Chapter 13*
Chastened Logicism?

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§13*.1 Logicism

The logicist programme is generally reckoned to have failed. Frege had hoped to found arithmetic on logic, but the intuitive theory of sets—the successors to his *Umfänge*—turned out to be inconsistent,[1] and when we abandoned intuitive set theory, and replaced it by some axiomatized version, it lost its innocent simplicity, and no longer seemed like logic. But once we distinguish transfinite arithmetic, which is much more a mathematical theory than an articulation of logical principles, from arguing with adjectives, which has a good claim to be just logic, the fundamental objection to the logicist programme is removed.

There remain other objections. Frege's own system was inconsistent because of his

$$\text{Axiom V } (\mathrm{A}F)(\mathrm{A}G)((\{\hat{x} : Fx\} = \{\hat{x} : Gx\}) \leftrightarrow (\mathrm{A}x)(Fx \leftrightarrow Gx)).$$

But George Boolos and Crispin Wright have re-examined Frege's original argument to see what can be salvaged, and have shown that Axiom V was not really needed.[2] There is room for argument

[1] See above, §12.2.

[2] G.Boolos, "The Consistency of Frege's *Foundations of Arithmetic*", in Judith Jarvis Thompson, ed., *On Being and Saying: Essays for Richard Cartwright*, Cambridge, Mass., 1987, pp.3-30; reprinted in W.D.Hart, ed., *The Philosophy of Mathematics*, Oxford, 1996, pp.185-202. Crispin Wright, *Frege's Conception of Numbers as Objects*, Aberdeen, 1983. In using his formulations of Hume's principle and Frege's Axiom V, I have transposed into my logical symbolism.

whether their alternative to Frege's Axiom V is a purely logical principle, but it is no longer a foregone conclusion that it is not.

Whitehead and Russell were trying to establish a Strong Logicist thesis, which claimed:[3]

1. The *concepts* of mathematics can be derived from logical concepts through explicit definitions. (compare constructivism)
2. The *theorems* of mathematics can be derived from logical axioms through purely logical deductions.

They succeeded in defining natural numbers and proving Peano's Postulates. But in order to do this, they needed axiomatic set theory, including the Axiom of Infinity and the Axiom of Choice, and a highly artificial and implausible Axiom of Reducibility to undo the damage done by their Theory of Types. In spite of great achievements, they had not carried through the Strong Logicist programme to a successful conclusion, and it became apparent that it was something they could not do. But it was only the Strong Logicist programme that had failed. We need to distinguish that from a chastened logicism, which, though less ambitious and less sharply defined, expresses important truths about mathematics.

Set theory was required, in the opinion of many of those who thought about the foundations of mathematics, in order to avoid recourse to second-order logic. Frege and Dedekind had availed themselves of second-order logic to generalise over concepts and over chains, and did not regard this as weakening their claim to be Logicists, but by the middle of the Twentieth Century it was widely held, largely at the behest of Quine, that logic, properly so called, should be restricted to "first-order logic", that is the logic of the sentential functors, $\neg$, $\rightarrow$, $\wedge$, $\vee$, *etc.*, together with identity and the existential and universal quotifiers restricted to quotifying only over individuals, and not anything else, such as qualities or quotities themselves. We need to consider what should be counted as logic, as well as the sense that should be given to the claim that mathematics is grounded in logic.

[3] Rudolf Carnap, "The Logicist Foundations of Mathematics", *Erkenntnis*, 1931; translated and reprinted in Paul Benacerraf and Hilary Putnam, eds., *The Philosophy of Mathematics*, 2nd ed., Cambridge, 1983, p.41.

§13*.2 What Is Logic?

In ordinary discourse the word 'logic' is used in a variety of senses, often taking its colour from what it is being contrasted with. Feminists are infuriated when men say that women are emotional rather than logical, and scientists talk of the logic of an experiment as opposed to the actual observations made. Historians often account for a statesman's actions in terms of the logic of the situation, as opposed to some personal predilection of his or the fortuitous result of happenstance. Philosophers used to talk of inductive logic at the same time as maintaining that logic was purely deductive. In these and many other contrasts, although the exact sense of the word 'logic' varies, the force of the contrast is the same: logic is topic-neutral, something that does not vary with personalities, empirical data, or the chance concatenation of events. It is, in the terminology of the Schoolmen, the "Universal Ordinary" studying patterns of inference that are not peculiar to any particular subject, but are common to all. It abstracts from particular instances, and considers only the general form.

But our concept of logic is under strain. Set against the drive for universality is the requirement of cogency. Hence the demand that logical arguments be so incontrovertible that it would be inconsistent to refuse to concede the conclusion, having admitted the premises. Logical arguments, on this showing, will be those that must be accepted, on pain of making oneself incommunicable with. They are valid, simply in virtue of the meaning of words, and without meaning there can be no communication. Granted that we want to be understood, there is maximal cogency in such inferences.

Nevertheless, it is unsatisfactory to characterize logic in terms of inconsistency alone. For one thing, there are many arguments valid in virtue of the meaning-rules of natural language, which it would be inconsistent to controvert, but which do not properly belong to the realm of logic: 'he is an uncle: so he is not an only child', 'today is Monday, so tomorrow is Tuesday' are maximally incontrovertible, but not general in their application. And, as we have seen, inconsistency is not the only sanction. A sceptic can deny an old-fashioned inductive argument if he wants, without being inconsistent or making himself unintelligible; rather, he denies himself all possibility of knowledge of general truths or of those not yet experienced, if he will not ever generalise or extrapolate from what he already knows to what he would like to know.

Faced with these difficulties in characterizing logic, philosophers can formalise, replacing non-deductive inferences by suitably stipulated axioms. Deductive inference, represented formally by the single turnstile, $\vdash$, then becomes a matter of abiding by the rules laid down, and the sanction for those who break the rules is simply that they are not playing the game.

This formal characterization of logic is attractive; we could reasonably hope to program a computer to act according to explicit rules concerned solely with the **syntax** of strings of symbols. But we still have to decide which symbols to have. Bureaucrats are very fond of acronyms, and use strings of letters rather than meaningful words, but it does make their discourse logical. Only rather few symbols deserve to be accounted logical symbols.

If we adopt the semantic approach, and seek to distil logic from various fields of discourse, as what is common to all valid argumentation, we reach a similar problem. We can form an idea of a "logical constant"[4] as that which is the same in all patterns of argument, but there is a suspicion of arbitrariness in deciding what exactly is a logical constant, and what not; is 'is identical with' a logical constant?

§13*.3 Boolean Plus

When we argue, we draw inferences because we disagree. If we did not disagree, we should not argue at all. As it is, being autonomous beings, we are capable of having our own ideas, each seeing things differently from others, so that when I give vent to my views, you need not accept them. You can say No, and I can say No to your No, thereby reasserting my original contention.[5]

But we do not merely contradict each other. We give reasons. Each appeals to considerations the other is likely to concede, and draws inferences from them to support his side of the dispute. I put forward a proposition, and from it infer the truth of another proposition: p **therefore** q, which we might symbolize as

$$p \;\Vdash q,$$

[4] The terms 'logical constant' and 'connective' are used indiscriminately by logicians. I shall use 'logical constant' in semantic contexts, 'connective' in syntactic ones, and 'functor' to cover both.

[5] See above, §7.5.

to avoid favouring either the syntactic $\vdash$ or the semantic $\models$. We do not just draw inferences, however; we may need to discuss them, in which case we may, as in the Deduction Theorem, consider the equivalent implication, arguing about the truth of the proposition $p \rightarrow q$, rather than the validity of the inference

$$p \;\; \Vdash q.$$

Equally, we may reverse the order of argument, and contend that q **because** p.[6]

Once we have some concept of negation and implication, we can define the other sentential connectives. Although $\neg$ renders the meaning of 'not' moderately well, $\rightarrow$ is significantly different from 'if . . , then . . ', as also $\wedge$ (or &) from 'and', while 'or' in ordinary English is ambiguous between the inclusive and exclusive sense, only the former being expressed by $\vee$. Nevertheless, they constitute a "regimented" version of the connectives used in ordinary language, and provide the logical constants for a topic-neutral formal logic.

The familiar sentential connectives are not the only ones. There are sixteen possible truth tables for binary connectives linking two variables, each of which can take either of two values, TRUE or FALSE. We often add $\leftrightarrow$ (or $\equiv$), for 'if and only if', and sometimes $\top$ for TRUE, and $\perp$ for FALSE. The latter can be used as a primitive instead of $\neg$. (Indeed, instead of having one of these together with $\rightarrow$ (or $\wedge$, or $\vee$), we can define both negation and all the binary sentential connectives in terms of just one, $|$ or $\downarrow$, non-conjunction or non-disjunction; but this is somewhat artificial.) Whatever primitives we choose, we have a Boolean Algebra, B_2, for propositional calculus, which goes a long way towards articulating the formal structure of argument in every field of discourse.

But propositional calculus is not the whole of logic. We often modify propositions. We can outline possibilities, or recognise necessities, or consider counter-factual conditionals. We can engage in fiction and tell stories, or consider obligations, or distinguish the conjectural from the well-established. We can hope, expect, fear, warn, promise or threaten about things to come, and can remember, ponder, relate, or explain, the past. All these activities

[6] See Gilbert Ryle, "If, So, Because", in M.Black, ed., *Philosophical Analysis*, Ithaca, N.Y., 1950.

have some propositional content—we can say what the content of our hopes, wishes, judgements or romances is, and pick out entailments and inconsistencies among them—but cannot be represented in terms of propositions alone.

Grammatically, modification of a proposition is often expressed in English by the use of an auxiliary verb, and in inflected languages by a change of mood or tense. But whatever the shift of tense or mood, and whether it is expressed by an auxiliary verb or some more complicated locution, such as 'it is possible that ...' or 'it was going to be the case that ...', the modified proposition stands in some relation to the original one, and is still something that can be agreed with or disagreed with, accepted or rejected, shared or repudiated. It is reasonable to regard the modified proposition as still being itself a proposition, and therefore to see the various modifiers as *unary operators*, or *unary functors*, which operate each on a single proposition to yield a single proposition.

The unary operator may stand for any one of a wide variety of modal or tense auxiliary verbs, or adverbs, or propositional phrases. It is not to be assumed in advance that every modal operator in ordinary speech can be expressed adequately in modal logic: but it is a useful exercise to see how far we can go in considering the formal possibilities enriching propositional calculus by a simple basic unary operator, and the constraints on the rules of inference and axioms it is reasonable to recognise. Having added to propositional calculus a unary operator, which we may symbolize as $\Box$, with the same formation rules as $\neg$, we need to consider possible rules of inference and axioms governing the use of $\Box$. There is a wide range of possibilities. But there are constraints: we must not have too many rules, or our operator will be degenerate, definable in terms of the ordinary sentential connectives, and our logic will be nothing more than ordinary propositional calculus; if we have too few, however, our operator will lose all contact with the connectives of ordinary logic, and our modified discourse will no longer be a logic at all. If we are to give sustained attention to a mode of discourse, words must have their ordinary meanings, and analytic propositions must hold as well within the modalised discourse as outside it; and therefore, as far as propositional calculus is concerned, tautologies must remain tautologies when modified. Since every tautology is a theorem of propositional calculus, and *vice versa*, we stipulate

If Γ is a theorem, so is $\Box\Gamma$, *i.e.*, If $\vdash \Gamma$ then $\vdash \Box\,\Gamma$.

This rule of inference is characteristic of all modal logics (logics with an additional unary modifying operator), and is known as the Rule of Necessitation.

The Rule of Necessitation ensures that logical theorems remain logical theorems when modalised, but does not by itself suffice to legitimise standard inferences in modalised discourse: it enables us to introduce $\vdash$ into a mode of discourse, but not to use it to make inferences within it. If we are to carry ordinary inferences over into modalised discourse we need the further rule

If $\Gamma \vdash \Delta$, then $\Box\,\Gamma \vdash \Box\,\Delta$.[7]

But in accordance with the tendency of modern logic to replace inferences by implications, the question whether modalised inferences are real inferences, that is whether $\Box\,(\Gamma \rightarrow \Delta), \Box\,\Gamma \;\vdash \Box\,\Delta$, becomes the question whether a modalised implication $\Box\,(\Gamma \rightarrow \Delta)$ yields a straightforward implication between the modalised parts of it, $\Box\,\Gamma$ and $\Box\,\Delta$. We therefore lay down as an essential axiom for modal logic

G $\quad\quad \Box\,(\Gamma \rightarrow \Delta) \rightarrow (\Box\,\Gamma \rightarrow \Box\,\Delta).$

The axiom G entitles us to infer $\Box\,\Delta$ from $\Box\,(\Gamma \rightarrow \Delta)$ and $\Box\,\Gamma$ in two steps of *Modus Ponens.*

The Rule of Necessitation together with the Axiom G ensures that modalised discourse is "**inferentially transparent**". Essentially what we require is that we should be able to make the same inferences within modalised discourse as in unmodalised discourse. If there is a good argument about kicking the ball—e.g. that in order to kick it, one must approach it, or that if one kicks it, the result will be that it moves—the same argument should hold within the context of obligatory kicking, alleged kicking, future kicking, or past kicking. Else modal discourse becomes inferentially opaque.

If modal discourse is to avoid logical chaos, it must be subject to the Rule of Necessitation and Axiom G, which together constitute the standard minimum system of modal logic: these, in effect, govern its interrelationships with $\vdash$ and $\rightarrow$. In order to place it as fully as possible in the context of propositional calculus, we need also to consider its interrelationship with $\neg$. As a first try, we might think that it would "commute" with negation, *i.e.*

$$\vdash \Box\,\neg p \leftrightarrow \neg\,\Box\,p,$$

[7] Aristotle, *Prior Analytics*, I, 15, $34^{a}22-24$.

but in that case the modal operator would become vacuous, so far as propositional calculus was concerned.

The non-theorem $\Box\neg p \leftrightarrow \neg\Box p$ consists of two conjuncts, $\Box\neg p \to \neg\Box p$ and $\neg\Box p \to \Box\neg p$. While we cannot have both, we can, and should hope to, have one; else our modal operator $\Box$ will have so little to do with the ordinary logical constants that there will scarce be a logic worth talking about. Although we could choose either, and the decision is, as we shall shortly see, in some sense arbitrary, we shall choose the former conjunct, $\Box\neg p \to \neg\Box p$. The reason is that we naturally want to secure a certain "modal consistency" for our operator. Consistency requires that no well-formed formula of the form $p \wedge \neg p$ can be a theorem. We naturally go further—though it *is* further—and lay down that not only is $p \wedge \neg p$ not a theorem, but that the negation of $p \wedge \neg p$ is a theorem, that is,

$$\vdash \neg\,(p \wedge \neg p).$$

This is a theorem of ordinary propositional calculus. In considering the relation between $\Box$ and $\neg$, we may reasonably look for a comparable stipulation, *viz.* $\vdash \neg(\Box p \wedge \Box\neg p)$, which is equivalent to

$$\vdash \Box\neg p \to \neg\Box p,$$

the former of the two conjuncts. This in turn is equivalent to

$$\vdash \Box p \to \neg\Box\neg p,$$

or, writing $\Diamond$ for $\neg\Box\neg$,

$$\vdash \Box p \to \Diamond p,$$

which is a characteristic thesis of modal logic, known as the axiom D.

Almost all interesting systems of modal logic have D as an axiom. It yields four out of the six possible interconnexions between $\Box$ and $\wedge$, $\vee$ and $\neg$, and we cannot add either of the others on pain of modal degeneracy. We can therefore argue for it as giving us as much, in the way of interconnexion between the modal operator $\Box$ and the connectives $\wedge, \vee$ and $\neg$ of propositional calculus, as we can hope to have. These rules for $\Box$ make it the **most highly structured non-trivial operator relative to the Boolean operators**.

There is a parallel with topology. Topology can be seen as an enrichment of the Boolean algebra of sets. We can, as we saw in Chapter Ten,[8] axiomatize topology in terms of a unary operator, the interior operator, which is a function from sets into sets, just as $\Box$ is a function from propositions into propositions; and then we find a close parallel between the axioms of topology and those of a particular modal logic, **S4**. The fact that in standard expositions of topology we pick out a family of sets—the open sets—which are distinguished by certain infinitistic properties suggests an approach to modal logic in which the often-remarked parallel between necessity and the universal quotifier, itself often regarded as an infinite conjunction (and between possibility and the existential quotifier, regarded as an infinite disjunction) is probed further.

§13*.4 Iterated Modalities

Although the system that contains D goes as far as possible in relating the modal operator with the sentential connectives of propositional calculus, it leaves other questions unanswered. It tells us nothing of the relation between modalised and unmodalised discourse, nor of any relations between iterated modal operators, which may be all of essentially the same sort, but may also be differentiated from one another. Axioms giving rules for such relations can be laid down, giving rise to different logics, according to what rules are adopted. The Aristotelian axiom T, $\vdash \sqcap p \rightarrow p$, 'what must be, is', specifies a relation between modalised and unmodalised discourse, and holds in most modal logics, though not in those concerned with ethics and what we ought to do: the axioms 4 and 5, which are typical of the systems **S4** and **S5**, specify relations between iterated modal operators of the same type; the axiom B, which is typical of the system **B**, does both; and the quotifiers can be usefully viewed as modal operators differentiated from one another by virtue of the variables they bind.

These different modal logics are intricate, and hard to make sense of, if we consider them only from a syntactical point of view. Kripke provided a semantics in terms of "possible worlds" which casts light on each of them, and their relations with one another. He considered each unmodalised logic as having its universe of discourse in a "possible world" that might, or might not, be related to other possible worlds by an "accessibility relation". The semantic

[8] §10.2.

definition of $\sqcap$ was given by the stipulation that $\sqcap p$ should be true in a possible world if and only if p was true in every possible world accessible from it. It is moderately easy to see, then, that where the accessibility relation is reflexive, if $\sqcap p$ is true, p must be true also, so that T holds in that modal logic. It further follows that T holds only in such cases. Similarly, the axiom 4, characteristic of **S4**, which lays down that $\vdash \sqcap p \rightarrow \sqcap \sqcap p$, will hold if and only if the accessibility relation is transitive. It is more tricky to assure oneself that the "Brouwerian" system **B** which has as its axiom B, $\sqcup \sqcap p \rightarrow p$, is associated with a symmetric accessibility relation, and that **S5** is associated with an equivalence accessibility relation, but once these points are taken, we understand why it is that the system obtained by adding B to **S4** turns out to be **S5**.

The accessibility relation of **S4** is transitive, but not symmetric.[9] Relations that are not symmetric have converses that are different. For each modal logic which does not have B as an axiom or theorem, there will be another modal logic with the converse accessibility relation. We could use an "inverse" modal operator, $\sqcap^{-1}$, but that usage has not been adopted. Prior's Tense Logic is the most interesting logic based on an asymmetric accessibility relation. Instead of $\sqcap$, and $\sqcap^{-1}$, he has H for 'it has always been the case that', and G for 'it is always going to be the case that'; likewise he has P for 'it was at sometime the case that', and F for 'it will at sometime be the case that', instead of $\sqcup$, and $\sqcup^{-1}$.[10]

Tense logic helps us to expand our conception of possible worlds. Normally we think of only a limited number of them, sharply separated. But the accessibility relation could give rise to a dense, or even a continuous, order, with infinitely many possible worlds, nestled close together. Indeed, if we contemplate the many-worlds interpretation of quantum mechanics, with the world bifurcating as each probability collapses into one certainty or another, there are $2^{(2^{\aleph_0})}$ possible worlds already. Even if we are reluctant to multiply

[9] **S4** is normally taken to have T as one of its axioms, in which case the accessibility relation will be antisymmetric, giving rise to a quasi-ordering (see §9.7): but T is not a thesis of Prior's tense logics, and there the accessibility relation is irreflexive, and gives rise to a strict ordering.

[10] For fuller and illuminating discussion of Kripke semantics, see Brian F. Chellas, *Modal Logic*, Cambridge, 1980, esp. chs. 1 and 3; and G.E.Hughes and M.J.Cresswell, *Introduction to Modal Logic*, London, 1968, ch.4.

universes to that extent, we can express in modal terms the requirement that the accessibility relation be not only transitive, but dense. In Prior's terminology the former is secured by stipulating that $FFp \to Fp$ and the latter by stipulating that $Fp \to FFp$, or in standard modal terms $\vdash \sqcap p \to \sqcap \sqcap p$ secures transitivity of the accessibility relation, and $\vdash \sqcap \sqcap p \to \sqcap p$ secures density of the accessibility relation. Somewhat surprisingly, we can even secure continuity. Prior cites an axiom of Cocchiarella's which we would write, using his operators, $(Gp \to HG(Gp \to PGp)) \to HGp$[11] or, in modal terms,

$$\vdash (\sqcap p \to \sqcap^{-1}(\sqcap p \to \sqcup^{-1} \sqcap p)) \to \sqcap^{-1} \sqcap p.$$

We can also characterize the global structure of accessibility relations in modal terms. **S4** has a transitive accessibility relation; **S4.2** has a transitive accessibility relation with a lattice structure; **S4.3** has a linear transitive accessibility relation. Tense logic is unlike most modal logics in which T (that is, $\sqcap p \to p$) holds, since it needs not to assimilate the present to either the future or the past. D follows from T, and is unremarkable in those systems. But if we have D without T, then the accessibility relation is not reflexive (or T would hold) and the effect of D is to require that it be serial. If a relation is serial, it has no maximum, but it does not follow that it has no minimum: the successor relation, for example, will always yield a greater natural number, but does not rule out there being a least. We see then that D may hold without its mirror image, $\sqcap^{-1}p \to \sqcup^{-1}p$, holding. Prior is thus able to express in his tense logic the *separate* possibilities of there being a beginning and an end to time.

Thus far the iterated modalities have all been of the same basic type. But, clearly, there could be more than one type. Often they will be entirely independent of one another. 'I know that' and 'It is reported in *The Times* that' do not mesh for non-*Times* readers. The only cases that will give rise to a logic developed from what we already have, is when the distinct operators are not all acting globally on well-formed formulae of propositional calculus, but selectively, some on one, and some on other, parts of a well-formed formula with Boolean sentential connectives. We shall need

[11] A.N. Prior, *Past, Present and Future*, Oxford, 1967, p.72. Cited above, §9.7.

to flag the operators and those terms they operate on. Instead of bare $\sqcap$ and $\sqcup$, we shall distinguish them by subscripts, $\sqcap_x$ and $\sqcup_y$, *etc.*, and shall need then to indicate the scope of their operation, and thus ultimately to assign subscripts to those primitive terms susceptible to their modalising influence. Thus the general form of such a well-formed formula will be something like
$\sqcap_x(\sqcup_y p_x \to (q_{x,y} \wedge r))$,
where p is susceptible only to $\sqcap_x$, q is susceptible to $\sqcap_x$ and to $\sqcup_y$, and r is susceptible to neither.

At this stage it becomes obvious that what we are doing is to re-invent the quotifiers: $\sqcap_x$ is the universal quotifier, standardly expressed by $(\forall x)$ and in this book by $(\mathrm{A}x)$; $\sqcup_y$ is the existential quotifier, standardly expressed by $(\exists y)$ and in this book by $(\vee y)$. Similarly we could write $p(x)$, $q(x, y)$, r, but actually write $F(x)$, $G(x, y)$, or, saving brackets, Fx, Gxy. The rule of necessitation becomes the rule of generalisation:

$$\text{If} \vdash \Gamma \text{ then} \vdash (\mathrm{A}x)\Gamma$$

The axiom G becomes

$$(\mathrm{A}x)(Fx \to Gx) \to ((\mathrm{A}x)Fx \to (\mathrm{A}x)Gx).$$

The thesis D, $(\mathrm{A}x)Fx \to (\vee x)Fx$ holds in all non-empty universes, and follows from the equivalent, postulated in most systems, of T, $(\mathrm{A}x)Fx \to Fx$ (or, in some systems, $(\mathrm{A}x)Fx \to Fa$, where a is an individual name).

Besides the well-known cases where we have quantifiers binding different variables, we can consider modal predicate calculus where a modal operator interacts with a quantifier. It is fairly easy to prove

$$\vdash \sqcap (\mathrm{A}x)Fx \to (\mathrm{A}x) \sqcap Fx,$$

but the converse implication,

$$(\mathrm{A}x) \sqcap Fx \to \sqcap (\mathrm{A}x)Fx,$$

known as the Barcan formula, is not a thesis of the predicate version of **T** nor of **S4** (though it is of mainline predicate versions of **S5**).[12] The universal quantifier thus differs from simple conjunction, since

$$\vdash (\sqcap p \wedge \sqcap q) \to \sqcap (p \wedge q),$$

[12] See, G.E.Hughes and M.J.Cresswell, *An Introduction to Modal Logic*, London, 1968, pp.142-144 and 178-182.

which is again reminiscent of the characterization of topology in terms of open sets, where any finite intersection of open sets is itself open, but an infinite intersection of open sets may not be open itself.[13]

We may choose not to admit polyadic predicates, $q(x, y)$, or $G(x, y)$, and to confine our extension of logic to monadic predicates, $p(x)$, or $F(x)$, alone. In that case we have the monadic predicate calculus, which, like propositional calculus, is (two-way) *decidable*. Given any well-formed formula, we can tell in a finite number of steps, whether *or not* it is a theorem. Monadic predicate calculus, however, is extremely restricted, and it is natural to extend logic to include polyadic predicate calculus (generally known as predicate calculus *simpliciter*), which allows two-place predicates (and more-than-two-place predicates) expressing relations. If two-place predicates are allowed, we are allowing the logic of relations[14] as part of logic, and can define one two-place predicate with the properties of the Successor, and hence formulate Peano's postulates, and wonder whether the implication from Peano's postulates to the Gödelian sentence is a theorem or not, and hence conclude that the polyadic predicate calculus is not two-way decidable,[15] though nonetheless generally accepted as part of logic.

We thus see how modal logic and quotificational logic are natural developments from a Boolean system we are led to adopt by the importance of inference and negation. The logic of relations plays a crucial role in displaying the interconnexions between different modal logics. It would be difficult on this showing to draw any profound distinctions, or claim that some, but not others, of these studies were to be properly accorded the title of "logic".

Quine disagrees. He urges us to confine the term to first-order logic, that is to say, first-order predicate calculus with identity, on the grounds that only first-order logical theories display "Law and Order", and he regards modal logic as belonging with witchcraft and superstition.[16] Quine's exclusion of modal and tense logic seems irrational, but there are arguments for insisting that quotification should be only over individual, and not over predicate, variables. Predicates are ontologically more suspect than individuals, and have a different logic. More immediately, second-order logic lacks the **completeness** that first-order logic has, and is liable to give rise to paradox and inconsistency.

[13] See above, §10.2.

[14] See above, ch.9.

[15] See above, §8.8.

[16] W.V.Quine, *Word and Object*, Cambridge, Mass., 1960, p.242.

§13*.5 Completeness

Whereas first-order logic is complete, second-order logic is not.[17] Many well-formed formulae can be formulated in it which are independent of the axioms but which are valid, that is, true under all natural interpretations.

It is often seen as a great merit of first-order logic that it is complete, and thus reconciles the syntactic and semantic approaches,[18] so that we need not worry about distinguishing them, so long as we stick to first-order logic, and can use $\vdash$ and $\models$ more or less interchangeably. On the syntactic approach we define the sentential connectives, $\wedge$, $\vee$, $\neg$, $\rightarrow$, corresponding to regimented versions of our familiar 'and', 'or', 'not', 'if', by their inference patterns; on the semantic approach we define the analogous logical constants by truth-tables. It is moderately easy thus far to see that every syntactic entailment is valid semantically:

$$\text{If} \quad P \vdash Q \quad \text{then} \quad P \models Q,$$

and *vice versa*

$$\text{If} \quad P \models Q \quad \text{then} \quad P \vdash Q.$$

In particular, analytic propositions are tautologies, and *vice versa*:

$$\text{If} \quad \vdash Q \quad \text{then} \quad \models Q,$$

and

$$\text{If} \quad \models Q \quad \text{then} \quad \vdash Q,$$

In virtue of the former we say that the propositional calculus is **sound**, and in virtue of the latter we say that the propositional calculus is **complete**.[19]

If we add the quotifiers, $(\mathrm{A}x)$ and $(\vee x)$, we have to be careful in specifying their inference-patterns, and especially, what exactly

[17] Gödel proved the completeness of first-order logic, using, it should be noted, infinitistic methods (cf. §15.7): Henkin proved *a* completeness theorem for second-order logic, which depends on admitting, besides the primary interpretations, some further secondary ones: but the secondary interpretations are unnatural, and not at all what we should expect on the normal semantic approach. See further below, §15.9.

[18] See above, §3.5.

[19] See above, §3.5.

should count as models. Provided we quotify only over individual variables, we can establish both soundness and completeness, and this still holds good if we extend the simple first-order predicate calculus, by adding, with appropriate rules, the further two-place predicate, =, representing identity. That is to say, if P, Q are any well-formed formulae of first-order logic,

$$\text{If} \quad P \vdash Q \quad \text{then} \quad P \models Q,$$

and

$$\text{If} \quad P \models Q \quad \text{then} \quad P \vdash Q;$$

or, more succinctly,

$$\vdash Q \quad \text{if and only if} \quad \models Q.$$

The completeness of first-order logic not only reconciles the syntactic and semantic approaches, but, more sophisticatedly, is seen as a token of the adequacy of our axiomatization: it shows that we have captured in our syntactic notion of theoremhood the desirable semantic property of being true under every interpretation. From this it follows that, since syntactic proof-procedures can be "mechanized", a computer could be programmed to churn out every theorem, and thus, thanks to the completeness theorem, every valid well-formed formula of first-order logic. First-order logic is thus "computer-friendly", whereas second-order logic is not, since there is no corresponding way of producing every one of its valid well-formed formulae. In first-order logic we have a positive (although not a negative[20]) test for any particular well-formed formula's being true under all interpretations. It is very tedious, but in the long run it will work: we simply program a computer to produce every theorem in a systematic way, and check whether or not it is identical with the well-formed formula in question; if it is, then that well-formed formula is valid, is true under all interpretations; if it is not, the computer grinds on and produces the next theorem. In second-order logic we cannot do this. Although we can still program a computer to generate every *theorem* in a systematic way, so that if a well-formed formula *is* a theorem, it will turn up sooner or later in the list the computer spews out,

[20] See above, §8.8.

not every valid well-formed formula is a theorem.[21] That is, there are some well-formed formulae which are true under all natural interpretations, and so reasonable candidates for being accounted logical truths, but are not theorems according to the axioms and rules of inference of the system, and so could not be discovered or identified by any computer search.

Most logicians have regarded this as a defect of second-order logic. But the argument is two-edged, and can be seen as showing not the adequacy of the axiomatization, but the limits of the formalisation. First-order logic is complete, but only in the way eunuchs are. Eunuchs are able to do everything they want to do, but cannot want to do what other men want to do. First-order logic can prove every well-formed formula that is expressible in first-order logic and is true under all interpretations, but cannot express many propositions that other logics can. It secures complete success in its ability to prove propositions by cutting down its ability to formulate them. We cannot say things in first-order logic we might naturally want to say—for example that a set of well-formed formulae is satisfiable in any finite domain, or that an ordering is well-ordered—and so, with its expressive ability thus truncated, it is not surprising that it can prove those relatively few well-formed formulae it can formulate.

The incompleteness of second-order logic can thus be regarded in another light, showing it to be more juicy than first-order logic, and thus capable of grounding more substantial truths. Certainly, in second-order logic we sometimes seem to be led to propound further truths, such as the Axiom of Choice, or various versions of the Axiom of Infinity, which were not in any sense already implicit in the axioms, but suggest themselves as being additional axioms.[22]

[21] George S. Boolos and Richard C. Jeffrey, *Computability and Logic*, Cambridge, 2nd ed., 1980, ch.16.

[22] Alonzo Church, *Introduction to Mathematical Logic*, Princeton, 1956, ch.V, §54, p.315.

§13*.6 Paradox

It is not only completeness that has told in favour of first-order logic. Consistency is in issue as well. Second-order logic is suspected of inconsistency. It is thought to be equivalent to set theory—to quotify over qualities is very like quotifying over sets—and it is possible to reproduce the "heterological paradox" in unbridled second-order logic: for if we can **both** quotify over predicates **and** allow predicate variables to occupy the same positions as individual variables, then we can consider those predicates that are self-applicable—those for which it is true that $F(F)$—and those that are not—those for which it is true that $\neg F(F)$—and define the latter as a predicate of predicates, that is define

$$H(F) \quad \text{iff} \quad \neg F(F);$$

We then ask whether the predicate H, thus defined, is, or is not, predicable of itself. Either answer leads to a contradiction: if $H(H)$ then $\neg H(H)$; if $\neg H(H)$, then the defining condition is satisfied, and so it is true that $H(H)$.

The argument stinks. Expressed in the unfamiliar terminology of sets, we may be led to accept the definition of extraordinary sets which are members of themselves,[23] but the concept of a "heterological property" of not being self-applicable is hard to take. Nor does the formalism of the predicate calculus encourage us to stifle our objections. Predicate variables are different from individual variables, and any definition involving $F(F)$ is manifestly ill-formed. We may quotify over qualities, but that does not make them the same as individual substances. Sometimes, no doubt, we can refer to particular qualities and talk about them, but that is not to say that they are exactly the same as the individuals that are normally talked about as possessing them.

The standard formalisations of predicate calculus are crude. They draw only one distinction, that between individual terms and predicate terms, and blur all other distinctions. Moreover, the individual variables are down-graded to doing hardly any logical work, being little more than glorified logical blanks,[24] like the x in $\int F(x)\,dx$. All the work is done by the predicates, and, except for sometimes requiring that the universe of discourse be non-empty,

[23] See above, §12.2.

[24] See W.V. Quine, *Methods of Logic*, Harvard, 1951, §12.

no consideration is given to range or the logical shape of the individual variable. A logic that permits us to contrapose 'all ravens are black' into 'all non-black things are non-ravens' is not a logic that is sensitive to type-distinctions. We need, as we have seen,[25] to take individual variables seriously. We need to register in our formalism the fact that not only are predicate variables ineligible to occupy the positions reserved for individual variables, but often one individual variable cannot occupy the place of another. It makes perfectly good sense to ask whether the square of a cricket pitch is green, but not whether the square of 22 is. Ryle pointed out that there were many "category distinctions" in our conceptual scheme as expressed in ordinary language,[26] and that many philosophical errors arose from neglecting them, but the distinctions are more fluid and difficult to discern than he made out. In some senses, geographical and institutional, the University of Oxford is of a different type from the Oxford Colleges, but in other respects they are on the same footing—I am perfectly happy to accept cheques from both. Instead of an absolute, possibly ontological distinction between one categorial type and another, there are many different distinctions depending on context, which any formal system will find it difficult to register.

Equally with predicates there are many type distinctions we observe in our ordinary thought: '   is green' can be predicated of cricket squares, '   is generous' cannot. When we quotify over predicates, there is an implicit restriction of the range of quotification to predicates of the appropriate type. If we say someone has all the properties of Napoleon, we do not consider whether he might be a perfect square, and on discovering that Napoleon was not a perfect square conclude that neither is the person we are talking about: being a perfect square is a property that Napoleon neither possesses nor does not possess. Unclarity on this score does not greatly matter so long as we are considering only the positive predicates, since if we were mistakenly to ask the improper question whether or not Napoleon possessed the property of being a perfect square, we should return a negative answer and not ascribe perfect

[25] §4.3, §12.3 and §12.8.

[26] Gilbert Ryle, "Categories", *Proceedings of the Aristotelian Society*, **49**, 1938-9; reprinted in *Logic and Language*, Series II, ed. A.G.N.Flew, Oxford, 1953, pp.65-81; and Gilbert Ryle, *The Concept of Mind*, London, 1949, pp.16-18.

squaredom to the man we were talking about. But once we are dealing with negation, and consider someone who does not have all the properties of Napoleon, we may be tempted to ascribe to him non-perfect-squaredom and our troubles begin. Individual terms have a different logic, less liable to lead us astray. Whereas predicates, like propositions, can be negated and disjoined, it makes no sense except in some wider, specified context, to talk of a non-man, or a man-or-irrational-number.

We make use of type distinctions in our ordinary thinking and speaking, but characteristically articulate them when some particular inference is in question instead of seeking to lay down general rules in advance. The formation rules we already have are enough to rule out 'heterological' as ill-formed. No formal system is likely to be sensitive to all the nuances of ordinary language, and rule out every sort of nonsense, but that, although a barrier to the complete formalisation of logic, is not a conclusive objection to every quotification over qualities or quotities. We are right to be as reluctant to allow a predicate's either being or not being applicable to itself, as we are to a set's either being or not being a member of itself, but we have to confess that we have not yet formulated an adequate formal theory of predicates. This is in line with the general incompletely formalisable nature of second-order logic, but it is a confession, not a boast.

§13*.7 Second-order Logic

Second-order logic differs from first-order logic in a number of other important ways:[27] they are listed, together with those already discussed, in the table on the next page.

Although the differences are real, they hardly justify excluding second-order logic from being part of logic. For one thing, many of the differences, though alleged to favour first-order logic, actually tell the other way; and, anyway, there is no compelling argument for picking on those differences as decisive, in comparison with others, not held to be decisive, and in the face of continuing similarities between second-order logic and other systems accepted without question.

[27] See G. Boolos and R. Jeffrey, *Computability and Logic*, 2nd ed., Cambridge, 1980, ch.18. See also C.D. Parsons, "Objects and Logic", *Monist*, **65**, 1982, pp.498-505; and J.B. Moss review in *British Journal for the Philosophy of Science*, **36**, 1985, pp. 437-455.

	First-order Logic	**Second-order Logic**
1.	Complete *i.e.* $\models A \Longrightarrow \vdash A$	Only *Henkin*-complete We cannot secure that if A is true under all *principal* interpretations, then $\vdash A$
2.	So if $\models A$ then we can, given enough time, prove $\models A$	No effective positive test for validity
3.	Compact	Not compact
4.	Löwenheim-Skolem theorem holds	Löwenheim-Skolem theorem does not hold
5.	Peano Arithmetic not monomorphic	Peano Arithmetic monomorphic
6.	Some well-formed formulae true, but not provable	All arithmetical truths provable from Peano's postulates

The first two of the differences listed in the table have already been shown to be really to the advantage of second-order logic.[28] Compactness—the feature that a set of well-formed formulae is consistent provided every finite subset is—has, like completeness, been taken as a virtue. It goes with first-order logic's being finitely axiomatizable, and such that a computer can be programmed to do it. First-order logic is, essentially, a finitistic calculus, in which, therefore, every valid well-formed formula that can be expressed in its formalism can be proved in a finite number of steps. But compactness is really a demerit. It trades on the finiteness of a proof-sequence, and is counter-intuitive, leading once again to there being non-standard models of arithmetic, though by a different route from that of Gödel's theorem.[29] Its finitistic features are purchased

[28] In §13.5.

[29] See above, §6.3.

at the price of our never being able to specify completely what we are talking about (4 & 5). The Löwenheim–Skolem theorem is really a liability rather than an asset, showing, as it does, that in first-order theories we cannot in general specify our models, or even their cardinality.[30] We have already explored the awkwardness of Peano's postulates not being monomorphic in first-order logic, and Gödel's incompleteness theorem is a notorious embarrassment. Even the concept of identity cannot be defined in ordinary first-order predicate calculus, but has to be characterized by extra *ad hoc* axioms. Second-order predicate calculus, by contrast, is able to define identity without special extra axioms, and is able to specify exactly what we are talking about.

Second-order logic thus seems a natural further development of logic from its Boolean core. It is hard to justify our jibbing at this particular step. If it is the incompleteness of second-order logic that debars it from being a proper logic, should not the undecidability of first-order logic tell equally against that? Once we go beyond propositional calculus, we are led to modal logics with iterated modalities, and to quotificational logic with intertwined quotifiers, and the logic of relations. Though the next step *is* a further step, with considerable further implications, it is one we should take. If we have free predicate variables, it would be unreasonable not to be able to quotify over them; otherwise, we should be in the position of being able to specify that a universe of discourse was infinite, but not that it was finite.

There are trade-offs. First-order logic is finitely axiomatizable, but cannot express finitude: second-order logic can express finitude, but is not finitely axiomatizable, and our axiomatization is always liable to turn out to be incomplete and inadequate for our purposes. If we formalise second-order logic incautiously and ill-advisedly, we run into paradox and inconsistency, whereas first-order logic has simple formation rules which secure us against any danger of meaninglessness—but at the cost of often being unable to express our meaning at all.

If second-order logic is admitted as being part of logic, important consequences follow. The fact that it is not complete means that there are well-formed formulae which are true under all reasonable interpretations, but cannot be proved from the axioms by means of the rules of inference. Truth, once again, outruns formal provability.[31] And, more important for our present purpose, cannot be explicated in terms of analyticity.

[30] See above, §12.5.

[31] See above, §8.11.

§13*.8 Analytic and *A Priori* Truth

Many philosophers have found it difficult to give an account of mathematical truth, because they have taken it for granted that mathematical truth must be either analytic or synthetic, and if synthetic empirical, and if analytic empty. Since it is difficult to take the Protagorean view that it is empirical, they have been forced to the view that the whole of mathematics is a vast set of tautologies.[32] Traditional Logicism supports that view. If all mathematical truths are, ultimately, truths of first-order logic, then they are theorems, and anyone who denies them is guilty of inconsistency. But once we give up the identification of logic with first-order logic, we are no longer impelled to downgrade mathematical truth to mere tautology.

Whereas first-order logic is axiomatizable and complete, and it can be reasonably maintained that its theorems are analytic, second-order logic is not completely axiomatizable, and seems to generate truths which are not mere reformulation of the original axioms. They are not analytic. They do not just tell us what was in the original axioms and rules of inference of second-order logic. The Axiom of Choice cannot even be formulated in first-order logic. It has some similarity to Peano's fifth postulate—and Poincaré argued that mathematics was not analytic just because it employed argument by recursion.[33] The reasoning which impels us to generalise from *Sorites* Arithmetic to Peano Arithmetic, impels us also to accept the Axiom of Choice as true. If so, it is in some sense synthetic—it is not a deductive consequence of our explicit axioms of set theory—but it is not *a posteriori* in Kant's sense. The Continuum Hypothesis is in the same case, and arguably the axioms of infinity. The axioms of geometry, again, are not analytic, and though the Riemannian geometry of the General Theory may be adopted *a posteriori* on the basis of empirical evidence, the arguments adduced in Chapter Two were altogether different. So, too, are the arguments for extending the natural numbers to the integers, and the integers to the rationals, the rationals to the reals, and the reals to the complex numbers. Such truths, if they are

[32] See, for example, A.J.Ayer, *Language, Truth and Logic*, London, 1936, ch.4; or Bertrand Russell, *History of Western Philosophy*, London, 1946, p.860, pbk., London, 1991, p.786: mathematical knowledge is all of the same nature as the "great truth" that there are three feet in a yard.

[33] See above, §6.5.

truths, and if we do know them, are known not on the strength of sense experience, and are in that sense known *a priori*. They are, as regards their logical status, synthetic propositions, and, as regards their epistemological status, known *a priori*, if known at all. How such synthetic *a priori* propositions are possible remains an open question, but once we launch out from the restricted compass of first-order logic, it is a question we not only have to face, but can hope to answer.

Non-Analytic Propositions

1. Peano's Fifth Postulate—it was on account of its employing argument by recursion that Poincaré argued that mathematics was not analytic. (See above, ch.6, §6.5.)
2. Various versions of the Axiom of Infinity.
3. The Axiom of Choice.
4. The Continuum Hypotheses, together with various other, more *recherché* axioms suggested for set theory.
5. The axiomatic extension of the natural numbers to include the negative integers, the rational numbers, the reals, and complex numbers.
6. The axioms of different geometries.

The justification of these is not uniform. It involves in part a reassessment of traditional ideas of what it is to know something, and in part a closer examination of the interpretations, principal or secondary, under which valid well-formed formulae come out as true.