Chapter 9A
What is Logic?

Contents

§9A.1 First-order Logic

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Thus far the logic out of which mathematics has developed has been First-order Predicate Calculus with Identity, that is the logic of the sentential functors, ¬, →, ∧, ∨, *etc.*, together with identity and the existential and universal quotifiers restricted to quotifying only over individuals, and not anything else, such as qualities or quotities themselves. Some philosophers—among them Quine—have held that this, First-order Logic, as it is often called, constitutes the whole of logic. But that is a mistake. It leaves out Second-order Logic, which we need if we are to characterize the natural numbers precisely, and pays scant attention to the logic of relations, especially transitive relations, which is the key to much of modern mathematics. Quine's argument for restricting logic to First-order Logice was based on the grounds that only First-order logical theories display "Law and Order" and himself regards modal logic as belonging with witchcraft and superstition.[1] Predicates are ontologically more suspect than individuals, and have a different logic, which is liable to give rise to paradox and inconsistency. Moreover, Second-order Logic lacks the **completeness** that First-order Logice has, which provides a pleasing parallel between syntactic and semantic notions, and argues for the analyticity of deductive logic.

[1] W.V.Quine, *Word and Object*, Cambridge, Mass., 1960, p.242.

§9A.2 Paradox

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Predicates have a different logic from individuals. They can be negated, conjoined and disjoined: it makes sense to say of something that it is not-white, that it is black-and-white, that it is red-or-yellow, but not to talk of a non-raven, a raven-and-woodcock, or a raven-or-woodcock. And although, admittedly, we can talk of all the colours of the rainbow, or of two things having some properties in common, such talk is liable to give rise to paradox and inconsistency. Second-order Logic is thought to be equivalent to set theory—to quotify over qualities is very like quotifying over sets—and it is possible to reproduce the "heterological paradox" in unbridled Second-order Logic: for if we can **both** quotify over predicates **and** allow predicate variables to occupy the same positions as individual variables, then we can consider those predicates that are self-applicable—those for which it is true that $F(F)$—and those that are not—those for which it is true that $\neg F(F)$—and define the latter as a predicate of predicates, that is define

$$H(F) \quad \text{iff} \quad \neg F(F);$$

We then ask whether the predicate H, thus defined, is, or is not, predicable of itself. Either answer leads to a contradiction: if $H(H)$ then $\neg H(H)$; if $\neg H(H)$, then the defining condition is satisfied, and so it is true that $H(H)$.

But the argument stinks. Expressed in the unfamiliar terminology of sets, we may be led to accept the definition of extraordinary classes which are members of themselves,[2] but the concept of a "heterological property" of not being self-applicable is hard to take. Nor does the formalism of the predicate calculus encourage us to stifle our objections. Predicate variables are different from individual variables, and any definition involving F(F) is manifestly ill-formed. We may quotify over qualities, but that does not make them the same as individual substances. Sometimes, no doubt, we can refer to particular qualities and talk about them, but that is not to say that they are exactly the same as the individuals that are normally talked about as possessing them.

The standard formalisations of predicate calculus are crude. They draw only one distinction, that between individual terms

[2] See below §*12.2*.[Russell's Paradox]

and predicate terms, and blur all other distinctions. Moreover, the individual variables are down-graded to doing hardly any logical work, being little more than glorified logical blanks,[3] like the x in $\int F(x)\,\mathrm{d}x$. All the work is done by the predicates, and, except for sometimes requiring that the universe of discourse be non-empty, no consideration is given to the logical shape or range of the individual variable. A logic that permits us to contrapose 'all ravens are black' into 'all non-black things are non-ravens' is not a logic that is sensitive to type-distinctions. We need, as we have seen,[4] to take individual variables seriously. We need to register in our formalism the fact that not only are predicate variables ineligible to occupy the positions reserved for individual variables, but often one individual variable cannot occupy the place of another. It makes perfectly good sense to ask whether the square of a cricket pitch is green, but not whether the square of 22 is. Ryle pointed out that there were many "category distinctions" in our conceptual scheme as expressed in ordinary language,[5] and that many philosophical errors arose from neglecting them, but the distinctions are more fluid and difficult to discern than he made out. In some senses, geographical and institutional, the University of Oxford is of a different type from the Oxford Colleges, but in other respects they are on the same footing—I am perfectly happy to accept cheques from both. Instead of an absolute, possibly ontological distinction between one categorial type and another, there are many different distinctions depending on context, which any formal system will find it difficult to register.

Equally with predicates there are many type distinctions we observe in our ordinary thought: '   is green' can be predicated of cricket squares, '   is generous' cannot. When we quotify over predicates, there is an implicit restriction of the range of quotification to predicates of the appropriate type. If we say someone has all the properties of Napoleon, we do not consider whether he might be a perfect square, and on discovering that Napoleon was not a perfect square conclude that neither is he: being a perfect square

[3] See W.V. Quine, *Methods of Logic*, Harvard, 1951, §12.

[4] §4.3: see further, §12.3 and §12.8.

[5] Gilbert Ryle, "Categories", *Proceedings of the Aristotelian Society*, **49**, 1938-9; reprinted in *Logic and Language*, Series II, ed. A.G.N.Flew, Oxford, 1953, pp.65-81; and Gilbert Ryle, *The Concept of Mind*, London, 1949, pp.16-18.

is a property that Napoleon neither possesses nor does not possess. Unclarity on this score does not greatly matter so long as we are considering only the positive predicates, since if we were mistakenly to ask the improper question whether or not Napoleon possessed the property of being a perfect square, we should return a negative answer and not ascribe perfect squaredom to the man we were talking about. But once we are dealing with negation, and consider someone who does not have all the properties of Napoleon, we may be tempted to ascribe to him non-perfect-squaredom and our troubles begin. Individual terms have a different logic, less liable to lead us astray. Whereas predicates, like propositions, can be negated and disjoined, it makes no sense except in some wider, specified context, to talk of a non-man, or a man-or-irrational-number.

We make use of type distinctions in our ordinary thinking and speaking, but characteristically articulate them when some particular inference is in question instead of seeking to lay down general rules in advance. The formation rules we already have are enough to rule out 'heterological' as ill-formed. No formal system is likely to be sensitive to all the nuances of ordinary language, and rule out every sort of nonsense, but that, although a barrier to the complete formalisation of logic, is not a conclusive objection to every quotification over qualities or quotities. We are right to be as reluctant to allow a predicate's either being or not being applicable to itself, as we are to a set's either being or not being a member of itself, but we have to confess that we have not yet formulated an adequate formal theory of predicates. This is in line with the general incompletely formalisable nature of Second-order Logic, but it is a confession, not a boast.

§9A.3 Completeness

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Whereas First-order Logic is complete, Second-order Logic is not.[6] Many well-formed formulae can be formulated in it which are independent of the axioms but which are valid, that is, true in under all natural interpretations.

It is often seen as a great merit of First-order Logic that it is complete, and thus reconciles the syntactic and semantic approaches,[7] so that we need not worry about distinguishing them, so long as we stick to First-order Logic, and can use $\vdash$ and $\models$ more or less interchangeably. On the syntactic approach we define the sentential connectives, $\wedge$, $\vee$, $\neg$, $\rightarrow$, corresponding to regimented versions of our familiar 'and', 'or', 'not', 'if', by their inference patterns: on the semantic approach we define the analogous logical constants by truth-tables.[8] It is moderately easy thus far to see that every syntactic entailment is valid semantically,

$$\text{If} \quad P \vdash Q \quad \text{then} \quad P \models Q,$$

and *vice versa*

$$\text{If} \quad P \models Q \quad \text{then} \quad P \vdash Q.$$

In particular, analytic propositions are tautologies, and *vice versa*:

$$\text{If} \quad \vdash Q \quad \text{then} \quad \models Q,$$

and

$$\text{If} \quad \models Q \quad \text{then} \quad \vdash Q,$$

[6] Gödel proved the completeness of First-order Logic, using, it should be noted, infinitistic methods (cf.§*15*.7): Henkin proved *a* completeness theorem for Second-order Logic, which depends on admitting, besides the primary interpretations, some further secondary ones: but the secondary interpretations are unnatural, and not at all what we should expect on the normal semantic approach. See further below §9A.5 in this chapter [analapri].

[7] See above, §3.5.

[8] The terms 'connective' and 'logical constant' are used indiscriminately by logicians. I shall use 'connective' in syntactic contexts, 'logical constant' in semantic ones, and 'operator' or 'functor' to cover both.

In virtue of the former we say that the propositional calculus is **sound**, and virtue of the latter we say that the propositional calculus is **complete**.[9]

If we add the quotifiers, $(\mathrm{A}x)$ and $(\vee x)$, we have to be careful in specifying their inference-patterns, and especially, what exactly should count as models. Provided we quotify only over individual variables, we can establish both soundness and completeness, and this still holds good if we extend the simple First-order predicate calculus, by adding, with appropriate rules, the further two-place predicate, $=$, representing identity. That is to say, if P, Q are any well-formed formulae of First-order Logic,

$$\text{If}\quad P \vdash Q \quad\text{then}\quad P \models Q,$$

and

$$\text{If}\quad P \models Q \quad\text{then}\quad P \vdash Q;$$

or, more succinctly,

$$\vdash Q \quad\text{if and only if}\quad \models Q.$$

The completeness of First-order Logic not only reconciles the syntactic and semantic approaches, but, more sophisticatedly, is seen as a token of the adequacy of our axiomatization: it shows that we have captured in our syntactic notion of theoremhood the desirable semantic property of being true under every interpretation. From this it follows that, since syntactic proof-procedures can be "mechanized", a computer could be programmed to churn out every theorem, and thus, thanks to the completeness theorem, every valid well-formed formula of First-order Logic. First-order Logic is thus "computer-friendly", whereas Second-order Logic, since there is no corresponding way of producing every one of its valid well-formed formulae, is significantly less computer-friendly. In First-order Logic we have a positive test for any particular well-formed formula's being true under all interpretations. It is very tedious, but in the long run it will work: we simply programme a computer to produce every theorem in a systematic way, and check whether or not it is identical with the well-formed formula in question; if it is, then that well-formed formula is valid, is true under all interpretations; if it is not, the computer grinds on and produces the next

9 See above §3.5.

theorem.[10] In Second-order Logic we cannot do this. Although we can still programme a computer to generate every *theorem* in a systematic way, so that if a well-formed formula *is* a theorem, it will turn up sooner or later in the list the computer spews out, not every valid well-formed formula is a theorem.[11] That is, there are some well-formed formulae which are true under all natural interpretations, and so reasonable candidates for being accounted logical truths, but are not theorems according to the axioms and rules of inference of the system, and so could not be discovered or identified by any computer search.

Most logicians have regarded this as a defect of Second-order Logic. But the argument is two-edged, and can be seen as showing not the adequacy of the axiomatization, but the limits of the formalisation. First-order Logic is complete, but only in the way eunuchs are. Eunuchs are able to do everything they want to do, but cannot want to do what other men want to do. First-order Logic can prove every well-formed formula that is expressible in First-order Logic and is true under all interpretations, but cannot express many propositions that other logics can. It secures complete success in its ability to prove propositions by cutting down its ability to formulate them. We cannot say things in First-order Logic we might naturally want to say—for example that a set of well-formed formulae is satisfiable in any finite domain, or that an ordering is well-ordered—and so with its expressive ability thus truncated, it is not surprising that it can prove those relatively few well-formed formulae it can formulate.

[10] This procedure gives only a positive test for theoremhood. IF the well-formed formula in question *is* a theorem, sooner or later we shall discover that it is. But if it is not, it will not tell us that it is not: however far we have gone without its turning up as the conclusion of a valid proof, we can never be sure that a proof of the well-formed formula in question will not turn up at some later stage. First-order Logic has only a one-way test for being a theorem, not a two-way decision-procedure.

[11] George S. Boolos and Richard C. Jeffrey, *Computability and Logic*, Cambridge, 2nd ed., 1980, ch.16.

§9A.4 Second-order Logic

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Second-order Logic differs from First-order Logic in a number of other important ways:[12] they are listed, together with those already discussed, in the table on the page opposite.[13]

Although the differences are real, they hardly justify excluding Second-order Logic from being part of logic. For one thing, many of the differences, though alleged to favour First-order Logic, actually tell the other way; and, anyway, there is no compelling argument for picking on those differences as decisive, in comparison with others, not held to be decisive, and in the face of continuing similarities between Second-order Logic and other systems accepted without question.

The first two of the differences listed in the table have already been shown to be really to the advantage of Second-order Logic.[14] Compactness—the feature that a set of well-formed formulae is consistent provided every finite subset is—has, like completeness, been taken as a virtue. It goes with First-order Logic's being finitely axiomatizable, and such that a computer can be programmed to do it. First-order Logic is, essentially, a finitistic calculus, in which, therefore, every valid well-formed formula that can be expressed in its formalism can be proved in a finite number of steps. But compactness is really a demerit. It trades on the finiteness of a proof-sequence, and is counter-intuitive, leading once again to there being non-standard models of arithmetic, though by a different route from that of Gödel's theorem.[15] Its finitistic features are purchased at the price of our never being able to specify completely what we are talking about (4 & 5). The Löwenheim-Skolem theorem is really a liability rather than an asset, showing, as it does, that in First-order theories we cannot in general specify our models,

[12] See G.Boolos and R.Jeffrey, *Computability and Logic*, 2nd ed., Cambridge, 1980, ch.18. See also C.D. Parsons, "Objects and Logic", *Monist*, **65**, 1982, pp.498-505; and J.B.Moss review in *British Journal for the Philosophy of Science*, **36**, 1985, pp. 437-455.

[13] [CHECK position]

[14] In §9A.3. [Completeness]

[15] See above, ch. 6, §6.3, pp.[TS8-9]

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	First-order Logic	**Second-order Logic**
1.	Complete *i.e.* $\models A \Longrightarrow \vdash A$	Only *Henkin*-complete We cannot secure that if A is true under all *principal* interpretations, then $\vdash A$
2.	So if $\models A$ then we can, given enough time, prove $\models A$	No effective positive test for validity
3.	Compact	Not Compact
4.	Löwenheim-Skolem theorem holds	Löwenheim-Skolem theorem does not hold
5.	Peano Arithmetic not monomorphic	Peano Arithmetic monomorphic
6.	Some well-formed formulae true, but not provable	All arithmetical truths provable from Peano's postulates

or even their cardinality.[16] We have already explored the awkwardness of Peano's postulates not being monomorphic in First-order Logic, and Gödel's incompleteness theorem is a notorious embarrassment. Even the concept of identity cannot be defined in ordinary First-order predicate calculus, but has to be characterized by extra *ad hoc* axioms. Second-order predicate calculus, by contrast, is able to define identity without special extra axioms, and is able to specify exactly what we are talking about.

Second-order Logic thus seems a natural further development of logic from its Boolean core. It is hard to justify our jibbing at this particular step. If it is the incompleteness of Second-order Logic that debars it from being a proper logic, should not the undecidability of First-order Logic tell equally against that? If we have free predicate variables, it would be unreasonable not to be able to quotify over them; otherwise, we should be in the position

[16] See below, ch.12. [Set Theory]

of being able to specify that a universe of discourse was infinite, but not that it was finite.

There are trade-offs. First-order Logic is finitely axiomatizable, but cannot express finitude: Second-order Logic can express finitude, but is not finitely axiomatizable, and our axiomatization is always liable to turn out to be incomplete and inadequate for our purposes. If we formalise Second-order Logic incautiously and ill-advisedly, we run into paradox and inconsistency, whereas First-order Logic has simple formation rules which secure us against any danger of meaninglessness—but at the cost of often being unable to express our meaning at all.[17]

If Second-order Logic is admitted as being part of logic, important consequences follow. The fact that it is not complete means that there are well-formed formulae which are true under all reasonable interpretations, but cannot be proved from the axioms by means of the rules of inference. The Axiom of Choice, or various versions of the Axiom of Infinity, are not in any sense already implicit in the axioms, but suggest themselves as being additional axioms.[18] Truth, once again, outruns formal provability. And, more important for our present purpose, Second-order Logic is more juicy than First-order Logic, being capable of grounding substantial truths, and so cannot be explicated in terms of analyticity alone.

[17] A clear and accessible account of Second-order Logic, and the reasons for accepting it a being genuinely logic, is given by David Bostock, "On Motivatind Higher-order Logic", in T.R.Baldwin and T.J.Smiley, eds., *Studies in the Philosophy of Logic and Knowledge*, Oxford, 2004, pp.277-291, esp. pp. 281-286.

[18] Alonzo Church, *Introduction to Mathematical Logic*, Princeton, 1956, ch.V, §54, p.315.

§9A.5 Analytic and *A Priori*

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Many philosophers have found it difficult to give an account of mathematical truth, because they have taken it for granted that mathematical truth must be either analytic or synthetic, and if synthetic empirical, and if analytic empty. Since it is difficult to take the Protagorean view that it is empirical, they have been forced to the view that the whole of mathematics is a vast set of tautologies.[19] Traditional logicism supports that view. If all mathematical truths are, ultimately, truths of First-order Logic, then they are theorems, and anyone who denies them is guilty of inconsistency. But once we give up the identification of logic with First-order Logic, we are no longer impelled to downgrade mathematical truth to mere tautology.

Whereas First-order Logic is axiomatizable and complete, and it can be reasonably maintained that its theorems are analytic, Second-order Logic is not completely axiomatizable, and seems to generate truths which are not mere reformulation of the original axioms. They are not analytic. They do not just tell us what was in the original axioms and rules of inference of Second-order Logic. The Axiom of Choice cannot even be formulated in First-order Logic, and is known to be independent of the other axioms of Zermelo-Frankel set theory, but has commended itself to mathematicians as being none the less true, as a natural extrapolation of our undisputed ability to make a finite number of arbitrary choices. There is some similarity to Peano's fifth postulate—and Poincaré argued that mathematics was not analytic just because it employed argument by recursion.[20] The reasoning which impels us to generalise from *Sorites* Arithmetic to Peano Arithmetic, impels us also to accept the Axiom of Choice as true. If so, it is in some sense synthetic—it is not a deductive consequence of our explicit axioms of set theory—but it is not *a posteriori* in Kant's sense. The continuum hypothesis is in the same case, and arguably the axioms of infinity. The axioms of geometry, again, are not analytic, and

[19] See, for example, A.J.Ayer, *Language, Truth and Logic*, London, 1936, ch.4; or Bertrand Russell, *History of Western Philosophy*, London, 1946, p.860[CHECK date and page/Bod 266 e.128]: mathematical knowledge is all of the same nature as the "great truth" that there are three feet in a yard.

[20] See above ch.6, §6.5, p.[116 of 89][Also check italics in fn 14 there].

though the Riemannian geometry of the General Theory may be adopted *a posteriori* on the basis of empirical evidence, the arguments adduced in Chapter 2 were altogether different. So, too, are the arguments for extending the natural numbers to the integers, and the integers to the rationals, the rationals to the reals, and the reals to the complex numbers. Such truths, if they are truths, and if we do know them, are known not on the strength of sense experience, and are in that sense known *a priori*. They are, as regards their logical status, synthetic propositions, and, as regards their epistemological status, known *a priori*. The argument that justified our accepting Peano's Fifth Postulate as true was an argument about argument, and in order to determine the proper boundaries of deductive logic, we need to review the nature of argument, and the different dialogues within which it takes place.

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Non-Analytic Propositions

1. Peano's fifth Postulate—it was on account of its employing argument by recursion that Poincaré argued that mathematics was not analytic. (See above ch.6, §6.5, p.[116 of 89][Also check italics in fn 14 there].)
2. Various versions of the Axiom of Infinity.
3. The Axiom of Choice
4. The Continuum Hypotheses, together with various other, more *recherché* axioms suggested for set theory.
5. The axiomatic extension of the natural numbers to include the negative integers, the rational numbers, the reals, and complex numbers.
6. The axioms of different geometries.

The justification of these is not uniform. It involves in part a reassessment of traditional ideas of what it is to know something, and in part a closer examination of the interpretations, principal or secondary, under which valid well-formed formulae come out as true.

§9A.6 What Is Logic?

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In ordinary discourse the word ‘logic’ is used in a variety of senses, often taking its colour from what it is being contrasted with. Feminists are infuriated when men say that women are emotional rather than logical, and scientists talk of the logic of an experiment as opposed to the actual observations made. Historians often account for a statesman’s actions in terms of the logic of the situation, as opposed to some personal predilection of his or the fortuitous result of happenstance. Philosophers used to talk of inductive logic at the same time as maintaining that logic was purely deductive. In these and many other contrasts, although the exact sense of the word ‘logic’ varies, the force of the contrast is the same: logic is topic-neutral, something that does not vary with personalities, empirical data, or the chance concatenation of events. It isthe “Universal Ordinary” studying patterns of inference that are not peculiar to any particular subject, but are common to all. It abstracts from particular instances, and considers only the general form.

But our concept of logic is under strain. Set against the drive for universality is the requirement of cogency. Hence the demand that logical arguments be so incontrovertible that it would be inconsistent to refuse to concede the conclusion, having admitted the premisses. Logical arguments, on this showing, will be those that must be accepted, on pain of making oneself incommunicable with. They are valid, simply in virtue of the meaning of words, and without meaning there can be no communication. Granted that we want to be understood, there is maximal cogency in such inferences.

Nevertheless, it is unsatisfactory to characterize logic in terms of inconsistency alone. For one thing, there are many arguments valid in virtue of the meaning-rules of natural language, which it would be inconsistent to controvert, but which do not properly belong to the realm of logic: ‘he is an uncle: so he is not an only child’, ‘today is Monday, so tomorrow is Tuesday’ are maximally incontrovertible, but not general in their application. And, as we have seen, inconsistency is not the only sanction. A sceptic can deny an old-fashioned inductive argument if he wants, without being inconsistent or making himself unintelligible; rather, he denies himself all possibility of knowledge of general truths or of those not yet experienced, if he will not ever generalise or extrapolate from what he already knows to what he would like to know.

Faced with these difficulties in characterizing logic, many philosophers have formalised, replacing non-deductive inferences by suitably stipulated axioms. Deductive inference, represented formally by the single turnstile, $\vdash$, then becomes a matter of abiding by the rules laid down, and the sanction for those who break the rules is simply that they are not playing the game. The formal characterization of logic is attractive; we could reasonably hope to program a computer to act according to explicit rules concerned solely with the syntax of strings of symbols. But we still have to decide which symbols to have, and which strings to regard as important. Bureaucrats are very fond of acronyms, and use strings of letters rather than meaningful words, but it does make their discourse logical. Only rather few symbols deserve to be accounted logical symbols.

If we adopt the semantic approach, and seek to distil logic from various fields of discourse, as what is common to all valid argumentation, we reach a similar problem. We can form an idea of a "logical constant" as that which is the same in all patterns of argument, but there is a suspicion of arbitrariness in deciding what exactly is a logical constant, and what not; is 'is identical with' a logical constant? A pure theory of deductive argument cannot depend on an arbitrary choice of logical constants, any more than it can be just a matter of playing the game by some stipulated rules. It must, rather, arise from the nature and aims of argument, and the constraints that the context and the aims impose.

When we argue, we draw inferences because we disagree. If we did not disagree, we should not argue at all. As it is, being autonomous beings, we are capable of having our own ideas, each seeing things differently from others, so that when I give vent to my views, you need not accept them. You can say No, and I can say No to your No, thereby reasserting my original contention.[21]

But we do not merely contradict each other. We give reasons. Each appeals to considerations the other is likely to concede, and draws inferences from them to support his side of the dispute. I put forward a proposition, and from it infer the truth of another proposition: p **therefore** q, which we might symbolize as

$$p \;\Vdash q,$$

to avoid favouring either the syntactic $\vdash$ or the semantic $\models$. We do not only draw inferences, however; we may need to discuss them,

[21] See above, ch.7, §7.5[p.20 of 1997].

in which case we may, as in the Deduction Theorem, consider the equivalent implication, arguing about the truth of the proposition $p \to q$, rather than the validity of the inference

$$p \;\Vdash q,$$

[It accords with the spirit of logic to distance ourselves from substantial issues about the validity of inferences in particular disciplines, which depend on the way the world is, or on various principles of legal, political, historical, or philosophical argument, and to cast all inferences, other than those which are simple substitutions or which can be shown to have purely tautological conclusions, into the *Modus Ponens* form

$$\Gamma \to \Delta,\ \Gamma \vdash \Delta$$

In this way the question is moved from a question of validity, whether $\Gamma \vdash \Delta$ is a *valid inference*, to a non-logical question of truth, whether $\Gamma \to \Delta$ is a *true implication.*][22] Equally, we may reverse the order of argument, and contend that q **because** p.[23]

Once we have some concept of negation and implication, we can define the other sentential connectives. Although $\neg$ renders the meaning of 'not' moderately well, $\to$ is significantly different from 'if . . , then . . ', as also $\wedge$ (or &) from 'and', while 'or' in ordinary English is ambiguous between the inclusive and exclusive sense, only the former being expressed by $\vee$. Nevertheless, they constitute a "regimented" version of the connectives used in ordinary language, and provide the logical constants for a topic-neutral formal logic.

The familiar sentential connectives are not the only ones. There are sixteen possible truth tables for binary connectives linking two variables, each of which can take either of two values, TRUE or FALSE. We often add $\leftrightarrow$ (or $\equiv$), for 'if and only if', and sometimes $\top$ for TRUE, and $\perp$ for FALSE. The latter can be used as a primitive instead of $\neg$. (Indeed, instead of having one of these together with $\to$ (or $\wedge$, or $\vee$), we can define both negation and all the binary sentential connectives in terms of just one, $\mid$ or $\downarrow$, non-conjunction or non-disjunction; but this is somewhat artificial.) Whatever primitives we choose, we have a Boolean Algebra, B_2, for propositional calculus, which goes a long way towards articulating the formal structure of argument in every field of discourse.

[22] [? delete passage: is it really necessary?]

[23] See Gilbert Ryle, "If, So, Because" RefReq

§9A.7 Modal Logic

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Propositional calculus is not the whole of logic. We often modify propositions. We can outline possibilities, or recognise necessities, or consider counter-factual conditionals. We can engage in fiction and tell stories, or consider obligations, or distinguish the conjectural from the well-established. We can hope, expect, fear, warn, promise or threaten about things to come, and can remember, ponder, relate, or explain, the past. All these activities have some propositional content—we can say what the content of our hopes, wishes, judgements or romances is, and pick out entailments and inconsistencies among them—but cannot be represented in terms of propositions alone.

Grammatically, modification of a proposition is often expressed in English by the use of an auxiliary verb, and in inflected languages by a change of mood or tense. But whatever the shift of tense or mood, and whether it is expressed by an auxiliary verb or some more complicated locution, such as 'it is possible that ...' or 'it was going to be the case that ...', the modified proposition stands in some relation to the original one, and is still something that can be agreed with or disagreed with, accepted or rejected, shared or repudiated. It is reasonable to regard the modified proposition as still being itself a proposition, and therefore to see the various modifiers as *unary operators*, or *unary functors*, which operate each on a single proposition to yield a single proposition.

The unary operator may stand for any one of a wide variety of modal or tense auxiliary verbs, or adverbs, or propositional phrases. It is not to be assumed in advance that every modal operator in ordinary speech can be expressed adequately in modal logic: but it is a useful exercise to see how far we can go in considering the formal possibilities enriching propositional calculus by a simple basic unary operator, and the constraints on the rules of inference and axioms it is reasonable to recognise. Having added to propositional calculus a unary operator, which we may symbolize as $\Box$, with the same formation rules as $\neg$, we need to consider possible rules of inference and axioms governing the use of $\Box$. There is a wide range of possibilities. But there are constraints: we must not have too many rules or axioms, or our operator will be degenerate, definable in terms of the ordinary sentential connectives, and our logic will be nothing more than ordinary propositional calculus; if we have too few, however, our operator will lose all contact with the connectives of ordinary logic, and our modified discourse will no longer be a logic at all. If we are to give sustained attention to a mode of discourse, words must have their ordinary meanings, and analytic propositions must hold as well within the modalised discourse as outside it; and therefore, as far as propositional calculus is concerned, tautologies must remain tautologies when modified. Since every tautology is a theorem of propositional calculus, and *vice versa*, we stipulate

If Γ is a theorem, so is $\Box\,\Gamma$,
i.e., If $\vdash \Gamma$ then $\vdash \Box\,\Gamma$.

This rule of inference is characteristic of all modal logics (logics with an additional unary modifying operator), and is known as the Rule of Necessitation.

The Rule of Necessitation ensures that logical theorems remain logical theorems when modalised, but does not by itself suffice to legitimise standard inferences in modalised discourse: it enables us to introduce $\vdash$ into a mode of discourse, but not to use it to make inferences within it. If we are to carry ordinary inferences over into modalised discourse we need the further rule

If $\Gamma \vdash \Delta$, then $\Box\,\Gamma \vdash \Box\,\Delta$.[24]

But in accordance with the tendency of modern logic to replace inferences by implications, the question whether modalised inferences are real inferences, that is whether $\Box\,(\Gamma \to \Delta), \Box\,\Gamma \;\vdash \Box\,\Delta$, becomes the question whether a modalised implication $\Box\,(\Gamma \to \Delta)$ yields a straightforward implication between the modalised parts of it, $\Box\,\Gamma$ and $\Box\,\Delta$. We therefore lay down as an essential axiom for modal logic

K $\qquad \Box\,(\Gamma \to \Delta) \to (\Box\,\Gamma \to \Box\,\Delta)$.

The axiom K entitles us to infer $\Box\,\Delta$ from $\Box\,(\Gamma \to \Delta)$ and $\Box\,\Gamma$ in two steps of *Modus Ponens*.

The Rule of Necessitation together with the axiom K ensure that modalised discourse is "**inferentially transparent**". Essentially what we require is that we should be able to make the same inferences within modalised discourse as in unmodalised discourse. If there is a good argument about kicking the ball—e.g. that in order to kick it, one must approach it, or that if one kicks it, the result will be that it moves—the same argument should hold within the context of obligatory kicking, alleged kicking, future kicking, or past kicking. Else modal discourse becomes inferentially opaque.

[24] Aristotle, *Prior Analytics*, I, 15, $34^a 22-24$.

If modal discourse is to avoid logical chaos, it must be subject to the Rule of Necessitation and axiom K, which together constitute the standard minimum system of modal logic: these, in effect, govern its interrelationships with $\vdash$ and $\rightarrow$. In order to place it as fully as possible in the context of propositional calculus, we need also to consider its interrelationship with $\neg$. As a first try, we might think that it would "commute" with negation, *i.e.*

$$\vdash \Box \neg p \leftrightarrow \neg \Box p,$$

but in that case the modal operator would become vacuous, so far as propositional calculus was concerned.

The non-theorem $\Box\neg p \leftrightarrow \neg\Box p$ consists of two conjuncts, $\Box\neg p \rightarrow \neg\Box p$ and $\neg\Box p \rightarrow \Box\neg p$. While we cannot have both, we can, and should hope to, have one; else our modal operator $\Box$ will have so little to do with the ordinary logical constants that there will scarce be a logic worth talking about. Although we could choose either, and the decision is, as we shall shortly see, in some sense arbitrary, we shall choose the former conjunct, $\Box\neg p \rightarrow \neg\Box p$. The reason is that we naturally want to secure a certain "modal consistency" for our operator. Consistency requires that no well-formed formula of the form $p \wedge \neg p$ can be a theorem. We naturally go further—though it *is* further—and lay down that not only is $p \wedge \neg p$ not a theorem, but that the negation of $p \wedge \neg p$ is a theorem, that is,

$$\vdash \neg\,(p \wedge \neg p).$$

This is a theorem of ordinary propositional calculus. In considering the relation between $\Box$ and $\neg$, we may reasonably look for a comparable stipulation, *viz.* $\vdash \neg(\Box p \wedge \Box\neg p)$, which is equivalent to

$$\vdash \Box\neg p \rightarrow \neg\Box p,$$

the former of the two conjuncts. This in turn is equivalent to

$$\vdash \Box\, p \rightarrow \neg\,\Box\,\neg p,$$

or, writing $\Diamond$ for $\neg\,\Box\,\neg$,

$$\vdash \Box\, p \rightarrow \Diamond\, p,$$

which is a characteristic thesis of modal logic, known as the axiom D. Axiom D must hold, but its converse $\nvdash \Diamond p \rightarrow \Box p$ must not be a theorem, on pain of degeneracy. To secure this, we need a stronger implication than $\rightarrow$ which looks like an irreflexive asymmetric strict

ordering relation, but is in fact reflexive and antisymmetric. (See §9B.1.) Ideally we ought replace $\rightarrow$ by $\overrightarrow{-}$, and keep $\rightarrow$ for the asymmetric implication; but it would be cumbersome and finicky to do this. Instead we use the symbol $\Rightarrow$ to represent an irreflexive asymmetric implication, where $\vdash p \Rightarrow \neg p$, and
$\vdash (p \Rightarrow q) \Rightarrow \neg(q \Rightarrow p)$, and write axiom D
$\vdash \Box p \Rightarrow \Diamond p$ to make this point.

Almost all interesting systems of modal logic have D as an axiom. It yields four out of the six possible interconnexions between $\Box$ and $\wedge$, $\vee$ and $\neg$, and we cannot add either of the others on pain of modal degeneracy. We can therefore argue for it as giving us as much, in the way of interconnexion between the modal operator $\Box$ and the connectives $\wedge, \vee$ and $\neg$ of propositional calculus, as we can hope to have. These rules for $\Box$ make it the **most highly structured non-trivial operator relative to the Boolean operators**.

§9A.8 Iterated Modalities

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Although the system that contains D goes as far as possible in relating the modal operator with the sentential connectives of propositional calculus, it leaves other questions unanswered. It tells us nothing of the relation between modalised and unmodalised discourse, nor of any relations between iterated modal operators, which may be all of essentially the same sort, but may also be differentiated from one another. Axioms giving rules for such relations can be laid down, giving rise to different logics, according to what rules are adopted. The Aristotelian axiom T, $\vdash \Box p \rightarrow p$, 'what must be, is', specifies a relation between modalised and unmodalised discourse, and holds in most modal logics, though not in those concerned with ethics and what we ought to do: the axiom 4 which is typical of the system **S4** specifies relations between iterated modal operators of the same form (either $\Box\Box$ or $\Diamond\Diamond$) the axiom 5, which is typical of the system **S5**, specifies relations between iterated modal operators of different forms ($\Box\Diamond$ and $\Diamond\Box$); the axiom B, which is typical of the system **B**, does specifies relations between both forms in connection with unmodalised discourse; and the quotifiers can be usefully viewed as modal operators differentiated from one another by virtue of the variables they bind.

These different modal logics are intricate, and hard to make sense of, if we consider them only from a syntactical point of view. For the present it is enough to note that T is postulated as an axiom in **S4**, **S5** and **B**, but there are important uses for systems having only the axioms 4, 5 and B. It is also noteworthy that provided we have the axiom D, **S5** can be based on the axioms 4 and B alone, without needing to postulate T separately.

Thus far the iterated modalities have all been of the same basic type. But, clearly, there could be more than one type. Often they will be entirely independent of one another. 'I know that' and 'It is reported in the *Times* that' do not mesh for non-*Times* readers. The only cases that will give rise to a logic developed from what we already have, is when the distinct operators are not all acting globally on well-formed formulae of propositional calculus, but selectively, some on one, and some on other, parts of a well-formed formula with Boolean sentential connectives. We shall need to flag the operators and those terms they operate on. Instead of bare $\Box$ and $\Diamond$, we shall distinguish them by subscripts, $\Box_x$ and $\Diamond_y$, etc., and shall need then to indicate the scope of their operation, and thus ultimately to assign subscripts to those primitive terms susceptible to their modalising influence. Thus the general form of such a well-formed formula will be something like
$\Box_x(\Diamond_y p_x \to (q_{x,y} \wedge r))$,
where p is susceptible only to $\Box_x$, q is susceptible to $\Box_x$ and to $\Diamond_y$, and r is susceptible to neither.

At this stage it becomes obvious that what we are doing is to re-invent the quotifiers: $\Box_x$ is the universal quotifier, standardly expressed by $(\forall x)$ and in this book by $(\mathrm{A}x)$; $\Diamond_y$ is the existential quotifier, standardly expressed by $(\exists y)$ and in this book by $(\vee y)$. Similarly we could write $p(x), q(x,y), r$, but actually write $F(x), G(x,y)$, or, saving brackets, Fx, Gxy. The rule of necessitation becomes the rule of generalisation:

$$\text{If} \vdash \Gamma \text{ then} \vdash (\mathrm{A}x)\Gamma$$

The axiom G becomes

$$(\mathrm{A}x)(Fx \to Gx) \to ((\mathrm{A}x)Fx \to (\mathrm{A}x)Gx).$$

The thesis D, $(\mathrm{A}x)Fx \to (\vee x)Fx$ holds in all non-empty universes, and follows from the equivalent, postulated in most systems, of T, $(\mathrm{A}x)Fx \to Fx$ (or, in some systems, $(\mathrm{A}x)Fx \to Fa$, where a is an individual name). Since a variable can be quotified only once, so that $(\mathrm{A}x)(\mathrm{A}x)Fx$ and $(\vee x)(\mathrm{A}x)Fx$ are each equivalent to $(\mathrm{A}x)Fx$, and $(\vee x)(\vee x)Fx$ and $(\mathrm{A}x)(\vee x)Fx$ are each equivalent to $(\vee x)Fx$, it is evident that the quotifiers are like the modal operators of **S5**.

Besides the well-known cases where we have quotifiers binding different variables, we can consider modal predicate calculus where a modal operator interacts with a quotifier. It is fairly easy to prove

$$\vdash \Box(\mathrm{A}x)Fx \to (\mathrm{A}x)\Box Fx,$$

but the converse implication,

$$(\mathrm{A}x)\Box Fx \to \Box(\mathrm{A}x)Fx,$$

known as the Barcan formula, is not a thesis of the predicate version of **T** nor of **S4** (though it is of mainline predicate versions of **S5**).[25] The universal quotifier thus differs from simple conjunction, since

$$\vdash (\Box p \wedge \Box q) \to \Box(p \wedge q),$$

which is again reminiscent of the characterization of topology in terms of open sets, where any finite intersection of open sets is itself open, but an infinite intersection of open sets may not be open itself.

We may choose not to admit polyadic predicates, $q(x, y)$, or $G(x, y)$, and to confine our extension of logic to monadic predicates, $p(x)$, or $F(x)$, alone. In that case we have the monadic predicate calculus, which, like propositional calculus, is (two-way) *decidable*. Given any well-formed formula, we can tell in a finite number of steps, whether *or not* it is a theorem. Monadic predicate calculus, however, is extremely restricted, and it is natural to extend logic to include polyadic predicate calculus (generally known as predicate calculus *simpliciter*), which allows two-place predicates (and more-than-two-place predicates) expressing relations, in which case, we are allowing the logic of relations[26] as part of logic. We thus see how modal logic and quotificational logic are natural developments from a Boolean system we are led to adopt by the importance of inference and negation. It will emerge that the logic of relations also plays a crucial role in displaying the interconnexions between different modal logics.[27] It would be difficult on this showing to draw any profound distinctions, or claim that some, but not others, of these studies were to be properly accorded the title of "logic".

[25] See, G.E.Hughes and M.J.Cresswell, *An Introduction to Modal Logic*, London, 1968, pp.142-144 and 178-182.

[26] See next chapter, ch.9B[TranRel].

[27] See below, [?ch.13, Chast Log?]