a useful characterization of mathematical operations. But we need to keep tight tabs on 'all' if we are to avoid inconsistency, and restrict its range to what has *already* been defined, and not allowed to range over ordinals or cardinals not yet precisely characterized.

[43] See below §12.8, §15.3.

In a dialogue neither side has, as a matter of right, the last word. Sometimes we can ascend to a meta-dialogue, and see that it would be counter-productive if one side persisted; but that cannot be taken for granted. For, as we shall see in the next chapter, proofs cannot be completely formalised, and however fully we have specified what it is to be a proof, some proofs can be devised which do not fall under our specification thus far, but nevertheless commend themselves to us as cogent. And so, when we engage in argument, our 'all's are only 'all, thus far's; and it is open to our respondent to go a step further, and make a new move, which we had not thought of before. We cannot talk of all cardinal numbers, all ordinal numbers, or all sets, because the very way we introduce these terms precludes our being able to exclude the possibility of further ones, not hitherto envisaged, and so not covered by our 'all' thus far.