Course Outline for A3 Dynamics and Kinematics (8 lectures)

Lecturer: Dr Amy Zavatsky **Michaelmas Term 2004**

Expanded syllabus:

Dynamics: Angular momentum (general definition), rigid body motion with rotation and translation. Kinematics: Velocity and acceleration; motion in rotating frames of reference. Mechanisms; instantaneous centres, calculation of velocity and acceleration (including velocity and acceleration diagrams), simple, compound and epicyclic gear trains (calculation of gear ratios and torques, effective inertia).

Lecture Synopsis:

1. *Dynamics of rigid bodies: plane kinematics*

Translation (rectilinear, curvilinear), rotation about a fixed axis, and general plane motion. Linear and angular velocity and acceleration. Absolute and relative velocity and acceleration. Instantaneous centre of rotation. Rate of change of a vector with respect to a rotating coordinate frame. General motion in a rotating frame of reference.

2. *Dynamics of rigid bodies: plane kinetics*

Force, mass, and acceleration. General equations of motion. Free-body diagram, equivalent force-couple, kinetic diagram. Angular momentum about a fixed point, about the mass centre, about an arbitrary point. Work and energy (translational and rotational), power, impulse and momentum (linear and angular). General problem formulation and review.

3. *Linkages and mechanisms: basic concepts*

Definitions: mechanism, linkage (or kinematic chain), types of motion transformation, open-loop versus closed-loop, lower and higher joints (or pairs), mobility (or degrees-of-freedom). Grübler's equation for mobility. The four-bar linkage (input, output, coupler, transmission angle) and Grashof's law. Kinematic inversion. Introduction to slider-crank mechanism and quick-return mechanism. Analysis versus synthesis.

4. *Linkages and mechanisms: position and velocity*

Relative velocity (link with revolute joint, link with revolute joint and slider). Velocity diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Velocity image. Instantaneous centres and centrodes. Kennedy's theorem for three instantaneous centres. Velocity pole method for finding angular velocities in linkages (four-bar linkage, slider-crank).

5. *Linkages and mechanisms: acceleration*

Relative acceleration (link with revolute joint, link with revolute joint and slider). Acceleration diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Acceleration image. Relative acceleration of coincident particles at the point of contact of rolling elements. Equivalent mechanisms.

6. *Linkages and mechanisms: computational analysis*

Kinematic analysis by complex numbers (link with revolute joint, link with revolute joint and slider). Loop closure equations (slider-crank, quick-return mechanism). Solution of the position problem using the Newton-Raphson method. Vector chains to describe motion of an arbitrary point on a mechanism. Introduction to generalised coordinates and constraint equations.

7. *Linkages and mechanisms: forces in linkages*

Review of two-force bodies, three-force bodies, and couples. Static forces (slider-crank, four-bar linkage). D'Alembert's principle (inertia force and concept of dynamic equilibrium) in contrast to direct "F=ma" approach. Dynamic analysis (four-bar linkage, slider-crank). Virtual work method to find forces and torques. Kinetic energy and equivalent inertia.

8. *Gear trains: simple, compound, and epicyclic gear trains*

Friction drives versus gears. Examples of spur gears, rack and pinion, bevel gears, and worm gears. Spur gear terminology. Fundamental law of gearing (circular gears). Gear trains: simple, compound, planetary. Equivalent inertia.

Examples Sheets: 3A3C, 3A3D

Learning Outcomes:

1. Ability to calculate absolute and relative positions, velocities, and accelerations for two-dimensional rigid body motions in fixed, translating, and rotating frames of reference.

2. Ability to calculate for a rigid body moving in two dimensions the angular momentum with respect to a fixed point, to the mass centre of the body, or to any arbitrary point.

3. Ability to solve dynamics problems for two-dimensional rigid body motions including both translation and rotation.

4. Ability to identify the type of motion produced by a mechanism and to calculate its mobility.

5. Ability to find the instantaneous centres for the various components of a mechanism.

6. Ability to calculate the relative positions, velocities and accelerations of the various components of a mechanism using vector methods, diagrammatic methods, and complex numbers / loop closure equations.

7. Ability to calculate the static and dynamic forces within a mechanism and the equivalent inertia for a mechanism.

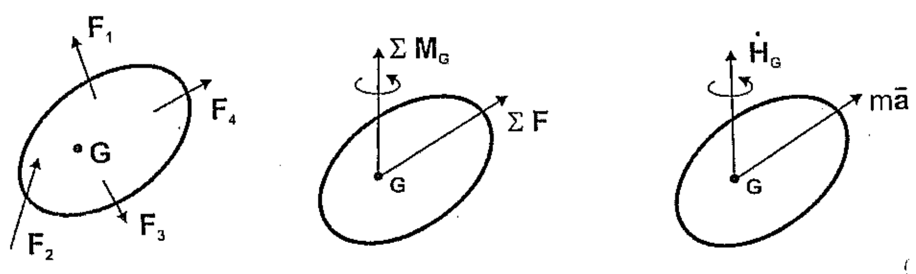
8. Ability to calculate the gear ratios, torques, and effective inertia for simple, compound, and planetary gear trains.

Plane Kinetics of Rigid Bodies

Force, Mass, and Acceleration

General equations of motion

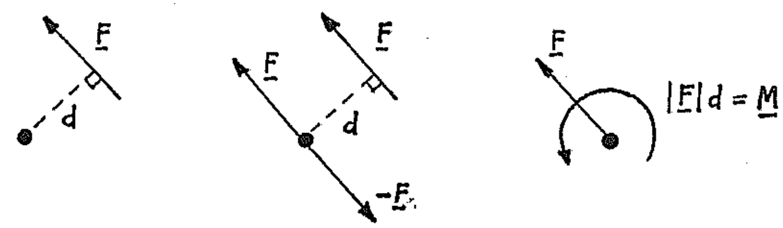
$\Sigma \mathbf{F} = m\,\bar{\mathbf{a}}$ $\qquad \Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$



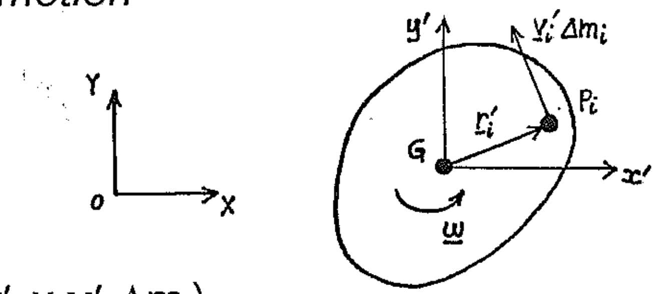
free-body diagram | equivalent force-couple | kinetic diagram

In general, the resultant of the external forces acting on the body does not pass through the mass centre of the body.

The external forces can be replaced by their equivalent force-couple system as follows:



Angular momentum of a rigid body in plane motion



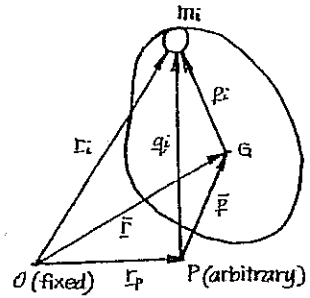
$\mathbf{H}_G = \Sigma\,(\mathbf{r}'_i \times \mathbf{v}'_i\,\Delta m_i)$

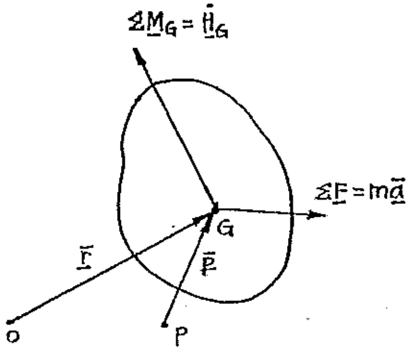
$\mathbf{H}_G = \Sigma\,(\mathbf{r}'_i \times (\omega \times \mathbf{r}'_i)\,\Delta m_i)$

Magnitude $= \omega\,\Sigma\,(r'_i)^2\,\Delta m_i = \bar{I}\,\omega$

Direction = same as ω

$\mathbf{H}_G = \bar{I}\,\omega$ and $\dot{\mathbf{H}}_G = \bar{I}\,\dot{\boldsymbol{\omega}} = \bar{I}\,\boldsymbol{\alpha} = \Sigma\,\mathbf{M}_G$





About O (fixed):

$$\mathbf{H}_o = \Sigma\, \mathbf{r}_i \times m_i\, \mathbf{v}_i$$

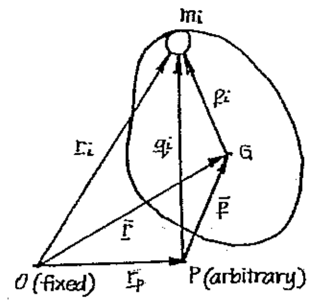
$$\dot{\mathbf{H}}_o = \Sigma\, \mathbf{M}_o$$

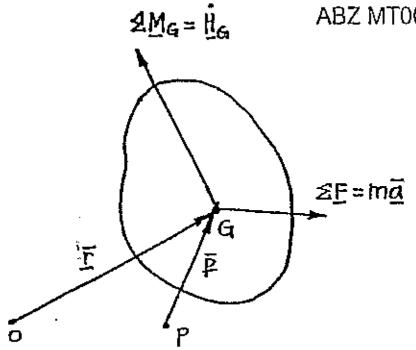
About G (mass centre):

$$\mathbf{H}_G = \Sigma\, \mathbf{p}_i \times m_i\, \dot{\mathbf{r}}_i$$
$$\mathbf{H}_G = \Sigma\, \mathbf{p}_i \times m_i\, \dot{\mathbf{p}}_i$$

$$\dot{\mathbf{H}}_G = \Sigma\, \mathbf{M}_G$$

$$\Sigma\, \mathbf{M}_o = \Sigma\, \mathbf{M}_G + \bar{\mathbf{r}} \times \Sigma\, \mathbf{F} = \dot{\mathbf{H}}_G + \bar{\mathbf{r}} \times m\bar{\mathbf{a}}$$





About P (arbitrary point):

$$\mathbf{H}_P = \Sigma\, \mathbf{q}_i \times m_i\, \dot{\mathbf{r}}_i$$
$$\mathbf{H}_P = \mathbf{H}_G + \bar{\mathbf{p}} \times m\bar{\mathbf{v}}$$

$$\Sigma\, \mathbf{M}_P = \Sigma\, \mathbf{M}_G + \bar{\mathbf{p}} \times \Sigma\, \mathbf{F} = \dot{\mathbf{H}}_G + \bar{\mathbf{p}} \times m\bar{\mathbf{a}}$$

$$(\mathbf{H}_P)_{rel} = \Sigma\, \mathbf{q}_i \times m_i\, \dot{\mathbf{q}}_i$$
$$(\mathbf{H}_P)_{rel} = \mathbf{H}_G + \bar{\mathbf{p}} \times m\bar{\mathbf{v}}_{rel}$$

$$\Sigma\, \mathbf{M}_P = (\dot{\mathbf{H}}_P)_{rel} + \bar{\mathbf{p}} \times m\mathbf{a}_P$$

$$\Sigma\, \mathbf{M}_P = (\dot{\mathbf{H}}_P)_{rel} \text{ if } \begin{cases} \mathbf{a}_P = 0 \\ \bar{\mathbf{p}} = 0 \\ \bar{\mathbf{p}} \parallel \mathbf{a}_P \end{cases}$$

Figure 7 shows a uniform circular disk (radius $R = 0.2$ m, mass $M = 25$ kg) mounted on a rotating bar (length $l = 0.4$ m, mass $m = 6$ kg) in a horizontal plane. The bar rotates about a vertical shaft through point O with a clockwise angular velocity of $\omega = 4$ rad s^{-1}. In case (a), the disk is welded to the bar. In case (b), the disk is pinned freely at A; it moves with curvilinear translation and thus has no rigid-body rotation. In case (c), the relative angle between the disk and the bar increases at the rate $\dot{\theta} = 8$ rad s^{-1}. Calculate the angular momentum of the disk and bar about point O for each case.

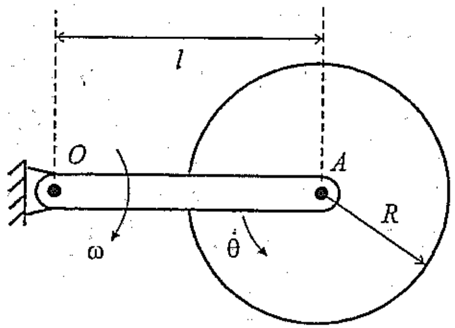


Figure 7

Bar $I_o = I_G + md^2 = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = m\ell^2\left(\frac{1}{12}+\frac{1}{4}\right) = \frac{1}{3}m\ell^2$

$H_o = I_o\omega = \frac{1}{3}(6)(0.4)^2(4) = 1.28$ kg m²/s

(a) Disk $I_o = \frac{1}{2}mr^2 + m\ell^2 = m\left(\frac{r^2}{2}+\ell^2\right)$

$I_o = 25\left[\frac{(0.2)^2}{2} + (0.4)^2\right] = 4.5$ kg m²

$H_o = I_o\omega = 4.5(4) = 18$ kg m²/s or $\underline{H}_o = \underline{H}_G + \underline{r}\times m\underline{v}$

Bar + Disk $H_o = 18 + 1.28 = 19.28$ kg m²/s

(b) Disk $\underline{H}_o = \underline{r}\times m\underline{v} \Rightarrow H_o = (0.4)(25)[(0.4)(4)] = 16$ kg m²/s

Bar + Disk $H_o = 16 + 1.28 = 17.28$ kg m²/s

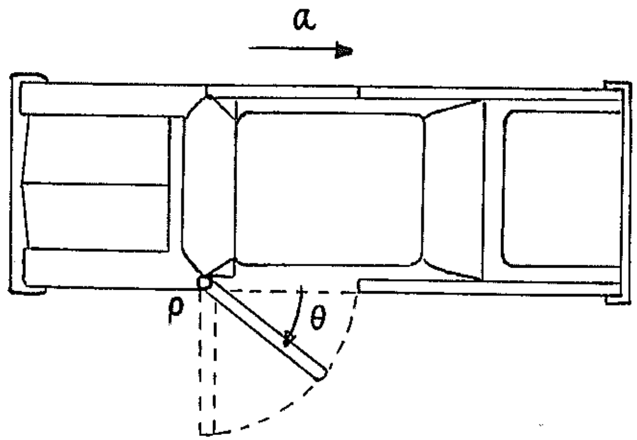
(c) Disk rotating at $\dot{\theta}$ $\underline{H}_o = \underline{H}_G + \underline{r}\times m\underline{v}$

$H_o = \left(\frac{1}{2}mr^2\right)(\omega - \dot{\theta}) + \ell m(\ell\omega)$

$H_o = \frac{1}{2}(25)(0.2)^2(4-8) + 0.4(25)(0.4)(4) = 14$ kg m²/s

Bar + Disk $H_o = 14 + 1.28 = 15.28$ kg m²/s

Example



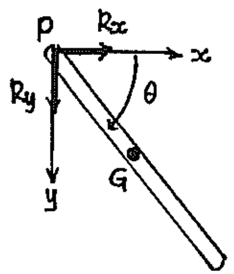
A car door is inadvertently left slightly open. As the brakes are applied to give the car a constant rearward acceleration a, the car door swings open.

Derive expressions for:

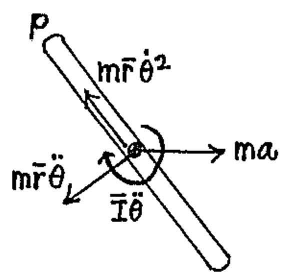
(a) the angular velocity of the door as it swings past the 90° position

(b) the components of the hinge reactions for any value of θ.

The mass of the door is m, its mass centre is a distance r from the hinge axis P, and the radius of gyration about P is k_P.



FREE-BODY DIAGRAM



KINETIC DIAGRAM

$$\underline{a}_G = \underline{a}_P + \underline{a}_{G/P}$$

$$\underline{a}_G = \underline{a} + \underbrace{\underline{\ddot{\theta}} \times \underline{r}_{G/P}}_{\bar{r}\ddot{\theta}} + \underbrace{\underline{\dot{\theta}} \times \underline{\dot{\theta}} \times \underline{r}_{G/P}}_{\bar{r}\dot{\theta}^2}$$

EQUATIONS OF MOTION :

$$\Sigma F_x = R_x = ma - m\bar{r}\dot{\theta}^2\cos\theta - m\bar{r}\ddot{\theta}\sin\theta \qquad (1)$$

$$\Sigma F_y = R_y = -m\bar{r}\dot{\theta}^2\sin\theta + m\bar{r}\ddot{\theta}\cos\theta \qquad (2)$$

It is convenient to take moments about point P (the hinge) since unknowns R_x and R_y will not appear in the equation.

$$\Sigma \underline{M}_P = \underline{0} = \underline{\dot{H}}_G + \underline{\bar{\rho}} \times m\underline{\bar{a}}$$

$$\Sigma \overset{+\curvearrowleft}{M_P} \quad 0 = \bar{I}\ddot{\theta} + \bar{r}(m\bar{r}\ddot{\theta}) - \bar{r}(ma\sin\theta) \qquad (3)$$

$I_P = \bar{I} + m\bar{r}^2$ PARALLEL-AXIS THEOREM

$$mk_P^2 = \bar{I} + m\bar{r}^2$$

$$\bar{I} = m(k_P^2 - \bar{r}^2) \qquad (4)$$

Substituting (4) into (3) gives an expression for $\ddot{\theta}\left(= \dfrac{a\bar{r}}{k_P^2}\sin\theta\right)$

To find $\dot{\theta}$,

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\,\frac{d\dot{\theta}}{d\theta} \qquad \ddot{\theta}\,d\theta = \dot{\theta}\,d\dot{\theta}$$

Substituting for $\ddot{\theta}$ and integrating from 0 to θ gives

$$\dot{\theta}^2 = \frac{2a\bar{r}}{k_P^2}(1-\cos\theta) \qquad \dot{\theta}^2 = \frac{2a\bar{r}}{k_P^2} \text{ at } \theta = 90°$$

Using expression for $\dot{\theta}^2$ above and (1), (2) gives R_x, R_y.

Note that the moment equation could also be written as :

$$\Sigma \underline{M}_P = (\underline{\dot{H}}_P)_{rel} + \underline{\bar{\rho}} \times m\underline{a}_P = 0$$

$$\Sigma \overset{+\curvearrowleft}{M_P} = (mk_P^2)\ddot{\theta} + (-ma\bar{r}\sin\theta) = 0$$

This also gives $\ddot{\theta} = \dfrac{a\bar{r}}{k_P^2}\sin\theta$

Work and Energy

- particularly good for solving problems involving velocity and displacement

- main advantage is that work and energy are scalars

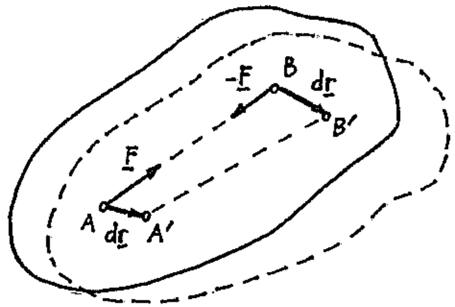
$U_{1\to2} = \Delta T + \Delta V_g + \Delta V_e$

$U_{1\to2}$ = work of all forces acting on the various particles of the body, (internal and external forces)

ΔT = change in total kinetic energy of the particles forming the rigid body

ΔV_g = change in gravitational potential energy

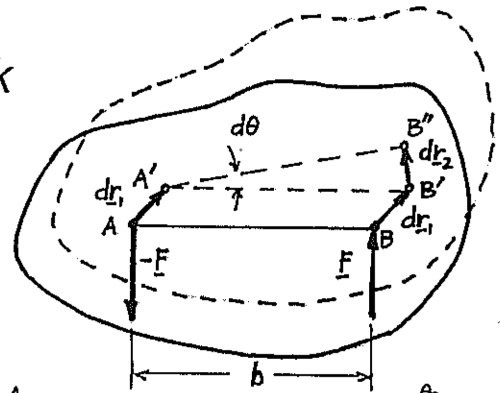
ΔV_e = change in elastic potential energy



The total work of the internal forces acting on the particles of a rigid body is zero. $U_{1\to2}$ reduces to the work of the external forces acting on the body during the displacement considered.

For motion of a single rigid body, the term ΔV_e also disappears.

Work



$|d\underline{r}_2| = b\,d\theta$

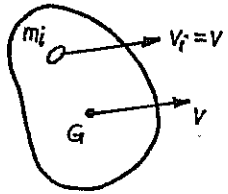
$dU = F\,b\,d\theta$

$dU = M\,d\theta$

$U = \int_{A_1}^{A_2} \mathbf{F} \bullet d\mathbf{r}$ $\qquad U = \int_{\theta_1}^{\theta_2} M\,d\theta$

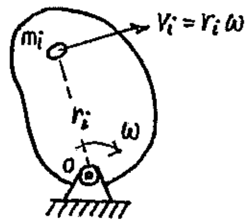
Kinetic Energy

Translation



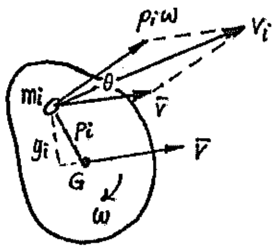
$T = \Sigma \frac{1}{2} m_i v^2$
$T = \frac{1}{2} v^2 \Sigma m_i$
$T = \frac{1}{2} mv^2$

Fixed-axis rotation



$T = \Sigma \frac{1}{2} m_i (r_i \omega)^2$
$T = \frac{1}{2} \omega^2 \Sigma m_i r_i^2$
$T = \frac{1}{2} I_o \omega^2$

General plane motion



$T = \Sigma \frac{1}{2} m_i v_i^2$
$T = \Sigma \frac{1}{2} m_i [\bar{v}^2 + (p_i \omega)^2 + (2\bar{v} p_i \omega \cos \theta)]$
using law of cosines

$$\tfrac{1}{2} (2\bar{v}\, \omega) \Sigma m_i p_i \cos\theta = \bar{v}\, \omega \Sigma m_i p_i \cos\theta$$
$$= \bar{v}\, \omega \Sigma m_i y_i$$
$$= \bar{v}\, \omega\, m \bar{y} = 0$$

$T = \Sigma \frac{1}{2} m_i [\bar{v}^2 + (p_i \omega)^2]$
$T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \bar{I} \omega^2$ since $\bar{I} = \Sigma m_i p_i^2$

The kinetic energy expressed in terms of the instantaneous centre (which has zero velocity) is $T = \frac{1}{2} I_C \omega^2$

Power

$$P = \frac{dU}{dt} = \frac{\mathbf{F} \bullet d\mathbf{r}}{dt} = \mathbf{F} \bullet \mathbf{v}$$

$$P = \frac{dU}{dt} = \frac{M\, d\theta}{dt} = M \omega$$

Impulse and Momentum

- particularly good for solving problems involving time and velocities
- the only practicable method for solving problems involving impulsive motion or impact

Linear momentum

$\mathbf{L} = m\,\overline{\mathbf{v}}$ $\qquad \Sigma\,\mathbf{F} = \dot{\mathbf{L}}$

$$\int_{t_1}^{t_2} \Sigma\,\mathbf{F}\,dt = \mathbf{L}_2 - \mathbf{L}_1$$

Angular momentum

$H_G = \bar{I}\,\omega$ $\qquad \Sigma\,M_G = \dot{H}_G$

$$\int_{t_1}^{t_2} \Sigma\,M_G\,dt = (H_G)_2 - (H_G)_1$$

Problem Formulation & Review

(a) *Identification of body or system*

Make an unambiguous decision as to which body or system of bodies is to be analysed.

Isolate the selected body or system by drawing its free-body diagram and its kinetic diagram.

(b) *Type of motion*

Identify the category of motion:

- rectilinear translation
- curvilinear translation
- fixed-axis rotation
- general plane motion

(c) *Coordinate system*

Choose an appropriate coordinate system. The geometry of the particular kinematics is usually the deciding factor.

Designate the positive sense for moment and force summations and be consistent with this choice.

(d) *Principle and method*

- Force-mass-acceleration

If the instantaneous relationship between the applied forces and the resulting accelerations is desired, then the equivalence between the free-body diagram ($\Sigma\mathbf{F}$, $\Sigma\mathbf{M}$) and the kinetic diagram ($m\mathbf{a}$, $I\alpha$) will give the most direct approach to a solution.

- Work-energy

When motion occurs over an interval of displacement, initial and final velocities can be related without having to calculate the accelerations.

- Impulse-momentum

If the interval of motion is specified in terms of time (rather than displacement), use linear and/or angular momentum principles.

(e) *Assumptions and approximations*

Assess the practical significance of any assumptions and approximations made.