Course Outline for A3 Dynamics and Kinematics (8 lectures)

Lecturer: Dr Amy Zavatsky **Michaelmas Term 2004**

Expanded syllabus:

Dynamics: Angular momentum (general definition), rigid body motion with rotation and translation. Kinematics: Velocity and acceleration; motion in rotating frames of reference. Mechanisms; instantaneous centres, calculation of velocity and acceleration (including velocity and acceleration diagrams), simple, compound and epicyclic gear trains (calculation of gear ratios and torques, effective inertia).

Lecture Synopsis:

1. *Dynamics of rigid bodies: plane kinematics*

Translation (rectilinear, curvilinear), rotation about a fixed axis, and general plane motion. Linear and angular velocity and acceleration. Absolute and relative velocity and acceleration. Instantaneous centre of rotation. Rate of change of a vector with respect to a rotating coordinate frame. General motion in a rotating frame of reference.

2. *Dynamics of rigid bodies: plane kinetics*

Force, mass, and acceleration. General equations of motion. Free-body diagram, equivalent force-couple, kinetic diagram. Angular momentum about a fixed point, about the mass centre, about an arbitrary point. Work and energy (translational and rotational), power, impulse and momentum (linear and angular). General problem formulation and review.

3. *Linkages and mechanisms: basic concepts*

Definitions: mechanism, linkage (or kinematic chain), types of motion transformation, open-loop versus closed-loop, lower and higher joints (or pairs), mobility (or degrees-of-freedom). Grübler's equation for mobility. The four-bar linkage (input, output, coupler, transmission angle) and Grashof's law. Kinematic inversion. Introduction to slider-crank mechanism and quick-return mechanism. Analysis versus synthesis.

4. *Linkages and mechanisms: position and velocity*

Relative velocity (link with revolute joint, link with revolute joint and slider). Velocity diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Velocity image. Instantaneous centres and centrodes. Kennedy's theorem for three instantaneous centres. Velocity pole method for finding angular velocities in linkages (four-bar linkage, slider-crank).

5. *Linkages and mechanisms: acceleration*

Relative acceleration (link with revolute joint, link with revolute joint and slider). Acceleration diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Acceleration image. Relative acceleration of coincident particles at the point of contact of rolling elements. Equivalent mechanisms.

6. *Linkages and mechanisms: computational analysis*

Kinematic analysis by complex numbers (link with revolute joint, link with revolute joint and slider). Loop closure equations (slider-crank, quick-return mechanism). Solution of the position problem using the Newton-Raphson method. Vector chains to describe motion of an arbitrary point on a mechanism. Introduction to generalised coordinates and constraint equations.

7. *Linkages and mechanisms: forces in linkages*

Review of two-force bodies, three-force bodies, and couples. Static forces (slider-crank, four-bar linkage). D'Alembert's principle (inertia force and concept of dynamic equilibrium) in contrast to direct "F=ma" approach. Dynamic analysis (four-bar linkage, slider-crank). Virtual work method to find forces and torques. Kinetic energy and equivalent inertia.

8. *Gear trains: simple, compound, and epicyclic gear trains*

Friction drives versus gears. Examples of spur gears, rack and pinion, bevel gears, and worm gears. Spur gear terminology. Fundamental law of gearing (circular gears). Gear trains: simple, compound, planetary. Equivalent inertia.

Examples Sheets: 3A3C, 3A3D

Learning Outcomes:

1. Ability to calculate absolute and relative positions, velocities, and accelerations for two-dimensional rigid body motions in fixed, translating, and rotating frames of reference.
2. Ability to calculate for a rigid body moving in two dimensions the angular momentum with respect to a fixed point, to the mass centre of the body, or to any arbitrary point.
3. Ability to solve dynamics problems for two-dimensional rigid body motions including both translation and rotation.
4. Ability to identify the type of motion produced by a mechanism and to calculate its mobility.
5. Ability to find the instantaneous centres for the various components of a mechanism.
6. Ability to calculate the relative positions, velocities and accelerations of the various components of a mechanism using vector methods, diagrammatic methods, and complex numbers / loop closure equations.
7. Ability to calculate the static and dynamic forces within a mechanism and the equivalent inertia for a mechanism.
8. Ability to calculate the gear ratios, torques, and effective inertia for simple, compound, and planetary gear trains.

Primary Reference 1 **Lectures 1, 2**

Engineering Mechanics: Dynamics
JL Meriam and LG Kraige.
John Wiley & Sons, 1998 (4^{th} ed.)

DYNAMICS OF PARTICLES (Prelims P4)

1 Introduction to Dynamics
2 Kinematics of Particles
3 Kinetics of Particles
  Force, Mass, Acceleration
  Work and Energy
  Impulse and Momentum
4 Kinetics of Systems of Particles

DYNAMICS OF RIGID BODIES
(Prelims P4, Finals A3)

5 Plane Kinematics of Rigid Bodies
  Absolute, relative velocities and accelerations
  Translating and rotating axes
6 Plane Kinetics of Rigid Bodies
  Force, Mass, Acceleration
  Work and Energy
  Impulse and Momentum

7 Intro to Three-Dimensional Rigid Body Dynamics
  (Finals B1)
8 Vibrations and Time Response (Finals A3)

Plane Kinematics of Rigid Bodies

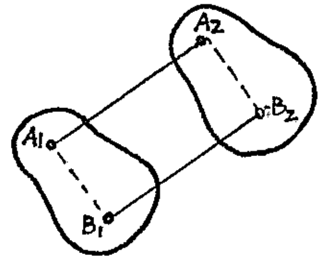
Three types of planar rigid-body motion

1. *Translation*

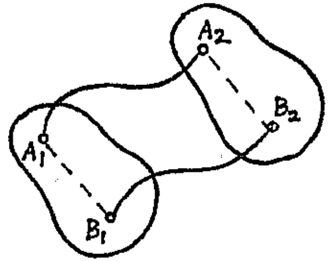
Any straight line inside the body keeps the same direction during the motion.

All the particles forming the body move along parallel paths.

Rectilinear translation: paths are straight lines



Curvilinear translation: paths are curved lines

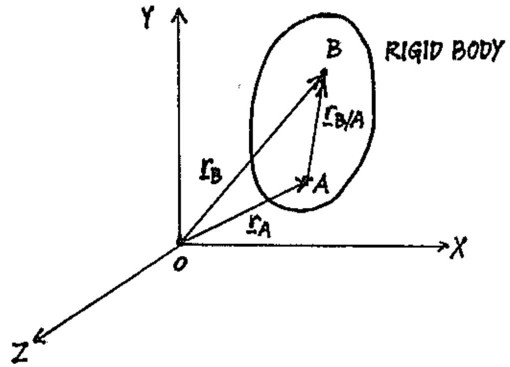


When a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant.

$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$

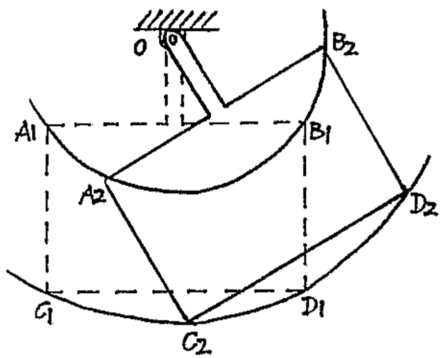
$\mathbf{v}_B = \mathbf{v}_A$

$\mathbf{a}_B = \mathbf{a}_A$

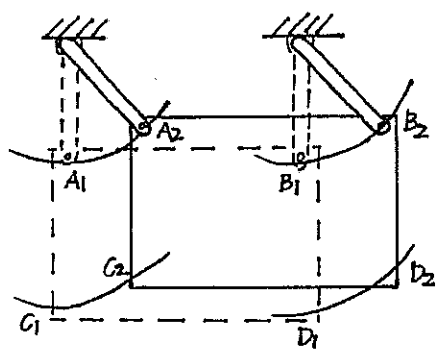


Remember that position, velocity, and acceleration are vectors and so have both magnitude and direction.

Rotation should not be confused with certain types of curvilinear translation.



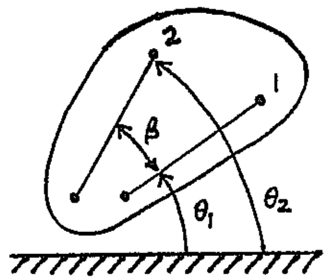
ROTATION



CURVILINEAR TRANSLATION

2. Rotation about a fixed axis

The rotation of a rigid body is described by its angular motion.



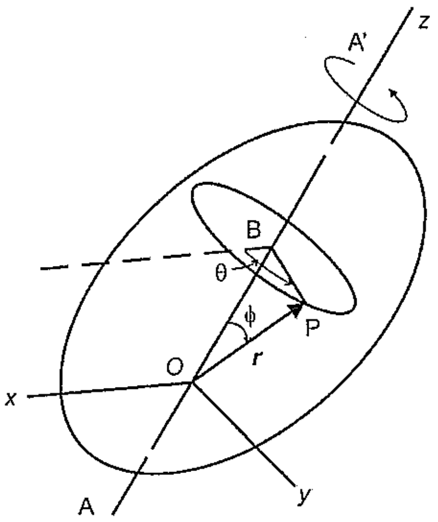
$$\theta_2 = \theta_1 + \beta$$

$$\dot{\theta}_2 = \dot{\theta}_1 \qquad \ddot{\theta}_2 = \ddot{\theta}_1 \qquad \Delta\theta_2 = \Delta\theta_1$$

All lines on a rigid body in its plane of motion have the same angular displace-ment, the same angular velocity, and the same angular acceleration.

During rotation about a fixed axis, the particles forming the body move in parallel planes along circles centred on the same fixed axis.

If the "axis of rotation" intersects the rigid body, then the particles located on the axis have zero velocity and zero acceleration.

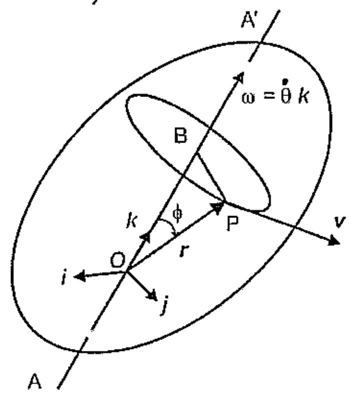


$\Delta s = (BP)\,\Delta\theta = (r \sin\phi)\,\Delta\theta$

Dividing by Δt *and taking the limit as* Δt *goes to zero,*

$v = r\,\dot{\theta}\sin\phi$

In terms of vectors,



Angular velocity is $\boldsymbol{\omega} = \omega\,\mathbf{k} = \dot{\theta}\mathbf{k}$.

$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$

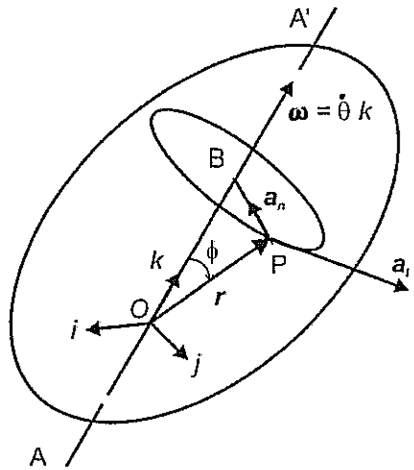
Magnitude $|\mathbf{v}| = |\boldsymbol{\omega}|\,|\mathbf{r}|\sin\phi$, *as before*
Direction - tangent to the circular path

Remember the right-hand rule for vector cross products.

$\mathbf{a} = \dot{\mathbf{v}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}$

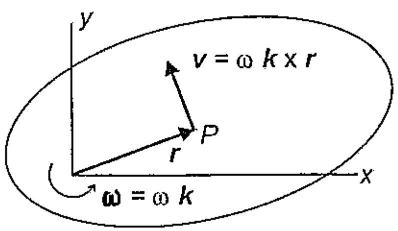
$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}$ *where* $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$

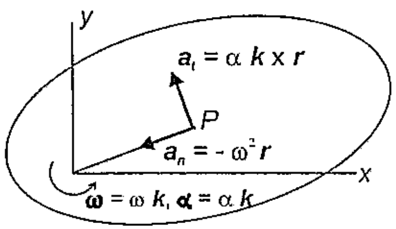
$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$



$$\mathbf{a} = \underbrace{\boldsymbol{\alpha} \times \mathbf{r}}_{\text{tangential component}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{normal component}}$$

The rotation of a rigid body about a fixed axis may be described by the motion of a representative slab in a reference plane perpendicular to the axis of rotation.



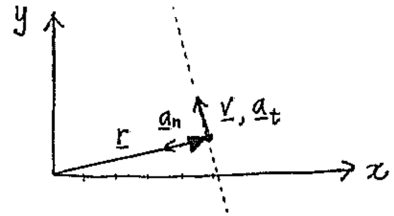


EXAMPLE

$\underline{\omega} = 4\underline{k}$

$\dot{\underline{\omega}} = \underline{\alpha} = -2\underline{k}$

$\underline{r} = 5\underline{i} + \underline{j}$



Velocity

$\underline{v} = \underline{\omega} \times \underline{r}$

$\underline{v} = 4\underline{k} \times (5\underline{i} + \underline{j}) = (4\underline{k} \times 5\underline{i}) + (4\underline{k} \times \underline{j})$

$\underline{v} = 20\underline{j} + 4(-\underline{i})$

$\underline{v} = -4\underline{i} + 20\underline{j}$

$$\underline{v} = \begin{vmatrix} i & j & k \\ 0 & 0 & 4 \\ 5 & 1 & 0 \end{vmatrix} = \underline{i}(0-4) - \underline{j}(0-20) + \underline{k}(0-0)$$

$\underline{v} = -4\underline{i} + 20\underline{j}$ tangential $\perp \underline{r}$

Acceleration

$\underline{a} = \underline{\alpha} \times \underline{r} + \underline{\omega} \times \underline{\omega} \times \underline{r} = \underline{\alpha} \times \underline{r} + \underline{\omega} \times \underline{v}$

$\underline{a} = -2\underline{k} \times (5\underline{i} + \underline{j}) + 4\underline{k} \times (-4\underline{i} + 20\underline{j})$

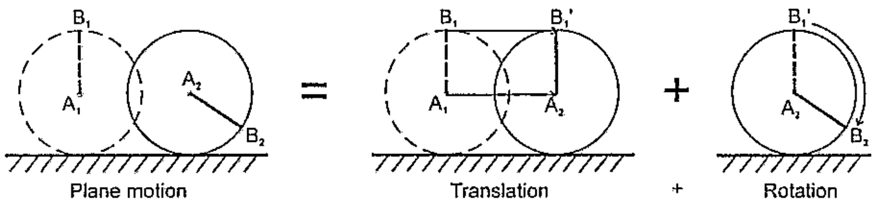
$\underline{a} = -10\underline{j} - 2(-\underline{i}) + (-16)\underline{j} + 80(-\underline{i})$

$$\underline{a} = \underbrace{2\underline{i} - 10\underline{j}}_{\text{tangential}} - \underbrace{80\underline{i} - 16\underline{j}}_{\text{radial}}$$

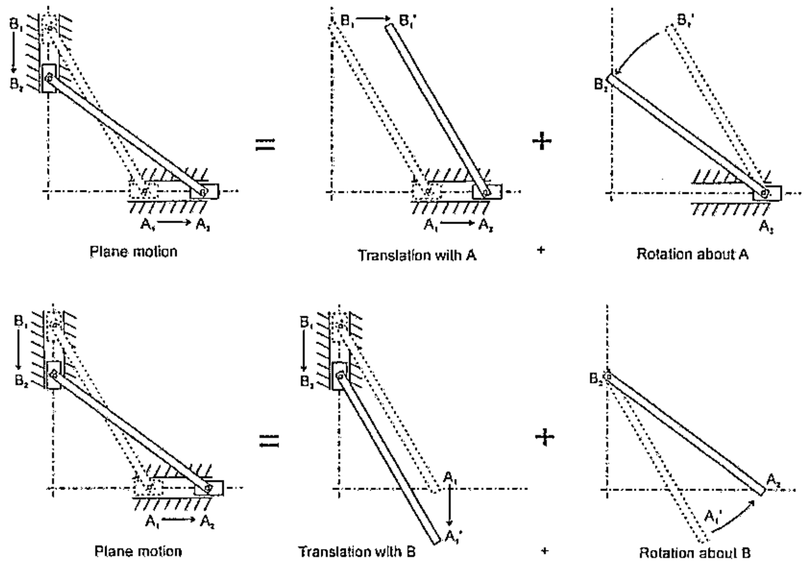
3. General plane motion

A general plane motion may always be considered as the sum of a translation and a rotation.

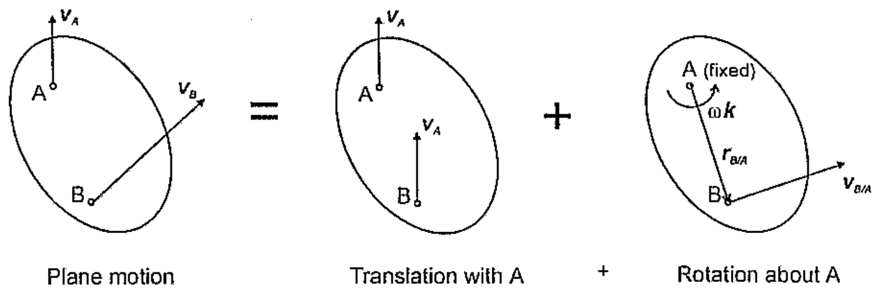
Wheel rolling on a straight track



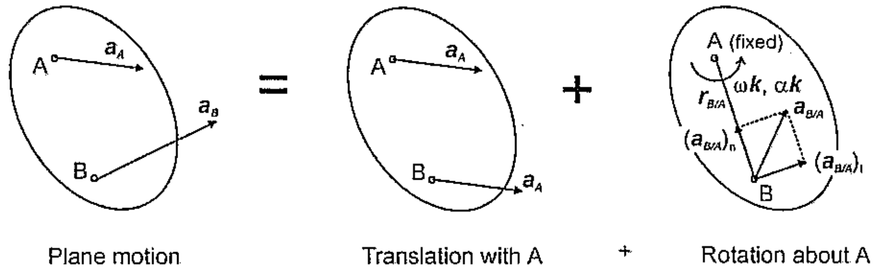
Rod whose ends slide, respectively, along a horizontal and a vertical track



Absolute and relative velocity



$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad \mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$



$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

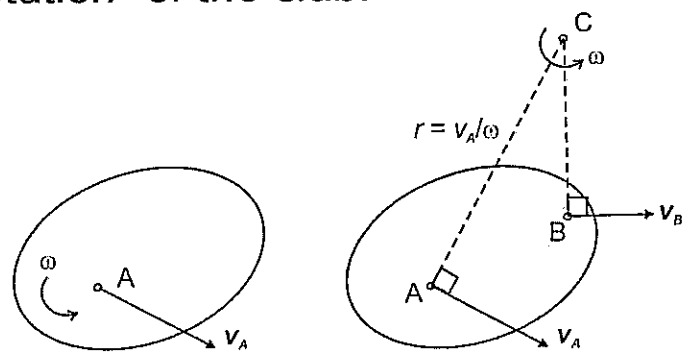
tangential $(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A}$ $\quad (a_{B/A})_t = r\alpha$

radial $(\mathbf{a}_{B/A})_r = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$ $\quad (a_{B/A})_r = \omega^2 r$

Instantaneous centre of rotation

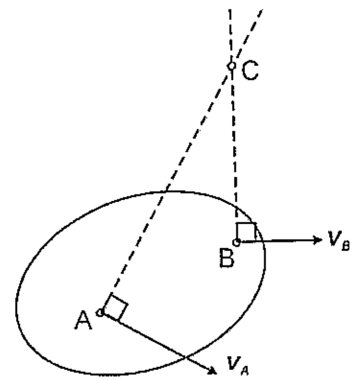
Consider the general plane motion of a "slab". At any given instant, the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis perpendicular to the plane of the slab, called the "instantaneous axis of rotation".

The axis intersects the plane of the slab at a point C, called the "instantaneous centre of rotation" of the slab.

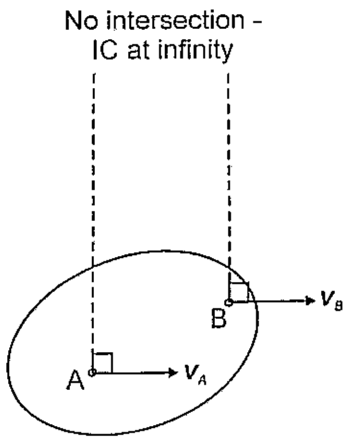


$v_A = (CA)\,\omega \qquad v_B = (CB)\,\omega$

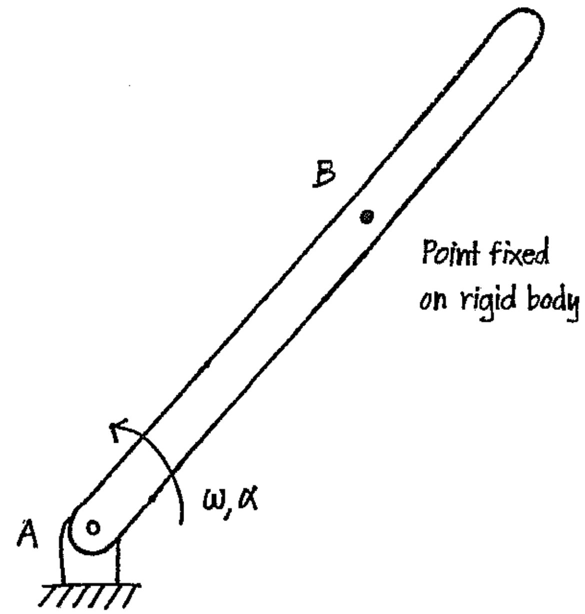
If the directions of the velocities of two particles A and B of the slab are known and if they are different, the instantaneous centre C is obtained by drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B and determining the point at which the two lines intersect.

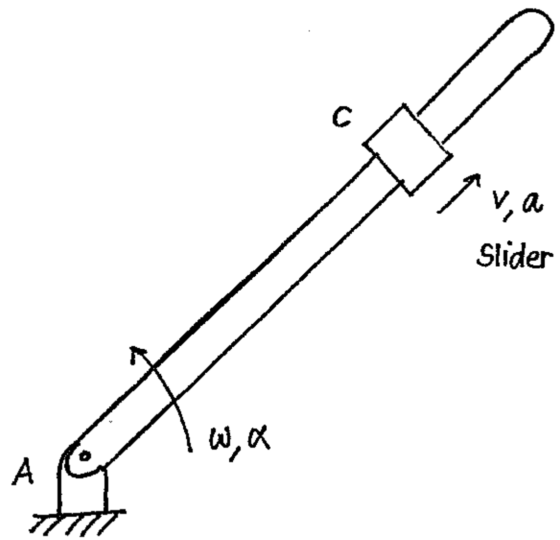


If $\mathbf{v}_A$ and $\mathbf{v}_B$ are parallel, the instantaneous centre C would be an infinite distance away and ω would be zero. All points on the slab would have the same velocity.



In general, the point C **does not** have zero acceleration, and therefore,
the accelerations of the various particles of the slab **cannot** be determined as if the slab were rotating about C.





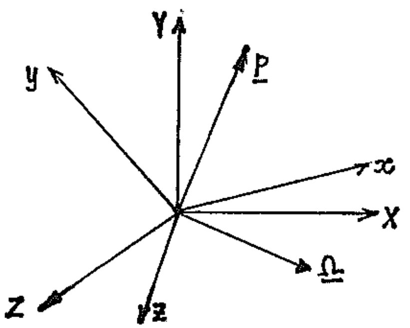
Rate of change of a vector with respect to a rotating frame

Fixed coordinate system XYZ
with unit vectors **I**, **J**, **K**

Moving coordinate system xyz
with unit vectors **i**, **j**, **k**

xyz rotates with angular velocity Ω

Consider the vector **P**, whose direction and magnitude change with time.



Relative to XYZ, $\mathbf{P} = P_X\,\mathbf{I} + P_Y\,\mathbf{J} + P_Z\,\mathbf{K}$

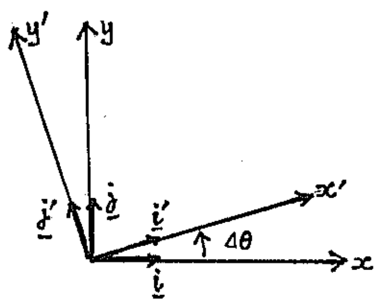
$$\dot{\mathbf{P}} = \dot{P}_X\,\mathbf{I} + \dot{P}_Y\,\mathbf{J} + \dot{P}_Z\,\mathbf{K} + \underbrace{P_X\,\dot{\mathbf{I}} + P_Y\,\dot{\mathbf{J}} + P_Z\,\dot{\mathbf{K}}}_{0}$$

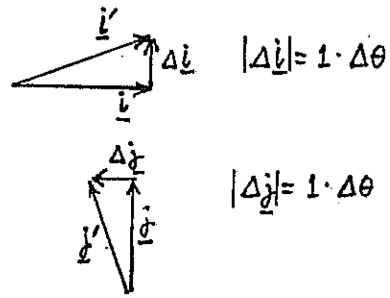
Relative to xyz,

$$\mathbf{P} = P_x\,\mathbf{i} + P_y\,\mathbf{j} + P_z\,\mathbf{k}$$

$$\dot{\mathbf{P}} = (\dot{P}_x\,\mathbf{i} + \dot{P}_y\,\mathbf{j} + \dot{P}_z\,\mathbf{k}) + (P_x\,\dot{\mathbf{i}} + P_y\,\dot{\mathbf{j}} + P_z\,\dot{\mathbf{k}})$$

$$\dot{\mathbf{P}} = \left(\frac{d\mathbf{P}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{P}$$

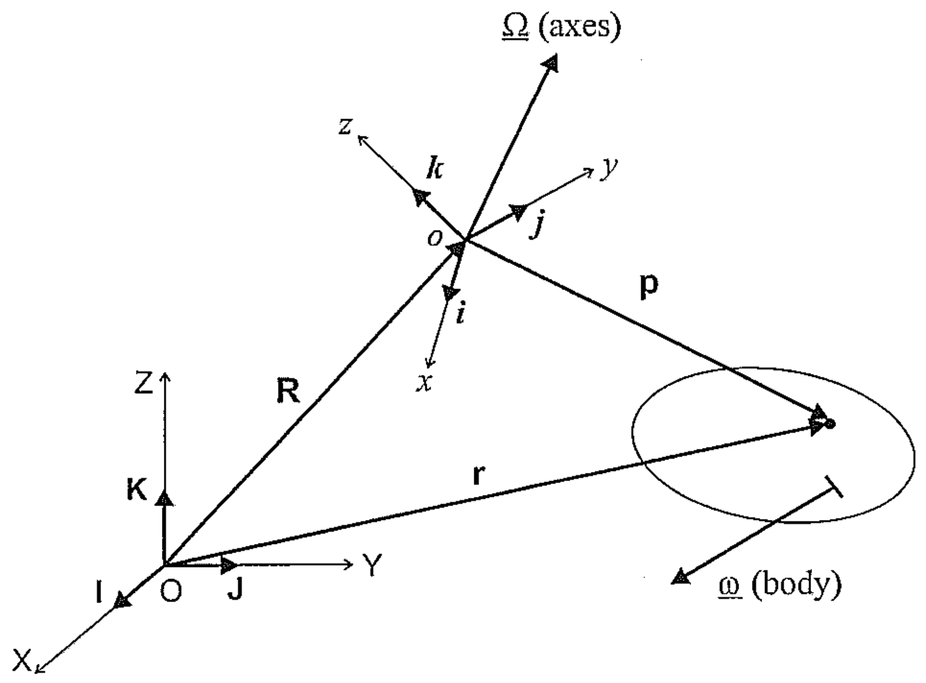




$$\left(\frac{d\mathbf{i}}{dt}\right) = \omega\,\mathbf{j} \qquad \left(\frac{d\mathbf{j}}{dt}\right) = -\,\omega\,\mathbf{i}$$

$$\dot{\mathbf{i}} = \omega\mathbf{k} \times \mathbf{i} \qquad \dot{\mathbf{j}} = \omega\mathbf{k} \times \mathbf{j}$$

Motion relative to a rotating frame



Fixed frame XYZ with unit vectors (**I J K**)
Moving frame *xyz* with unit vectors (*i j k*)

$\underline{\Omega}$ = angular velocity of *xyz* frame
$\underline{\omega}$ = angular velocity of body

r = R + p

$$\underline{r} = \underline{R} + \underline{\rho}$$

$$\frac{d\underline{r}}{dt} = \frac{d\underline{R}}{dt} + \left(\frac{d\underline{\rho}}{dt}\right)_{XYZ}$$

$$\underbrace{\frac{d\underline{r}}{dt}}_{\dot{r}} = \underbrace{\frac{d\underline{R}}{dt}}_{\dot{R}} + \underbrace{\left(\frac{d\underline{\rho}}{dt}\right)_{xyz}}_{\dot{\rho}} + \underbrace{\underline{\Omega}\times\underline{\rho}}_{\Omega\rho}$$

$$\frac{d^2\underline{r}}{dt^2} = \frac{d^2\underline{R}}{dt^2} + \frac{d}{dt}\left[\left(\frac{d\underline{\rho}}{dt}\right)_{xyz}\right]_{XYZ} + \frac{d}{dt}\left[\underline{\Omega}\times\underline{\rho}\right]_{XYZ}$$

$$\frac{d^2\underline{r}}{dt^2} = \frac{d^2\underline{R}}{dt^2} + \left(\frac{d^2\underline{\rho}}{dt^2}\right)_{xyz} + \underline{\Omega}\times\left(\frac{d\underline{\rho}}{dt}\right)_{xyz} + \dot{\underline{\Omega}}\times\underline{\rho} + \underline{\Omega}\times\left(\frac{d\underline{\rho}}{dt}\right)_{XYZ}$$

$$\frac{d^2\underline{r}}{dt^2} = \frac{d^2\underline{R}}{dt^2} + \left(\frac{d^2\underline{\rho}}{dt^2}\right)_{xyz} + \underline{\Omega}\times\left(\frac{d\underline{\rho}}{dt}\right)_{xyz} + \dot{\underline{\Omega}}\times\underline{\rho} + \underline{\Omega}\times\left[\left(\frac{d\underline{\rho}}{dt}\right)_{xyz} + \underline{\Omega}\times\underline{\rho}\right]$$

$$\underbrace{\frac{d^2\underline{r}}{dt^2}}_{\ddot{r}} = \underbrace{\frac{d^2\underline{R}}{dt^2}}_{\ddot{R}} + \underbrace{\left(\frac{d^2\underline{\rho}}{dt^2}\right)_{xyz}}_{\ddot{\rho}} + \underbrace{\dot{\underline{\Omega}}\times\underline{\rho}}_{\dot{\Omega}\rho} + \underbrace{2\underline{\Omega}\times\left(\frac{d\underline{\rho}}{dt}\right)_{xyz}}_{\substack{\text{Coriolis} \\ 2\Omega\dot{\rho}}} + \underbrace{\underline{\Omega}\times\underline{\Omega}\times\underline{\rho}}_{\Omega^2\rho}$$

If the moving coordinate system is fixed in the rigid body,

$\underline{\Omega} = \underline{\omega}$ and $|\underline{\rho}|$ = constant.

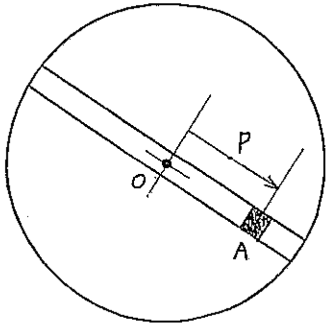
The equations become

$$\underline{r} = \underline{R} + \underline{\rho}$$

$$\dot{\underline{r}} = \dot{\underline{R}} + \underline{\omega}\times\underline{\rho}$$

$$\ddot{\underline{r}} = \ddot{\underline{R}} + \dot{\underline{\omega}}\times\underline{\rho} + \underline{\omega}\times\underline{\omega}\times\underline{\rho}$$

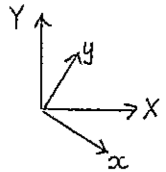
Disk with radial slot rotates about O with a counterclockwise angular velocity of 4 rad/s which is decreasing at the rate of 10 rad/s².



The motion of slider A is separately controlled and at this instant $p = 150$ mm, $\dot{p} = 125$ mm/s, and $\ddot{p} = 2025$ mm/s².

Determine the absolute velocity and acceleration of A for this position.

(Meriam & Kraige)



XY fixed with origin at O

xy moving, origin at O, aligned along slot

$\underline{r}_A = \underline{R} + \underline{p}_A = \underline{0} + 150\,\underline{i} = 150\,\underline{i}$ (mm)

Velocity

$\dot{\underline{r}} = \dot{\underline{R}} + \dot{\underline{p}} + \underline{\Omega} \times \underline{p} = \underline{0} + 125\,\underline{i} + (4\,\underline{k} \times 150\,\underline{i}) = 125\,\underline{i} + 600\,\underline{j}$ (mm/s)

Acceleration

$\ddot{\underline{r}} = \ddot{\underline{R}} + \ddot{\underline{p}} + \dot{\underline{\Omega}} \times \underline{p} + \underline{\Omega} \times \underline{\Omega} \times \underline{p} + 2\,\underline{\Omega} \times \dot{\underline{p}}$

$\ddot{\underline{R}} = \underline{0}$

$\ddot{\underline{p}} = 2025\,\underline{i}$

$\dot{\underline{\Omega}} \times \underline{p} = -10\,\underline{k} \times 150\,\underline{i} = -1500\,\underline{j}$

$\underline{\Omega} \times \underline{\Omega} \times \underline{p} = 4\,\underline{k} \times 4\,\underline{k} \times 150\,\underline{i} = 4\,\underline{k} \times 600\,\underline{j} = -2400\,\underline{i}$

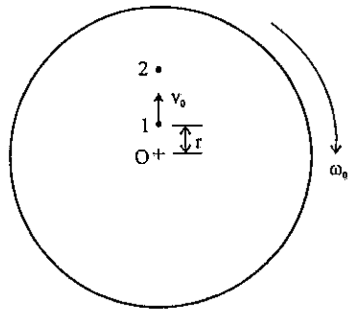
$2\,\underline{\Omega} \times \dot{\underline{p}} = 2(4\,\underline{k}) \times 125\,\underline{i} = 1000\,\underline{j}$

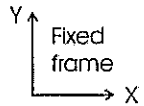
$\ddot{\underline{r}} = -375\,\underline{i} - 500\,\underline{j}$ (mm/s²)

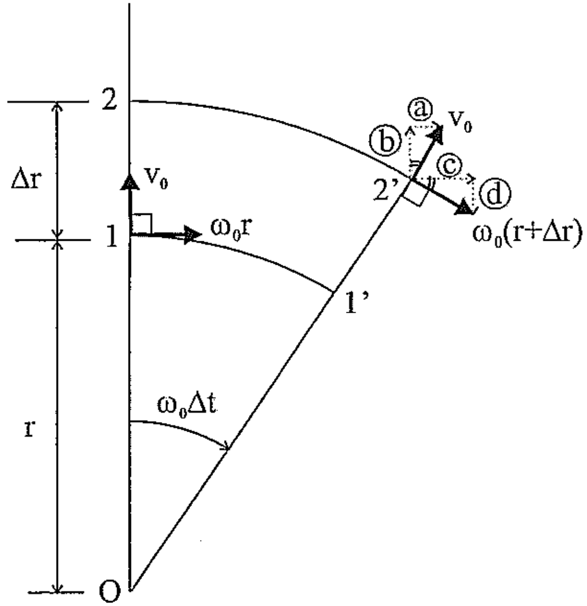
ABZ

Centripetal and Coriolis Acceleration

- A turntable is rotating at a constant angular velocity ω_0 about its fixed centre O.
- An ant (modelled as a point) is walking along a radius of the turntable at a constant velocity v_0 relative to the turntable.
- The ant is currently at point 1 and wants to go to the nearby point 2.
- We wish to find the acceleration of the ant relative to a fixed reference frame XYZ.







- JH Williams "Fundamentals of Applied Dynamics"

Acceleration in the radial direction (the fixed Y direction)

ΔV_{radial} = [New radial velocity] - [Old radial velocity]

ΔV_{radial} = [Term (b) + Term (d)] - $[V_0]$

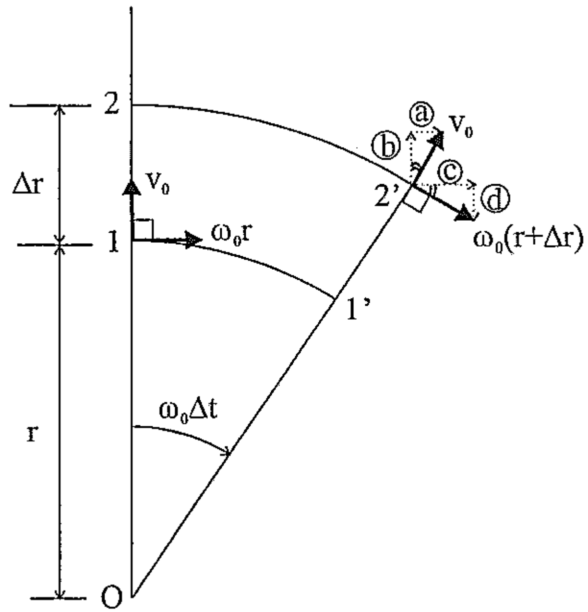
$\Delta V_{radial} = [V_0 \cos(\omega_0 \Delta t) - \omega_0 (r+\Delta r) \sin(\omega_0 \Delta t)] - [V_0]$

For a small time increment Δt, $\sin(\omega_0 \Delta t) \approx \omega_0 \Delta t$ and $\cos(\omega_0 \Delta t) \approx 1$

$\Delta V_{radial} = [V_0 - \omega_0^2 r \Delta t - \omega_0^2 \Delta r \Delta t] - [V_0]$

$\Delta V_{radial} = -\omega_0^2 r \Delta t - \omega_0^2 V_0 (\Delta t)^2$ since $\Delta r = V_0 \Delta t$

$a_{radial} = \lim_{\Delta t \to 0} \dfrac{\Delta V_{radial}}{\Delta t} = -\omega_0^2 r$ CENTRIPETAL ACCELERATION (note direction: towards O)



Acceleration in the tangential direction (the fixed X direction)

$\Delta V_{tangential}$ = [New tangential velocity] - [Old tangential velocity]

$\Delta V_{tangential}$ = [Term (a) + Term (c)] - $[\omega_0 r]$

$\Delta V_{tangential} = [V_0 \sin(\omega_0 \Delta t) + \omega_0 (r+\Delta r) \cos(\omega_0 \Delta t)] - [\omega_0 r]$

For a small time increment Δt, $\sin(\omega_0 \Delta t) \approx \omega_0 \Delta t$ and $\cos(\omega_0 \Delta t) \approx 1$

$\Delta V_{tangential} = [V_0 \omega_0 \Delta t + \omega_0 r + \omega_0 \Delta r] - [\omega_0 r]$

$\Delta V_{tangential} = V_0 \omega_0 \Delta t + \omega_0 V_0 \Delta t$ since $\Delta r = V_0 \Delta t$

$\Delta V_{tangential} = 2 \omega_0 V_0 \Delta t$

$a_{tangential} = \lim_{\Delta t \to 0} \dfrac{\Delta V_{tangential}}{\Delta t} = 2 \omega_0 V_0$ CORIOLIS ACCELERATION

- Note that the Coriolis acceleration is independent of the position r.
- Note that the effect of ω_0 in changing the ORIENTATION of V_0 (Term (a)) is exactly the same as the effect of V_0 in carrying $\omega_0 r$ to a different radius, changing its MAGNITUDE (Term (c)).