# Chapter 10B

# Prototopology[0]

Contents



## §10B.1 Nextitude and Togetherness

The two partial orderings developed in the previous chapter are paradigms of the superlative and the comparative ideals. But the two need each other. We talk about the next moment, but feel uneasy, because however small moments of time might be, we could imagine an even smaller interval, less than that between one moment and the next. And equally, when talking about portions or regions, we identify some as being neighbours, sharing the same boundary or limit, and so being next to each other. The two ideals ought to be complementary, but seem to be opposed.

The tension between the comparative and superlative, the dense and the discrete ideals of ordering admits a profound resolution: dense orderings may be not just dense, but also **continuous**. Continuous orderings combine, in a manner of speaking, the virtues of both the dense and the discrete. They are dense—between any two elements there is a third, so that there is, in a sense, always more ordering going on—but they have the virtue of a discrete ordering in maintaining some sort of next-ness.

[0] Throughout this chapter I have drawn heavily on the work, and the patience, of David Bostock, who has let me see much unpublished work of his own, and on innumerable occasions has pointed out the loopholes in my latest arguments and definitions.

*Prima facie* this is a contradiction. But as often in mathematics, *prima facie* contradictions are resolved by a shift in meaning.[1] In this case it is the word 'next' that has to change its meaning. Although it is not possible for two *elements*[2] to be next each other—that is tantamount to discreteness—an element can be next to, that is to say, a "limit point of", a *set* of elements, or of a *portion*, or of a *region*, or of an *extension*.

It is a difficult distinction, and one that Hume missed. His arguments against the continuity of time and space are not only epistemological. Besides Zeno's argument against infinite divisibility, he adduces an argument from succession.

> 'Tis a property inseparable from time, and which in a manner constitutes its essence, that each of its parts succeeds another, and that none of them, however contiguous, can ever be co-existent. For the same reason, that the year 1737 cannot concur with the present year 1738, every moment must be distinct from, and posterior or antecedent to another. 'Tis certain then, that time, as it exists, must be compos'd of indivisible moments. For if in time we could never arrive at an end of division, and if each moment, as it succeeds another, were not perfectly single and indivisible, there would be an infinite number of co-existent moments, or parts of time; which I believe will be allow'd to be an arrant contradiction.[3]

Three words are crucial: 'succeeds', 'another' and 'contiguous'. The word 'succeeds' could mean a one–one relation—that is, a relation in which to each moment there is exactly one moment that succeeds it, and exactly one moment that it succeeds—or a many–many relation—that is, we could say of every moment in 1738 that it succeeds every moment in 1737. In the former sense it defines a discrete ordering. The successor to any moment is the next after it, and there is no moment after it and before its successor. In the latter sense, however, it is compatible with an ordering's being not discrete but dense. Hume equivocates between the two. At the conclusion of his argument he is taking 'succeeds' as being equivalent to 'posterior' and correlative with 'antecedent'. In that

[1] See above, §7.3 and §7.10.

[2] Compare Aristotle, *Physics*, $\Delta, 10, 218^a 18$.

[3] *A Treatise of Human Nature*, Bk I, Part II, Sect II; in Selby-Bigge edn. p.31. Compare Aristotle, *Physics*, E, 1, $231^b 10 - 18$.

sense it is an arrant contradiction to suppose time to be composed of an infinite number of moments which are neither posterior nor antecedent to one another. But at the start of his argument he is taking 'succeeds' as a one–one relation, like the 'successor' of modern mathematics. Only so will it establish his contention that time is discrete. But there is no warrant for supposing that the moments of time must be successive in that sense. For, the fact that given only two distinct moments of time one must be after the other, it does not follow that one must be next after the other. Hume has smuggled in an assumption of nextness by his double use of 'succeeds'.

The equivocation is made more compelling by a parallel one in the use of 'another', which may mean 'some other' or may mean 'a particular other'. The former sense does not pick out a particular moment as *the* moment that a given moment succeeds. In that sense all that Hume is saying is that for each moment there is at least one (and in fact many) moments which it succeeds. And this is true, and carries with it no suggestion of discreteness. But the use of the singular—'another'—carries with it the implication of uniqueness, that for each moment there is one and only one moment which it succeeds. And this would yield the conclusion that time was discrete.

The word 'contiguous' comes from the Latin *tango*, I touch. Each moment is contiguous to another if it is in contact with it, and touches it. But if it touches it, there cannot be any moment between, or it would separate them. So contiguity as a relation between moments implies discreteness. And contiguity seems called for, else we separate moments (and similarly points in space) into a discontinuum. Some sort of togetherness is called for if we are to characterize the unity of time (and similarly of space), and contiguity seems to express this requirement. And so it does, but not as a relation between moments. Instead of one moment being contiguous to another moment—which must yield discreteness—one moment is contiguous to an interval, and *vice versa*. Every moment is contiguous to those intervals of which it is the limit, or bound, or boundary, and conversely every interval is contiguous to those moments that are its limits or bounds. In this way we can characterize the togetherness of the continuum without reducing it to a discrete "contiguum".

Continuous orderings reconcile the comparative merits of denseness, shown by the rational numbers, where it is always possible to squeeze another in, and we never reach an end of divisibility, with the superlative merits of the natural numbers, which know their next-door neighbours, and have no gaps in their ranks. They achieve this, by changing the focus, and not concentrating on individual elements alone, but considering other, less atomized, entities as well.

Next-ness is important, because only if neighbours are next each other can they be together, and constitute one unseparated whole of which they are parts. Togetherness is a key notion in philosophy generally as well as in mathematics. The Great Chain of Being binds the disparate levels of Reality into One Coherent Whole, and togetherness is a mark of being integrated into a single unity. Although the *part of* relation is a simple mereological one, the concept of *being a whole* is stronger than a simple correlative of *being a part*, and plays an essential role in the theory of measurement, which depends on there being some extensive magnitudes for which the measure of the whole is the sum of the measures of the parts. Togetherness is the chief concern of topology, and is taken for granted in geometry. Plato had many dark sayings in his later dialogues, the *Parmenides* and especially the *Philebus*, about *τὸ πέρας* (*to peras*) and *τὸ ἄπειρον* (*to apeiron*) and about the more and the less and the Great and the Small, and Aristotle seems sometimes to be engaged in what we might describe as prototopology.[4] He and his successors puzzled long over the continuum, but never quite succeeded in givng an account of irrational numbers. Kant recognised the importance of *Analysis Situs*[CHECK]. But it was the German mathematicians of the Nineteenth Century, concerned to give a rigorous treatment of functions and of irrational numbers, who finally elucidated the concepts of continuity and connectedness, the key concepts of topology, which can be taken as the rigorous explication of togetherness.

[4] See especially, Kenneth M.Sayre, *Plato's Late Ontology*, Princeton University Press, 1983; David Bostock, "Aristotle on Continuity in *Physics* VI," in L.Judson, ed., Aristotle's *Physics*: A Collection of Essays, Oxford, 1991, pp.180-188, esp. p.186; and M.J.White, *The Continuous and the Discrete*, Oxford, 1992.[Perhaps add Pythagoras; see Taylor and Sayre]

## §10B.2 The Continuum

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Eudoxus almost succeeded in giving a satisfactory account of the real-number continuum, but it was not until Dedekind and Cantor that a really rigorous treatment was finally achieved.

Dedekind's account turned on the least upper bound of certain sets of rational numbers: Cantor's on nested intervals of rational numbers converging to a limit. Dedekind considered the rational numbers in their natural order, and particular partitions of them into "Dedekind cuts". A Dedekind cut was a partition of the rational numbers in which every member of the left-hand set was less than any member of the right-hand set. A real number was defined as a Dedekind cut of rational numbers. Since the rational numbers are dense, there could not be both a greatest member of the right-hand set and a least member of the left-hand set, for then there would be some number between them, which would not be a member of either, so that they would not together constitute a partition after all. The three remaining posssibilities are that the left-hand set should have a greatest member, or that the right-hand set should have a least member, or that neither left-hand set should have a greatest member, nor the right-hand set should have a least member. In either of the two former cases, the Dedekind cut would correspond to a rational number; that is, it would define a rational real number: and in the last case would constitute a specification of an irrational real number. Thus if we define a Dedekind cut by saying that every rational number whose square is less than 2 is to belong to the left-hand set, every rational number whose square is greater than 2 is to belong to the right-hand set, there will be no rational number that is the greatest of the left-hand set, or least of the right-hand set, and we can *identify* the Dedekind cut *as* the irrational real number $\sqrt{2}$.

In a very similar spirit Cantor defined a real number as the limit of a sequence of nested intervals of rational numbers. Cantor's approach is closer to our normal way of representing irrational numbers by an infinite decimal. Essentially, instead of depending on a least upper bound, it invokes a limiting lower bound which is, once again, of a different ontological type from the members of the sequence, according to our normal reckoning, but is identified with that sequence for the purpose of definition. In either case, real numbers are defined as special classes of rational numbers, themselves equivalence classes of positive rationals, which in turn are equivalence classes of natural numbers. Real numbers can thus be grounded in the natural numbers, which, the logicists hoped, could be completely defined in terms of formal logic alone. In any case we have analysed magnitude, *μέγεθος* (*megethos*), expressed by real numbers in terms of the natural numbers, *ἀριθμοὶ* (*arithmoi*), as well as providing an arithmetical model of a continuous ordering.

But the continuum is not in itself an arithmetical or algebraic concept. It has been thought about most commonly in relation to time, where the sense of order and connectedness are paramount, and that of number remote, if not altogether absent. Thus, although the arithmetization of analysis was quite proper for defining real numbers and freeing analysis from all spatial and temporal intuition, it brings in extraneous numerical concepts, and obscures the conceptual links we need to reveal between the foundations of mathematics and logical concepts other than the quotifiers. In order to reveal the conceptual links between the foundations of mathematics and logical concepts other than the quotifiers, we need to carry through, with no disrespect to Cantor and Dedekind, a counter-programme of the In a sense, therefore, we need to carry through, with no disrespect to Cantor and Dedekind, a counter-programme of the De-arithmetization of Analysis, in which we seek a more austere approach in terms of ordering relations alone, without reference to numbers, rational or real.

The rational numbers are not essential for characterizing a continuous ordering. They provided an order, and a guarantee of denseness, but so does time. The Greeks had a sense of the present, *τὸ νῦν*, (*to nun*), *both* dividing the past from the future, *and* joining them together in a single united time.[5] Although the past and future are disjoint, with the present instant between them, yet they are connected, because the present instant is next them both. The continuum, because it is gapless, has that sense of connectedness, which a merely dense ordering lacks; *συνεχῆ μὲν ὧν τὰ ἔσχατα ἕν* (*suneche men hon ta eschata hen*), those things are connected whose extremities are one.[6]

Prior cites from Cocchiarella a postulate that secures that a dense temporal order be continuous;[7] he gives it in the Polish notation:

$$CGp\,CHGCGpPGpHGp;$$

[5] *Physics*, IV, 11, 220a4-13.

[6] *Physics* VI, 1, 231a20.

[7] A.N.Prior, *Past, Present and Future*, Oxford, 1967, p.72. See also §13.4 below.

which in a more familiar notation is:

$$(Gp \rightarrow (HG(Gp \rightarrow PGp) \rightarrow HGp)),$$

where $G$ means 'it always will be the case that ... ' $H$ means 'it always has been the case that ... ' $P$ means 'it sometime was the case that ... '.[8] The key is the back step from $Gp$ to $PGp$; if any situation which is always going to obtain must have been going to do so at some earlier time, then in continuous time it must always have held good, though not necessarily in a merely dense temporal ordering. Thus in a merely dense temporal ordering a situation might obtain for all instants after $\sqrt{2}$, and at each of those instants $Gp$ would be the case, and it would be true also that $PGp$, without its having to be the case that $p$ or $Gp$ at any instant before $\sqrt{2}$. But if $\sqrt{2}$ is admitted as a temporal instant, then since $p$ obtains *ex hypothesi* at all later instants, $Gp$ holds at $\sqrt{2}$ and hence $PGp$ at earlier instants. A merely dense ordering has gaps, such as $\sqrt{2}$, which prevent the start of a situation's obtaining being shunted ever further back to extend over all time. A continuous ordering, by contrast, is, like a discrete ordering, "gapless".

We can then, up to a point, give an account of continuity in terms of ordering relations alone.[9] It should not surprise us that a Least Upper Bound[10] characterization of continuity combines the superlative merit of discrete orderings with the comparative merit of dense ones, by picking on the unique limit which is greater than all the (infinitely) many members of a set, and which, therefore, can be said to be next-them-all, though not next any one of them. It is reasonable, therefore, to see continuous orderings as the resolution of the two paradigms outlined in the previous chapter.

[8] Tense logic can be seen as a special case of modal logic, in which the Browerian axiom does not hold (nor axiom T), and we can therefore distinguish between the modal operators and their inverses. Cocchiarella's Continuity Postulate then becomes $(\Box\, p \rightarrow (\Box^{-1}\Box\,(\Box\, p \rightarrow \Diamond^{-1}\Box\, p) \rightarrow \Box^{-1}\Box\, p))$.

[9] E.V.Huntington, *The Continuum and Other Types of Serial Order*, Cambridge, Mass., 1917; pbk, New York, 1955, ch.V, §54-§65, pp.44-57; and E.V.Huntington, "A Set of Postulates for Real Algebra, Comprising Postulates for a One-Dimensional Continuum and for the Theory of Groups", *Transactions of the American Mathematical Society*, **6**, 1905, pp.18-22; and O.Veblen, "Definition in Terms of Order Alone in the Linear Continuum and in Well-ordered Sets", *Transactions of the American Mathematical Society*, **6**, 1905, pp.165ff.

[10] [Decide between 'least[' and 'lowest' and standardise]

## §10B.3 De-arithmetization and De-axiomatization

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We should naturally like to generalise. The continuum of the real numbers provide the paradigm "standard topology", but is essentially one-dimensional: it would be good to have continuity and connectedness defined in universes of many dimensions and none. We could then give a foundational account of topology, together with an exegesis of magnitude and extension, which would yield a robust theory of measurement.

But we are asking too much. As we shall see in the next section, this promising programme runs into the sands. In any case, so far as measurement is concerned, we cannot expect an entirely non-numerical account to be available. Even if we have succeeded in ordering magnitudes (such as weights, lengths, angles, velocities) continuously, a further "Archimedean" condition is required to secure measurability. In metrical terms it is that the scale should be uniform, so that any magnitude, no matter show small, will, if doubled a sufficient number of times, exceed any other magnitude, however large. In purely ordinal terms the "Archimedean" condition is most easily understood by contrast with orderings that are non-Archimedean. The points on the square, ordered by putting those with a lower $x$-coordinate before those with a larger one, and when the $x$-coordinates were the same, by putting those with a lower $y$-coordinate before those with a larger one is a non-Archimedean ordering. Although we normally take it as indisputable that time is ordered Archimedeanly, some mystics and writers of fiction have envisaged a non-Archimedean time, in which aeons of understanding or experience are "injected" into a single instant of our ordinary time.[11] Archimedes distinguished such non-Archimedean orderings from ordinary linear ones by means of the metrical criterion. Excluding non-Archimedean orderings without benefit of measurements is difficult. Huntington cites a postulate of linearity for the bounded real numbers originally due to Cantor.

*The class K contains a denumerable subclass R in such a way that between any two elements of the given class K there is an element of R.*[12]

[11] See more fully, J.R.Lucas, *A Treatise on Time and Space*, London, 1973, §7, esp. pp.38-40.

[12] E.V.Huntington, *The Continuum and other types of serial order*, Cambridge, Mass., 1917; pbk, New York, 1955, ch.5, §54, p.44; O.Veblen, *Transactions of the American Mathematical Society*, **6**, 1905, pp.165-171.

This condition is easily seen to be satisfied by the real line, since the rational numbers furnish a sufficient separation of the real numbers, while being, somewhat surprisingly, denumerable. It is too arithmetical to be entirely satisfactory; but, then, the condition it is intended to exclude is one involving measures and numbers. Nevertheless, we should like to be able to characterize linear orderings as being in this *recherché* sense uniform, and in the absence of a more abstract and intuitive characterization, the de-arithmetization of analysis remains incomplete.

We should like to generalise to topological spaces. Here the program would be as much one of de-axiomatization as de-arithmetization. In most modern treatments topology is introduced axiomatically, in terms of open sets. Topology is presented as an enrichment of Set Theory, to which we add some further term, such as *open set*, and further axioms governing its use. The open sets form a family of sets, closed under union and finite intersection. This approach has the virtue of conceptual simplicity—essentially it adds to set theory only the monadic property of a set's being open—and the rules, though complicated, have the merit of underscoring the distinction between finite and infinite. It also would appeal to a Leibnizian with a strong sense of the importance of positive monadic predicates, though not regarding negative predicates as referring to real qualities. Finite conjunctions of positive predicates would refer to complex qualities, instantiated by some open set, potentially capable of an infinite membership; but infinite conjunctions of positive predicates would individuate some particular unique monad. Disjunctions of positive predicates, whether finite or infinite, would specify open, though perhaps not very natural, sets, clearly capable of infinite membership. A Leibnizian approach would be suitable for a jurisconsult, considering legal cases, some of which very closely resembled one another, while others were fairly far from them, and clearly separate. More generally, the distinction between finite intersections of open sets, which are themselves open, and infinite intersections of open sets, which may be closed, parallels a distinction in the logic of argument, which countenances any finite number of objections or reformulations, but regards a critic who goes on raising objections or shifting his stance *ad infinitum* as merely captious, a sceptic to be ignored rather than a substantial opponent to be taken seriously.[13]

[13] See J.R.Lucas, "The Lesbian Rule", *Philosophy*, **30**, 1955, pp.195-213; and "Philosophy and Philosophy Of", *Proceedings of the British Academy*, **72**, 1986, pp.261-262.

The axiomatic approach brings out clearly the separate assumptions we commonly make about togetherness. Nevertheless, it fails to characterize topology adequately. The basic axioms highlight some features but leave out others, and have to be supplemented by a succession of further, often somewhat *ad hoc*, axioms of separation whose rationale is far from self-evident. And even then, we find that the characterization is still too wide, and we have to specify that it is the "standard topology" we have in mind. It would be better if, instead of postulating axioms, we could develop the crucial concepts of topology in terms of ordering relations alone, as the general reconciliation of the comparative ideal of denseness with the superlative one of nextness.[14]

### §10B.4 Whitehead's Programme

In the second decade of the Twentieth Century Whitehead attempted to ground topology and geometry in mereology, using "Extensive Abstraction" to define nearness.[15] If he had succeeded, we would have had a fourth volume of *Principia Mathematica*.

Whitehead worked with an informal version of the mereology outlined at the end of the previous chapter.[16] He laid down postulates which secure that $\succ$ and $\prec$ are dense and serial, and that $\succ$ is (lime-)tree-like.[17] "Abstractive classes" were a generalisation of Cauchy sequences and a predecessor of modern filters and directed sets, which could be used to define limits of regions in the same

[14] For an early plea to develop topology on these lines, see K.Menger, "Topology Without Points", *Rice Institute Pamphlets*, **27**, 1940, pp.80-107.

[15] A.N.Whitehead, *The Principles of Natural Knowledge*, Cambridge, 1919, §30, pp.101-106. I am indebted to T.J.Smiley, Fellow of Clare College, Cambridge, for drawing my attention to this.

[16] See above, §9.12; Whitehead's system is presented concisely and rigorously by P. Simons, *Parts*, Oxford, 1987, ch.2, §2.9.1, pp.81-86. Whitehead himself talks of 'events', $a, b, c, \ldots$ *etc*. But this term, though appropriate in view of his empiricist programme, is awkward for the present purpose, and will be replaced by 'region'; similarly, Whitehead's relation *extends over*, symbolized as $aKb$. which is naturally interpreted as *has as a proper part of itself*, $\succ$, will be rephrased in terms of $\prec$.

[17] See above, §9.6.

way as Cantor used nested intervals to define point-like real numbers. Whitehead defined an "abstractive class" as a set[18] of nested regions. That is, a set of regions was called an abstractive class, when

1. of any two of its members one was a proper part of the other, and
2. there was no region which was a proper part of every extension of the set.[19]

Part of Whitehead's motive in promoting the method of Extensive Abstraction was metaphysical. He was at that time working very closely with Russell, and hoped that abstractive classes would serve the purpose of grounding the mathematical concepts of point, line and surface, in four-dimensional regions—"events"—which could be understood in safely empiricist terms. But there were further, conceptual, goals his programme might well have been able to achieve, which are of greater concern to us here.

Abstractive classes converged to limits, which might be boundaries: boundaries would not have to be postulated as a disinct sort of entity, but could be defined in terms of regions—and in particular the common boundaries which two extensive magnitudes must share if they were to constitute together one unseparated whole.[20] Abstractive classes offered also a prospect of distinguishing limit elements of different dimensionality, and thus of defining dimensions, and distinguishing points, lines and surfaces, and carrying out suggestion (ii) in §2.5 of grounding geometrical concepts topologically. And granted that we could define points by means of abstractive classes, we should have a natural definition of a point's being in a region, namely that some member of an abstractive class converging on it as its limit, was itself part of the region in question; similarly, of a point's being outside a region, namely that some member of an abstractive class converging on it as its limit, did not itself overlap the region in question; and of a point's being on the boundary of a region, namely that every member of an abstractive class converging on it as its limit, overlapped the region in question without being a part of it. Abstractive classes, indeed,

[18] Whitehead does not explicitly require that the set be countable (*i.e.* of cardinality $\aleph_0$), though he implicitly assumes it.

[19] p.104.

[20] See below, §11.4. [AddRule]

could do duty for neighbourhoods, or open sets, and thus enable us to develop standard point-set topology.

Whitehead's key notion is that of one abstractive class's "covering" another abstractive class. If $\Phi$ and $\Psi$ are abstractive classes, $\Phi$ is said to cover $\Psi$ iff every member of $\Phi$ has some member of $\Psi$ being part of it; in symbols

$$\Phi \text{ covers } \Psi \quad \text{iff} \quad (\mathrm{A}x)(x\varepsilon\Phi \;\rightarrow\; (\mathrm{V}y)(y\varepsilon\Psi \wedge y \prec x)).$$

Clearly covering is a transitive and reflexive relation. We shall symbolize it by $\supseteq$.[21] When $\Phi$ covered $\Psi$ and $\Psi$ covered $\Phi$, they were said to be mutually covering each other, and mutual covering was an equivalence relation. Thus limit elements, such as points, lines, or surfaces, could be defined as equivalence classes of abstractive classes that mutually cover one another; and among such equivalence classes of abstractive classes, points could be characterized as those associated with abstractive classes which did not cover any abstractive class that did not also cover them.

Defining linear limits would be somewhat fiddly. An implicit definition would characterize a linear limit, $\mathcal{L}$, as an equivalence class of abstractive classes which covered punctiform abastractive classes (hereafter called points) without being covered by them, and which satisfied the further conditions:

1. for every pair of distinct points, $x$ and $y$, there was another point, $z$, such that:
   - (i) there was a linear limit which was itself covered by $\mathcal{L}$, and covered $x$ and $z$, but not $y$;
   - (ii) there was a linear limit which was itself covered by $\mathcal{L}$, and covered $y$ and $z$, but not $x$;
2. for every $z$ such that
   - (i) there was a linear limit which was itself covered by $\mathcal{L}$, and covered $x$ and $z$, but not $y$;
   - (ii) there was a linear limit which was itself covered by $\mathcal{L}$, and covered $y$ and $z$, but not $x$, and for every $w$ such that
   - (i) there was a linear limit which was itself covered by $\mathcal{L}$, and covered $x$ and $w$, but not $y$;

[21] Here we depart from our normal rule of having "less than" ordering relations, in order to avoid the risk of confusion with the set-theoretical relation $\subseteq$.

(ii) there was a linear limit which was itself covered by $\mathcal{L}$, and covered $y$ and $w$, but not $x$, either $w$ was covered by a linear limit which was itself covered by $\mathcal{L}$, and covered $x$ and $z$, but not $y$, or $w$ was covered by a linear limit which was itself covered by $\mathcal{L}$, and covered $y$ and $z$, but not $x$.

and similarly surfaces, solids, and higher-dimensional regions..

Since any two distinct points are defined by different equivalence classes of abstractive classe, each point will be in a region that does not contain the other point, and so we might hope to have the points constituting a Hausdorff space (or $T_2$-space)[22] and to be able to define two regions being pointwise connected if there were some abstractive class such that every one of its members overlapped with both regions, and two regions being completely connected if there were some abstractive class such that some of its members did not have as a part any region that did not overlap with either region. In this way we could reasonably hope to give definitions of limit elements, dimension and connectedness, in terms of generalised Cauchy sequences of regions (or some other sort of extensions) using just the relation of *part of* or its converse *having as a part of*. We should have achieved our aim of giving a non-arithmetical foundation to analysis and topology, in terms of mereology alone, itself characterized as a relational structure exemplifying a paradigm dense, serial, tree-like ordering relation.

Unfortunately Whitehead's programme failed. His definition did not exclude "pathological abstractive classes" which converge to a point that is outside, or on the boundary of, the regions of the abstractive class. The best example is the set of open intervals $(0, \delta_n)$, where $\delta_n$ tends to 0; this set satisfies Whitehead's definition, but fails to characterize the point [0] uniquely, since the set of open intervals $(-\delta_n, 0)$, also converges to 0, although no member of either set is a part of any member of the other.

There are many other examples of pathological abstractive classes which converge to a point that is outside, or on the boundary of, the regions of the abstractive class. With the loss of punctiformity[23] goes uniqueness. Different equivalence classes of mutually covering abstractive classes can be associated with the same point. Instead of a one-one relation, we have a many-one

[22] CHECK

[23] Whitehead speaks of abstractive classes being punctual: but punctuality is in contemoray usage a virtue of princes, not a property of points.

relation, and we therefore can no longer use abstractive classes to define points, because there is no longer a unique way of associating equivalence classes of abstractive classes with particular points. Besides the difficulty of *identifying* points, we are faced with a further difficulty of *characterizing* them. We have lost minimality. With genuine, that is to say non-pathological, abstractive classes, we can distinguish those equivalence classes that are associated with single points from those that are associated with other limit elements by their not covering any abstractive class that does not in turn cover them. But once we admit pathological abstractive classes, we can, except when confined to only one dimension, construct ever further ones, converging to the same point, which are covered by, but do not cover, other abstractive classes that also converge to the same point. We lose not only our criterion for distinguishing points from lines and higher-dimensional limit elements, but our criterion for distinguishing an abstractive class that converges to a single point from one that converges to two separate ones, and thus our ability to use abstractive classes to pick out connected regions from disconnected ones.

Whitehead may have been thinking in a circle. He seems to take it for granted that regions are connected. He does not discuss the problem of pathological extensive abstractive classes, but having introduced Extensive Abstraction, he does not use it to define connectedness; instead, he defines two regions' being connected in terms of a third region's overlapping them both without having any part disjoint from them both,[24] and later adopted an approach of de Laguna,[25] The fourth volume of *Principia Mathematica* was never written.

[24] "Two events (regions in my terminology) $x$ and $y$ are 'joined' when there is a third event (region) $z$ such that (i) $z$ intersects both $x$ and $y$, and (ii) there is a dissection of $z$ of which each member is a part of $x$, or of $y$, or of both." §29.1, p.102.

[25] T. de Laguna, "Point, Line and Surface as Sets of Solids", *Journal of Philosophy*, **19**, 1922, pp.449-461.

### §10B.5 Alternative Approaches

We can try to retrieve the situation in many ways. Whitehead later adopted a further suggestion of de Laguna, and reformulated his mereology in terms of a new primitive "extensive connection'.[26] Granted this concept, he could define the *part of* relation, since $x$ is a part of $y$ if and only if every $z$ that is connected with $x$ is also connected with $y$. He could thus recover the whole of his mereology, and could, furthermore, distinguish pathological abstractive classes—in which the successive regions are only "tangentially included" in their predecessors—from genuinely punctiform abstractive classes—in which every region that is connected with a member of the class overlaps its predecessors. Once pathological abstractive classes are ruled out, Whitehead's original programme could be re-instated, although acctually in *Process and Reality* he goes on to develop foundations of geometry and a theory of measurement in accordance with his special philosophical outlook. In a similar way, Bostock is able to define two regions being connected at only one point, and thence the concept of a point as the unique connexion of two regions that are pathwise connected at only one point.[27] Having defined points, he can go on to develop topology as the General Theory of Bounded Extensions (*i.e.* Connected, or Unseparated Regions). Somewhat similarly, Clark and Röper add a new primitive—$x$ *connects with* $y$, which Röper symbolizes by $\infty$—to the concepts of orthodox mereology, and postulates governing its use.[28]

All these approaches are useful in articulating what our concept of a region really involves, and interesting in showing how points can be introduced and identified in an entirely new way. But in invoking the concept of connectedness, they are presupposing some concept of togetherness or nextitude which it is the purpose of

[26] A.N.Whitehead, *Process and Reality*, Cambridge, 1929, Part IV, Chapter II, pp.416ff.

[27] In an unpublished monograph "Points".

[28] Bowman L.Clark CHECK, "A Calculus of Individuals Based on Connection", *Notre Dame Journal of Formal Logic,*, **22**, 1981, pp.204-218, esp. pt I, pp.204-208; and "Individuals and Points", *Notre Dame Journal of Formal Logic,*, **26**, 1985, pp.61-75. A useful summary is given in P. Simons, *Parts*, Oxford, 1987, §2.10.2, pp.94-98. Peter Röper "Region-based Topology", Forthcoming.[Should have forthcome by now: ask David]

this chapter to elucidate. They do succeed in grounding topoology in mereology, but develop a new discipline of "mereotopology", a discipline of considerable merits, but not providing foundational insights.[29]

We might be able to exclude pathological abstractive classes by strengthening the *part of* relation to be *embedded in* or *firmly within* or *nested in*,[30] and note that although it implies the *part of* relation, it is not conversely transitive: it may be the case that every portion that is *firmly within* $x$ is *firmly within* $y$, without $x$'s being either *firmly within* $y$ or identical with it. [Consider including a diagram.] But we lack a proof that this characterizes the relation uniquely, and an informal explication is open to the same objection that it assumes what we are seeking to define.

R.L. Moore sets out to provide a structure of regions, or "pieces" as he calls them, which will perform something of the same function as the family of open sets postulated by topologists.[31] He works with a transitive irreflexive relation, *being embedded in*, together with an axiom postulating a sequence of sets of portions, subject to conditions which impose a granular structure with a succession of sets of ever more finely grained portions, so that every portion, no matter how small, has some grain as a part of it, and provided we choose a set of sufficiently small grains, if one portion is embedded in another, all the grains that overlap the smaller will be themselves embedded in the larger. It is clear that this last postulate rules out pathological abstractive classes; each portion is given, as it were, a coating of ball bearings, which separate its boundary from that of any portion in which it is embedded. Whitehead's abstractive classes defined grains of ever smaller size, but

[29] See http://en.wikipedia.org/wiki/Mereotopology for an accessible account. See further A.C.Cohn and A.C.Varzi, "Mereotopological Connection", *Journal of Philosophical Logic*, Dordrecht, Boston, London, **32**, 2003, pp.357-390. for an overview of a wide variety of mereotopological theories.

[30] J.E.Tiles, *Things that Happen*, Aberdeen, 1981, §8, n.4. CHECK pp.40ff. and 53ff. A useful summary is given in P. Simons, *Parts*, Oxford, 1987, §2.10.2, pp.93-94. See also K.Menger, "Topology Without Points," *Rice Institute Pamphlets*, **27**, 1940, §§3-5, pp.84-96; J.Nicod, *Foundations of Geometry and Induction*, London, 1930, ch.4, pp.36-49. Wald, *Ergebnisse v. Math. Kolloquium, Vienna*, **3**, 1932, p.6.[CHECK]

[31] R.L.Moore, "A Set of Axioms for plane analysis situs", *Fundamenta Mathematicae*, **25**, 1935, 99.13-28.

not uniformly over the whole space. Moore has smaller and smaller portions everywhere, *and* sets of them, the sets being more or less ordered by "size", so that sufficiently small grains are everywhere available, to come between any portion and another it is embedded in. Moore's account has the virtue of strengthening Whitehead's programme just enough to make it work, but again, we are not really grounding the concepts of continuity and connectedness in simpler and more basic ones.

### §10B.6 Bool Plus

The attraction of Whitehead's programme was that it needed only the Boolean concepts of mereology and the quotifiers. But the quotifiers, we have seen,[32] are special cases of modal **S5** operators, and it seems reasonable to see whether some comparable enrichment of a Boolean system would serve our purpose. Kuratowski's axioms of closure are analogous to those of **S4**, and a mereology enriched by an operator governed by these axioms is as economical as Whitehead's mereology enriched by quotifiers, governed by **S5**-ish axioms.

In Kuratowski's approach, instead of a special property of sets—that of their being open—there is a special closure operator, which operates on sets to form sets, perhaps the same, perhaps different, which are said to be the closure of the original sets. The rules governing the closure operator show a striking similarity to those governing the modal operators in **S4**.[33] If we take a set's being included in another set as being like one proposition's implying another, then the closure operator is like the $\Diamond$ of modal logic, or what comes to the same thing, necessity, $\Box$, is like the interior operator, which is, so to speak, the dual of the closure operator.[34]

[32] Ch.9A, §9A.8.

[33] See above, ch.9A, §9A.8.

[34] Some care is needed, because the topological analogue of implication, $\rightarrow$, is being-a-subset-of, $\subseteq$, not being-a-superset-of, $\supseteq$. Since the interior of a set is always included in it, and a set is always included in its closure, $\Box$ has to be compared with the interior operator, and not, as might have been supposed, the closure operator. [Is this fn necessary? Was moved from §14.4, but the text itself may give sufficient guidance.][I think it is (12.3.08)]

If we think of the necessity operator as cutting out the penumbra of uncertainty, the parallel with the interior operator is illuminating.[35]

Admittedly, the rules governing the interior operator (or, equally, those governing the closure operator) are axioms, as are those governing the modal operator in **S4**. But, then, so are those governing quotifiers in standard First-order Logic. Minimal Boolean algebra needs some supplementation even for Mereology: to supplement similarly for topology is reasonable, provided the supplementation is similar and simple. And it is. Standard Boolean algebra has one monadic operator, complementation (or in Propositional Calculus, negation) with the rule that the complement of the complement of a portion is the original portion (compare the Rule of Double Negation in Propositional Calculus): the interior (or closure) operator is idempotent, that is to say, the interior of the interior of a portion is just the interior (or, the closure of its closure is just its closure), which is the other simplest rule for a monadic operator.

### §10B.7 How Different?

Once we can distinguish open and closed Boolean objects, we can re-instate Whitehead's programme. If we have an infinite sequence of regions, such the closure of each is a proper part of the interior of its predecessor, we avoid their all sharing a common boundary (or being only "tangentially included" in their predecessors). Each is firmly within its predecessor, and they all constitute a nested sequence with no region being part of them all.[36] We can once again distinguish punctiform abstactive classes from those converging to lines, surfaces or any other sort of boundary, by the minimality condition, that a punctiform abstactive class does not cover any abstractive class that it is not equally covered by. This allows that there can be different abstractive classes converging to the same point, but distinguishes those that converge to a single point from those that converge to more than one point, or to a line, or a surface. A point can now be securely defined in terms of the equivalence class of those abstractive classes that cover one another, and do not cover any abstractive class that does not cover them: it is

[35] See further, ch.14, §§14.3, 14.4. [ReCheck][ doesn't look right: Pattern recognition and Lakatos (12.3.08)]

[36] Compare Whitehead's definition 10, on p.421 of *Process and Reality*, Cambridge, 1929.

the limit to which each such abstractive class converges. An abstractive class converging to a line covers innumerable points, but is not covered by those converging to any of them. We thus are able both to introduce points and to have a criterion of dimensionality. Points are defined by equivalence classes of abstractive classes that themselves do not cover any abstractive class not coverd by them. And dimensions are defined recursively, that an abstractive class is of one higher dimensionality than that of the abstractive classes of the highest dimensionality which it covers, but is not covered by. We also have a criterion of point-wise connectedness: two portions are point-wise connected iff there is a punctiform abstractive class all of whose members overlap with both portions. We can strengthen this to two portions sharing a boundary of higher dimensionality; two portions are connected along a linear boundary iff there is an abstractive class all of whose members overlap with both portions, and which covers punctiform abstractive classes without being covered by them.

It might seem otiose to return to Whitehead, and to develop topology via abstractive classes, using the interior and closure operators solely to secure genuine punctiformity, rather than develop it directly from the axioms governing the prooperties of the closure operator, as Kuratowski himself does. We can. But Whitehead's original approach had the merit of basing topology not on some adventitious operator or family of sets, but on near and next. Whitehead's abstractive classes were "neighbourly", and gave a mereological account of portions being nearer and next to one another, and this legitimised his name of "regions", with their spatial overtones. Whitehead's account was intuitive, defined points in the same way as Cantor defines a real number, and offered a view of topology as a natural generalisation of analysis. It ought to work. It did not because of a lack of linearity. The rational numbers are linearly ordered, and so we can secure that one set is nested within another by specifying the two end-points individually, and that its lower one is greater, and its upper one is less, than the corresponding end-points of the other. But the *part of* relation gives only a partial ordering, and in the absence of any other available linear ordering, we couldnot specify a finite number of possible boundary points, and lay down that at none of them does the smaller region extend that far. Hence we couldnot secure, by means of the *part of* relation, that if one portion was a proper part of another, it was nested within it.

A similar difficulty besets Dedekind's approach. Where Cantor goes for the limiting smallest, Dedekind cinsiders the least upper bound. The rational numbers are linearly ordered, and so there is a unique least among the upper bounds; but in the absence of any additional linear ordering, and with only the partial ordering of *part of*, there may be many minimal upper bounds, without there being a single minimum one, and where there is a lowest upper bound, it may not be everywhere different from the lowest upper bound of another portion of which it is a proper part. So, once again, we are unable, in the absence of a further linear ordering, to secure complete difference between the boundaries of a portion and those of its proper parts.

The effect of the closure and interior operators is to to make up for the absence of linearity, and to provide each portion with a rim, so to speak, which enables us to specify that the outside of one will go into the inside of the other, and be properly nested within it. For this to work, it is necessary that the closure and interior operators have some ordering properties with respect to the *part of* relation. They do, but minimally so. There is a close analogue with the point made in §9A.7 about Axiom D. $\mathcal{I}$ and $\mathcal{C}$ have ordering properties with respect to the antisymmetric *improper part of* relation, that is *equal to or proper part of* relation, $\preceq$: $\mathcal{I}\S \preceq \S$ and $x \preceq \mathcal{C}\S$. Often the equality holds: with open regions in the former case, and closed ones in the latter. But they do not both hold. In all cases $\mathcal{I}\S \prec \mathcal{C}\S$. This parallels the necessary constraint in Modal logic, that Axiom D $\vdash \Box p \rightarrow \Diamond p$ must hold, but its converse $\not\vdash \Diamond p \rightarrow \Box p$ must not, on pain of degeneracy.[37] In Modal logic, if we introduce an extra monadic operator, $\Box$, we need to specify how it interacts with implication. Whether or not we have an axiom T, which can be expressed with the antisymmetric implication $\rightarrow$, we need to have D: $\vdash \Box p \Rightarrow \Diamond p$. This is to say that $\Box$ and $\Diamond$ have implicatory force, but minimally so, since it is weaker than it would be if T' $\vdash \Box p \Rightarrow p$ and T*' $\vdash p \Rightarrow \Diamond p$. Implicationally, $\Box p$ is a sort of *least upper bound* of $\Diamond p$. Similarly with the (proper) *part of* relation, $\prec$, $\mathcal{I}\S \prec \mathcal{C}\S$, and $\mathcal{C}\S$ is is a sort of *least larger region* of $\mathcal{I}\S$. The operators $\mathcal{C}$ and $\mathcal{I}$, like the operators $\Box$ and $\Diamond$, make a difference, but in an important sense, the least possible difference consistent with its being a difference at all.

[37] This is where it is again easy to be confused by $\rightarrow$ which looks like an irreflexive asymmetric ordering relation, but is in fact reflexive and antisymmetric. As in §9A.7 the symbol $\Rightarrow$ represents an irreflexive asymmetric implication, where $\vdash p \Rightarrow \neg p$, and $\vdash (p \Rightarrow q) \Rightarrow \neg(q \Rightarrow p)$.