Course Outline for A3 Dynamics and Kinematics (8 lectures)

Lecturer: Dr Amy Zavatsky **Michaelmas Term 2004**

Expanded syllabus:

Dynamics: Angular momentum (general definition), rigid body motion with rotation and translation. Kinematics: Velocity and acceleration; motion in rotating frames of reference. Mechanisms; instantaneous centres, calculation of velocity and acceleration (including velocity and acceleration diagrams), simple, compound and epicyclic gear trains (calculation of gear ratios and torques, effective inertia).

Lecture Synopsis:

1. *Dynamics of rigid bodies: plane kinematics*

Translation (rectilinear, curvilinear), rotation about a fixed axis, and general plane motion. Linear and angular velocity and acceleration. Absolute and relative velocity and acceleration. Instantaneous centre of rotation. Rate of change of a vector with respect to a rotating coordinate frame. General motion in a rotating frame of reference.

2. *Dynamics of rigid bodies: plane kinetics*

Force, mass, and acceleration. General equations of motion. Free-body diagram, equivalent force-couple, kinetic diagram. Angular momentum about a fixed point, about the mass centre, about an arbitrary point. Work and energy (translational and rotational), power, impulse and momentum (linear and angular). General problem formulation and review.

3. *Linkages and mechanisms: basic concepts*

Definitions: mechanism, linkage (or kinematic chain), types of motion transformation, open-loop versus closed-loop, lower and higher joints (or pairs), mobility (or degrees-of-freedom). Grübler's equation for mobility. The four-bar linkage (input, output, coupler, transmission angle) and Grashof's law. Kinematic inversion. Introduction to slider-crank mechanism and quick-return mechanism. Analysis versus synthesis.

4. *Linkages and mechanisms: position and velocity*

Relative velocity (link with revolute joint, link with revolute joint and slider). Velocity diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Velocity image. Instantaneous centres and centrodes. Kennedy's theorem for three instantaneous centres. Velocity pole method for finding angular velocities in linkages (four-bar linkage, slider-crank).

5. *Linkages and mechanisms: acceleration*

Relative acceleration (link with revolute joint, link with revolute joint and slider). Acceleration diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Acceleration image. Relative acceleration of coincident particles at the point of contact of rolling elements. Equivalent mechanisms.

6. *Linkages and mechanisms: computational analysis*

Kinematic analysis by complex numbers (link with revolute joint, link with revolute joint and slider). Loop closure equations (slider-crank, quick-return mechanism). Solution of the position problem using the Newton-Raphson method. Vector chains to describe motion of an arbitrary point on a mechanism. Introduction to generalised coordinates and constraint equations.

7. *Linkages and mechanisms: forces in linkages*

Review of two-force bodies, three-force bodies, and couples. Static forces (slider-crank, four-bar linkage). D'Alembert's principle (inertia force and concept of dynamic equilibrium) in contrast to direct "F=ma" approach. Dynamic analysis (four-bar linkage, slider-crank). Virtual work method to find forces and torques. Kinetic energy and equivalent inertia.

8. *Gear trains: simple, compound, and epicyclic gear trains*

Friction drives versus gears. Examples of spur gears, rack and pinion, bevel gears, and worm gears. Spur gear terminology. Fundamental law of gearing (circular gears). Gear trains: simple, compound, planetary. Equivalent inertia.

Examples Sheets: 3A3C, 3A3D

Learning Outcomes:

1. Ability to calculate absolute and relative positions, velocities, and accelerations for two-dimensional rigid body motions in fixed, translating, and rotating frames of reference.
2. Ability to calculate for a rigid body moving in two dimensions the angular momentum with respect to a fixed point, to the mass centre of the body, or to any arbitrary point.
3. Ability to solve dynamics problems for two-dimensional rigid body motions including both translation and rotation.
4. Ability to identify the type of motion produced by a mechanism and to calculate its mobility.
5. Ability to find the instantaneous centres for the various components of a mechanism.
6. Ability to calculate the relative positions, velocities and accelerations of the various components of a mechanism using vector methods, diagrammatic methods, and complex numbers / loop closure equations.
7. Ability to calculate the static and dynamic forces within a mechanism and the equivalent inertia for a mechanism.
8. Ability to calculate the gear ratios, torques, and effective inertia for simple, compound, and planetary gear trains.

Linkages and Mechanisms: Computational Analysis

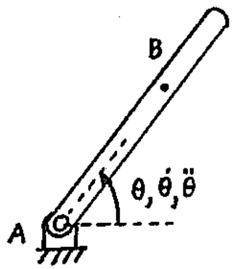
Conventional graphical methods of mechanism analysis have two main drawbacks:

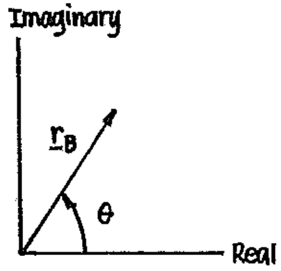
(1) Accuracy is limited (but can be increased if trigonometry is calculated)
(2) It is only possible to look at one position of the mechanism at a time, making analysis of a full 360° cycle very time-consuming.

Some other methods more suited to full cycle calculations and to implementation on computer are:

(1) direct approach using geometry (slider-crank mechanism)
(2) loop closure equations (using complex numbers)
(3) generalised co-ordinates and constraint equations

Kinematic Analysis By Complex Numbers





$\underline{r}_B = r_B(\cos\theta + i\sin\theta)$

$\underline{r}_B = r_B e^{i\theta}$

$r_B = |\underline{r}_B| = constant$

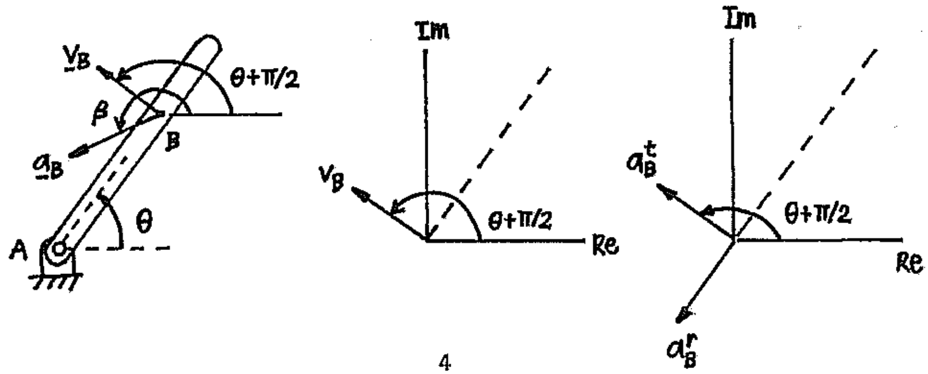
Velocity

$\underline{v}_B = \dot{\underline{r}}_B = r_B\dot{\theta}(ie^{i\theta}) = r_B\dot{\theta}e^{i(\theta+\pi/2)}$

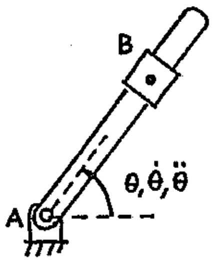
Acceleration

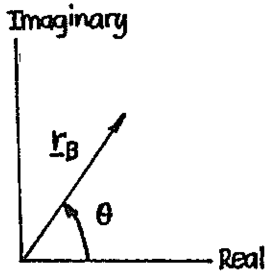
$\underline{a}_B = \ddot{\underline{r}}_B = r_B\ddot{\theta}(ie^{i\theta}) + r_B\dot{\theta}(i^2\dot{\theta}e^{i\theta})$

$\underline{a}_B = \ddot{\underline{r}}_B = r_B\ddot{\theta}e^{i(\theta+\pi/2)} - r_B\dot{\theta}^2e^{i\theta} = a_B e^{i\beta}$



Kinematic Analysis By Complex Numbers (continued)





$\underline{r}_B = r_B(\cos\theta + i\sin\theta)$

$\underline{r}_B = r_B e^{i\theta}$

$r_B = |\underline{r}_B| \neq$ constant

Velocity

$$\underline{v}_B = \dot{\underline{r}}_B = \dot{r}_B e^{i\theta} + r_B\dot\theta\,(ie^{i\theta}) = \dot r_B e^{i\theta} + r_B\dot\theta\, e^{i(\theta+\pi/2)}$$

Acceleration

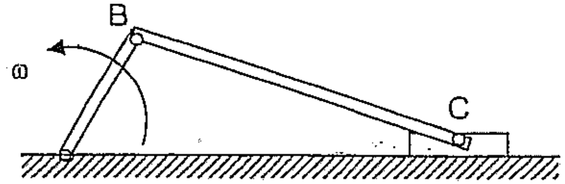
$$\underline{a}_B = \ddot{\underline{r}}_B = \ddot r_B e^{i\theta} + \dot r_B\dot\theta\,(ie^{i\theta}) + \dot r_B\dot\theta\,(ie^{i\theta}) + r_B\ddot\theta\,(ie^{i\theta}) + r_B\dot\theta\,(i^2\dot\theta e^{i\theta})$$

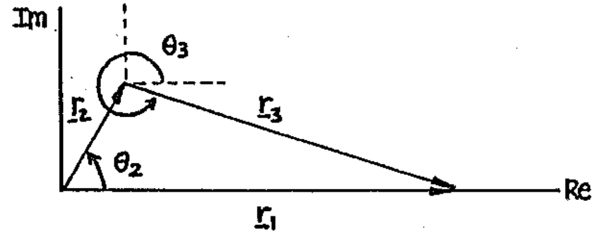
$$\underline{a}_B = \ddot{\underline{r}}_B = \left(\ddot r_B - r_B\dot\theta^2\right)e^{i\theta} + \left(r_B\ddot\theta + 2\dot r_B\dot\theta\right)e^{i(\theta+\pi/2)}$$

Sliding, Centripetal, Tangential, Coriolis

Loop Closure Equations

Slider-crank





Loop closure equation

$\underline{r}_1 = \underline{r}_2 + \underline{r}_3$ $\qquad r_1 e^{i\theta_1} = r_2 e^{i\theta_2} + r_3 e^{i\theta_3} = r_1$ since $\theta_1 = 0$

Velocity $\quad \dot r_1 = r_2\dot\theta_2\,(ie^{i\theta_2}) + r_3\dot\theta_3\,(ie^{i\theta_3})$

Acceleration $\quad \ddot r_1 = r_2\,(i\ddot\theta_2 - \dot\theta_2^2)\,e^{i\theta_2} + r_3\,(i\ddot\theta_3 - \dot\theta_3^2)\,e^{i\theta_3}$

Known quantities : $r_2, r_3, \theta_2, \dot\theta_2, \ddot\theta_2$

Unknown quantities : $r_1, \dot r_1, \ddot r_1, \theta_3, \dot\theta_3, \ddot\theta_3$

Six unknowns – Six equations (real, imaginary parts of position, velocity, acceleration equations)

Loop Closure Equations

Slider-crank (continued)

From the position equation,

$$r_1 = r_2(\cos\theta_2 + i\sin\theta_2) + r_3(\cos\theta_3 + i\sin\theta_3)$$

$$r_1 = r_2\cos\theta_2 + r_3\cos\theta_3 \qquad \text{REAL}$$

$$0 = r_2\sin\theta_2 + r_3\sin\theta_3 \qquad \text{IMAGINARY}$$

The latter two equations can be solved for θ_3, r_1.

From the velocity equation,

$$\dot{r}_1 = r_2\dot{\theta}_2(i\cos\theta_2 - \sin\theta_2) + r_3\dot{\theta}_3(i\cos\theta_3 - \sin\theta_3)$$

Equations for REAL and IMAGINARY parts may be solved for $\dot{\theta}_3$, $\dot{r}_1$.

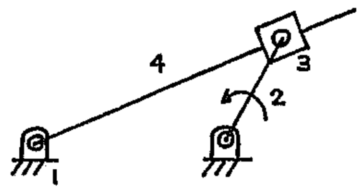
From the acceleration equation,

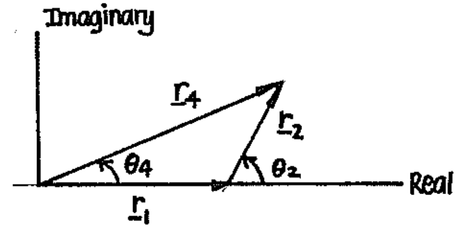
$$\ddot{r}_1 = r_2(i\ddot{\theta}_2 - \dot{\theta}_2^2)(\cos\theta_2 + i\sin\theta_2) + r_3(i\ddot{\theta}_3 - \dot{\theta}_3^2)(\cos\theta_3 + i\sin\theta_3)$$

Equations for REAL and IMAGINARY parts may be solved for $\ddot{\theta}_3$, $\ddot{r}_1$.

Loop Closure Equations

Quick-return





Loop closure equation

$$\underline{r}_4 = \underline{r}_1 + \underline{r}_2 \qquad r_4 e^{i\theta_4} = r_1 + r_2 e^{i\theta_2}$$

Velocity $\quad r_4\dot{\theta}_4(ie^{i\theta_4}) + \dot{r}_4 e^{i\theta_4} = r_2\dot{\theta}_2(ie^{i\theta_2})$

Acceleration $\quad r_4\dot{\theta}_4^2(i^2 e^{i\theta_4}) + r_4\ddot{\theta}_4(ie^{i\theta_4}) + 2\dot{r}_4\dot{\theta}_4(ie^{i\theta_4}) + \ddot{r}_4 e^{i\theta_4} = r_2\dot{\theta}_2^2(i^2 e^{i\theta_2}) + r_2\ddot{\theta}_2(ie^{i\theta_2})$

Known quantities : $r_1, r_2, \theta_2, \dot{\theta}_2, \ddot{\theta}_2$

Unknown quantities : $r_4, \dot{r}_4, \ddot{r}_4, \theta_4, \dot{\theta}_4, \ddot{\theta}_4$

Six equations : real and imaginary parts of position, velocity, acceleration equations

SOLUTION OF THE POSITION PROBLEM

$$r_4 e^{i\theta_4} = r_1 + r_2 e^{i\theta_2}$$

$$\left.\begin{aligned} r_4 \cos\theta_4 &= r_1 + r_2\cos\theta_2 \\ r_4 \sin\theta_4 &= r_2 \sin\theta_2 \end{aligned}\right\} \quad \theta_4, r_4 \text{ are unknowns}$$

Re-write equations as:

$$f_1(\theta_4, r_4) = r_1 + r_2\cos\theta_2 - r_4\cos\theta_4 = 0$$

$$f_2(\theta_4, r_4) = r_2\sin\theta_2 - r_4\sin\theta_4 = 0$$

non-linear, transcendental $\Rightarrow$ cannot use linear matrix methods

Use Newton-Raphson to solve numerically.

$$\theta_4 = \bar{\theta}_4 + \Delta\theta_4$$

$$\underset{\text{solution}}{r_4} = \underset{\text{estimate}}{\bar{r}_4} + \underset{\text{correction factor}}{\Delta r_4}$$

Do Taylor series expansion of f_1, f_2

Neglect higher order terms

Re-write in matrix form and solve for correction factors:

$$\begin{bmatrix} \Delta\theta_4 \\ \Delta r_4 \end{bmatrix} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial\theta_4}\right|_{\bar{\theta}_4,\bar{r}_4} & \left.\frac{\partial f_1}{\partial r_4}\right|_{\bar{\theta}_4,\bar{r}_4} \\ \left.\frac{\partial f_2}{\partial\theta_4}\right|_{\bar{\theta}_4,\bar{r}_4} & \left.\frac{\partial f_2}{\partial r_4}\right|_{\bar{\theta}_4,\bar{r}_4} \end{bmatrix}^{-1} \begin{bmatrix} -f_1(\bar{\theta}_4,\bar{r}_4) \\ -f_2(\bar{\theta}_4,\bar{r}_4) \end{bmatrix}$$

- This works as long as the derivatives can be found.
- Must also be able to invert the matrix.

$$\begin{bmatrix} \Delta\theta_4 \\ \Delta r_4 \end{bmatrix} = \begin{bmatrix} \bar{r}_4\sin\bar{\theta}_4 & -\cos\bar{\theta}_4 \\ -\bar{r}_4\cos\bar{\theta}_4 & -\sin\bar{\theta}_4 \end{bmatrix}^{-1} \begin{bmatrix} -f_1(\bar{\theta}_4,\bar{r}_4) \\ -f_2(\bar{\theta}_4,\bar{r}_4) \end{bmatrix}$$

Estimate solution based on educated guess: $\bar{\theta}_4, \bar{r}_4$

↓

Use estimate to solve for corrections $\Delta\theta_4, \Delta r_4$

↓

Add correction factors to estimates to come up with new estimates $\bar{\theta}_4 = \bar{\theta}_4 + \Delta\theta_4$, $\bar{r}_4 = \bar{r}_4 + \Delta r_4$

↓

Evaluate the functions $f_1(\bar{\theta}_4,\bar{r}_4)$ and $f_2(\bar{\theta}_4,\bar{r}_4)$ at the new values of $\bar{\theta}_4, \bar{r}_4$

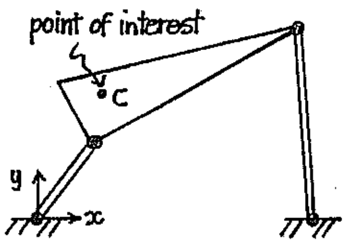
↓

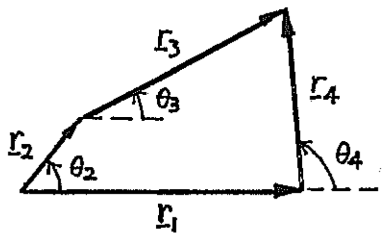
If f_1, f_2 close to zero, then terminate solution. Else recompute estimate

Use solutions (θ_4, r_4) as the educated guess for the next position of the mechanism.

VECTOR CHAINS TO DESCRIBE MOTION OF AN ARBITRARY POINT

Example : 4 bar linkage





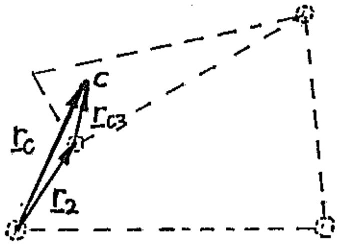
Loop-closure equation :

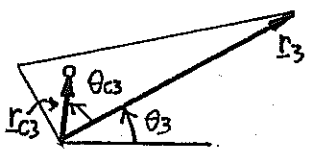
$\underline{r}_1 + \underline{r}_4 = \underline{r}_2 + \underline{r}_3$

Known quantities : $r_1, r_2, r_3, r_4, \theta_2, \dot{\theta}_2, \ddot{\theta}_2$

Unknown quantities : $\theta_3, \dot{\theta}_3, \ddot{\theta}_3, \theta_4, \dot{\theta}_4, \ddot{\theta}_4$

Six unknowns – six equations





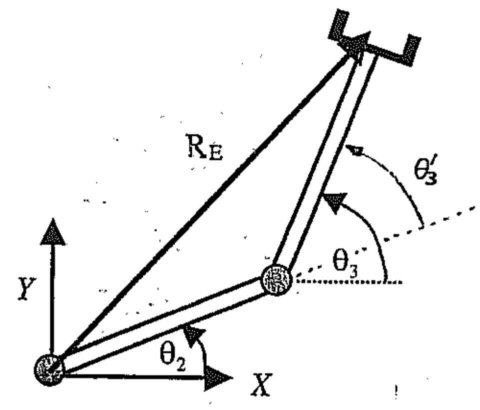
$\underline{r}_C = \underline{r}_2 + \underline{r}_{C3}$

$x_C = r_2 \cos\theta_2 + r_{C3} \cos(\theta_3 + \theta_{C3})$

$y_C = r_2 \sin\theta_2 + r_{C3} \sin(\theta_3 + \theta_{C3})$

Differentiate x_C, y_C to find velocity and acceleration.

TWO-LINK PLANAR ROBOT



$\underline{R}_E = \underline{R}_2 + \underline{R}_3$

$x_E = r_2 \cos\theta_2 + r_3 \cos\theta_3$

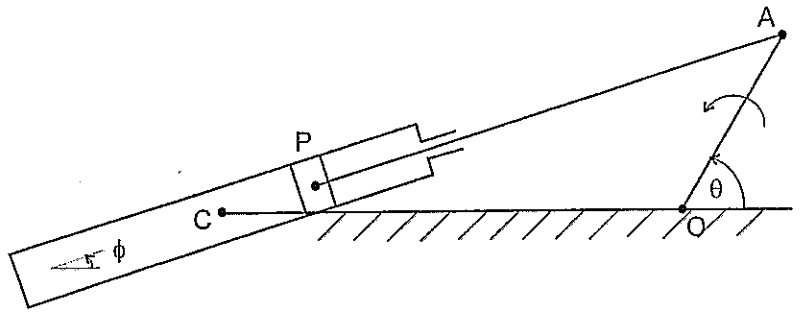
$y_E = r_2 \sin\theta_2 + r_3 \sin\theta_3$

$$\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \end{bmatrix} = \underbrace{\begin{bmatrix} -r_2 \sin\theta_2 & -r_3 \sin\theta_3 \\ r_2 \cos\theta_2 & r_3 \cos\theta_3 \end{bmatrix}}_{\text{Jacobian transformation}} \begin{bmatrix} \omega_2 \\ \omega_3 \end{bmatrix}$$

ROBOTICS uses different angle convention, i.e. joint angles $(\theta_2' = \theta_2,\ \theta_3')$.

The figure below shows an "oscillating cylinder" engine mechanism in which the cylinder pivots about the fixed point C and the crank OA rotates at a constant angular velocity of 10 rad s^{-1} anti-clockwise.

The dimensions of the mechanism are: OA = 0.5 m, AP = 1.3 m, OC = 1.2 m.



(i) Use an analytical approach (loop-closure equation) to find the values of $\dot{\phi}$, $\ddot{\phi}$ and the velocity and acceleration of the piston P when $\theta = 0°, 90°, 180°$.

(ii) Draw the corresponding velocity and acceleration diagrams for $\theta = 0°, 90°, 180°$.

ABZ
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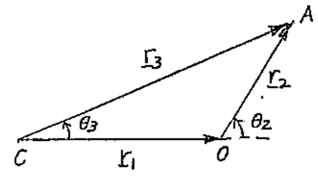


$r_1 = OC = 1.2\,m$ $\qquad \theta_2 = \theta$

$r_2 = OA = 0.5\,m$ $\qquad \theta_3 = \phi$

$\underline{r}_1 + \underline{r}_2 = \underline{r}_3$

$r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = r_3 e^{i\theta_3}$

(1) $r_1 + r_2 e^{i\theta_2} = r_3 e^{i\theta_3}$ $\quad$ since $\theta_1 = 0$

Differentiate to get velocities $(\dot{r}_3, \dot{\theta}_3)$

(2) $r_2(i\dot{\theta}_2)e^{i\theta_2} = \dot{r}_3 e^{i\theta_3} + r_3(i\dot{\theta}_3)e^{i\theta_3}$ $\qquad$ Note $\dot{r}_1 = \dot{r}_2 = 0$

Differentiate again to get accelerations $(\ddot{r}_3, \ddot{\theta}_3)$

$r_2(i\dot{\theta}_2)^2 e^{i\theta_2} = \ddot{r}_3 e^{i\theta_3} + \dot{r}_3(i\dot{\theta}_3)e^{i\theta_3} + \dot{r}_3(i\dot{\theta}_3)e^{i\theta_3} + r_3(i\ddot{\theta}_3)e^{i\theta_3} + r_3(i\dot{\theta}_3)^2 e^{i\theta_3}$

(3) $-r_2\dot{\theta}_2^2\, e^{i\theta_2} = \ddot{r}_3 e^{i\theta_3} + 2i\dot{r}_3\dot{\theta}_3 e^{i\theta_3} + i r_3\ddot{\theta}_3 e^{i\theta_3} - r_3\dot{\theta}_3^2 e^{i\theta_3}$ $\qquad$ Note $\ddot{\theta}_2 = 0$

Equation (1) must be solved to find r_3, θ_3

$r_1 + r_2(\cos\theta_2 + i\sin\theta_2) = r_3(\cos\theta_3 + i\sin\theta_3)$

Taking real and imaginary parts gives

$r_1 + r_2\cos\theta_2 = r_3\cos\theta_3$

$r_2\sin\theta_2 = r_3\sin\theta_3$

These two equations can either be solved explicitly or numerically (Newton-Raphson).

Equation (2) must be solved to find $\dot{r}_3, \dot{\theta}_3$

$i r_2\dot{\theta}_2(\cos\theta_2 + i\sin\theta_2) = \dot{r}_3(\cos\theta_3 + i\sin\theta_3) + i r_3\dot{\theta}_3(\cos\theta_3 + i\sin\theta_3)$

Taking real and imaginary parts gives

$-r_2\dot{\theta}_2\sin\theta_2 = \dot{r}_3\cos\theta_3 - r_3\dot{\theta}_3\sin\theta_3$

$r_2\dot{\theta}_2\cos\theta_2 = \dot{r}_3\sin\theta_3 + r_3\dot{\theta}_3\cos\theta_3$

(2)

$$\begin{bmatrix} -r_3 \sin\theta_3 & \cos\theta_3 \\ r_3 \cos\theta_3 & \sin\theta_3 \end{bmatrix} \begin{bmatrix} \dot\theta_3 \\ \dot r_3 \end{bmatrix} = \begin{bmatrix} -r_2 \dot\theta_2 \sin\theta_2 \\ r_2 \dot\theta_2 \cos\theta_2 \end{bmatrix}$$

$r_3, \theta_3, r_2, \theta_2, \dot\theta_2$ are known. Use Matlab to solve for $\dot\theta_3, \dot r_3$

Equation (3) must be solved to find $\ddot r_3, \ddot\theta_3$

$$-r_2\dot\theta_2^2(\cos\theta_2 + i\sin\theta_2) = \ddot r_3(\cos\theta_3 + i\sin\theta_3) + i\,2\dot r_3\dot\theta_3(\cos\theta_3 + i\sin\theta_3)$$
$$+ i r_3\ddot\theta_3(\cos\theta_3 + i\sin\theta_3) - r_3\dot\theta_3^2(\cos\theta_3 + i\sin\theta_3)$$

Taking real and imaginary parts gives

$$-r_2\dot\theta_2^2\cos\theta_2 = \ddot r_3\cos\theta_3 - 2\dot r_3\dot\theta_3\sin\theta_3 - r_3\ddot\theta_3\sin\theta_3 - r_3\dot\theta_3^2\cos\theta_3$$
$$-r_2\dot\theta_2^2\sin\theta_2 = \ddot r_3\sin\theta_3 + 2\dot r_3\dot\theta_3\cos\theta_3 + r_3\ddot\theta_3\cos\theta_3 - r_3\dot\theta_3^2\sin\theta_3$$

$$\begin{bmatrix} -r_3\sin\theta_3 & \cos\theta_3 \\ r_3\cos\theta_3 & \sin\theta_3 \end{bmatrix}\begin{bmatrix} \ddot\theta_3 \\ \ddot r_3 \end{bmatrix} = \begin{bmatrix} -r_2\dot\theta_2^2\cos\theta_2 + 2\dot r_3\dot\theta_3\sin\theta_3 + r_3\dot\theta_3^2\cos\theta_3 \\ -r_2\dot\theta_2^2\sin\theta_2 - 2\dot r_3\dot\theta_3\cos\theta_3 + r_3\dot\theta_3^2\sin\theta_3 \end{bmatrix}$$

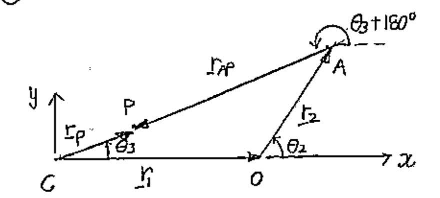
$r_3, \theta_3, r_2, \theta_2, \dot\theta_2, \dot r_3, \dot\theta_3$ are known. Use Matlab to solve for $\ddot\theta_3, \ddot r_3$.

$\theta_2 = 0°$	$\theta_3 = 0$	$\dot\theta_3 = 2.9412$ rad/s	$\ddot\theta_3 = 0$
	$r_3 = 1.7$ m	$\dot r_3 = 0$	$\ddot r_3 = -35.294$ m/s^2
$\theta_2 = 180°$	$\theta_3 = 0$	$\dot\theta_3 = -7.1429$ rad/s	$\ddot\theta_3 = 0$
	$r_3 = 0.7$ m	$\dot r_3 = 0$	$\ddot r_3 = 85.714$ m/s^2
$\theta_2 = 90°$	$\theta_3 = 22.6°$	$\dot\theta_3 = 1.4781$ rad/s	$\ddot\theta_3 = -25.0115$ rad/s^2
	$r_3 = 1.3$ m	$\dot r_3 = -4.6161$ m/s	$\ddot r_3 = -16.3747$ m/s^2

$\theta_3 = \phi$, $\dot\theta_3 = \dot\phi$, and $\ddot\theta_3 = \ddot\phi_3$

BUT the position, velocity, and acceleration of the slider are not (necessarily) given by r_3, $\dot r_3$, and $\ddot r_3$. Must write another position equation and differentiate:

(3)



$r_1 = OC = 1.2$ m
$r_2 = OA = 0.5$ m
$r_{AP} = AP = 1.3$ m

$$\underline{r}_P = \underline{r}_1 + \underline{r}_2 + \underline{r}_{AP}$$

$$x_P = r_1 + r_2\cos\theta_2 + r_{AP}\cos(\theta_3 + 180)$$

(4) $$x_P = r_1 + r_2\cos\theta_2 - r_{AP}\cos\theta_3$$

$$y_P = r_2\sin\theta_2 + r_{AP}\sin(\theta_3 + 180)$$

(5) $$y_P = r_2\sin\theta_2 - r_{AP}\sin\theta_3$$

Differentiate (4) and (5) to get velocities:

(6) $$\dot x_P = -r_2\dot\theta_2\sin\theta_2 + r_{AP}\dot\theta_3\sin\theta_3$$

(7) $$\dot y_P = r_2\dot\theta_2\cos\theta_2 - r_{AP}\dot\theta_3\cos\theta_3$$

$$V_P^2 = \dot x_P^2 + \dot y_P^2$$

Differentiate (6) and (7) to get accelerations:

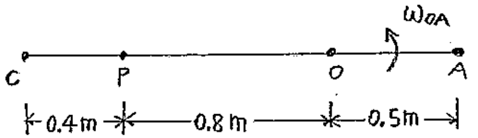
$$\ddot x_P = -r_2\dot\theta_2^2\cos\theta_2 + r_{AP}\ddot\theta_3\sin\theta_3 + r_{AP}\dot\theta_3^2\cos\theta_3$$
$$\ddot y_P = -r_2\dot\theta_2^2\sin\theta_2 - r_{AP}\ddot\theta_3\sin\theta_3 + r_{AP}\dot\theta_3^2\sin\theta_3$$
$$a_P^2 = \ddot x_P^2 + \ddot y_P^2$$

So, the position, velocity, and acceleration of the slider are:

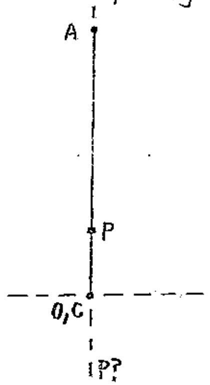
$\theta_2 = 0°$	$x_P = 0.4$ m	$\dot x_P = 0$	$\ddot x_P = -38.75$ m/s^2
	$y_P = 0$	$\dot y_P = 1.1764$ m/s	$\ddot y_P = 0$
		$v_P = 1.1764$ m/s	$a_P = 38.75$ m/s^2
$\theta_2 = 180°$	$x_P = -0.6$ m	$\dot x_P = 0$	$\ddot x_P = 116.33$ m/s^2
	$y_P = 0$	$\dot y_P = 4.286$ m/s	$\ddot y_P = 0$
		$v_P = 4.286$ m/s	$a_P = 116.33$ m/s^2
$\theta_2 = 90°$	$x_P = 0$	$\dot x_P = -4.262$ m/s	$\ddot x_P = -9.873$ m/s^2
	$y_P = 0$	$\dot y_P = -1.774$ m/s	$\ddot y_P = -18.890$ m/s^2
		$v_P = 4.616$ m/s	$a_P = 21.31$ m/s^2

④ VELOCITY AND ACCELERATION DIAGRAMS

CRANK ANGLE = 0°



Velocity Diagram:



$V_A = \omega_{OA}(OA) = 10(0.5) = 5$ m/s ↑

$V_{P/A}$ must be ↑ or ↓

$V_{P/C}$ could have ↑↓ and ⇄ components, but horizontal component must be zero for velocity diagram to work. This matches with $\dot{r}_3 = 0$ on page ②.

$V_{A/C} = \omega_{CA}(CA) = \omega_{CA}(1.7) = V_A = 5$ so $\omega_{CA} = 2.941$ rad/S ↻

$V_{P/C} = \omega_{CA}(CP) = (2.941)(0.4) = 1.176$ m/s ↑

This matches with $\dot{y}_P$ on page ③.

Acceleration Diagram:

$a_A = (\omega_{OA})^2(OA) = 10^2(0.5) = 50$ m/s ←

$a_{P/A}$ radial $= (\omega_{AP})^2(AP) = (\omega_{CA})^2(AP) = (2.941)^2(1.3) = 11.24$ m/s →

$a_{P/A}$ tangential $= \dot{\omega}_{AP}(AP)$

$a_{P/C}$ radial $= (\omega_{CP})^2(CP) + \ddot{r}_{P/C} = (\omega_{CA})^2(CP) + \ddot{r}_{P/C} = 3.46$ m/s ← $+ \ddot{r}_{P/C}$

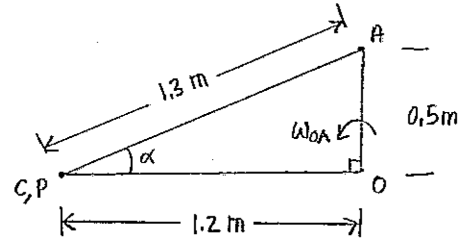
$a_{P/C}$ tangential $= \dot{\omega}_{CP}(CP) + 2\omega_{CP}\dot{r}_{CP} = \dot{\omega}_{CP}(CP)$ since $\dot{r}_{CP} = 0$

$\ddot{r}_{P/C} = 50 - (11.24 + 3.46) = 35.3$ m/s ← Compare $\ddot{r}_3$ on page ②.

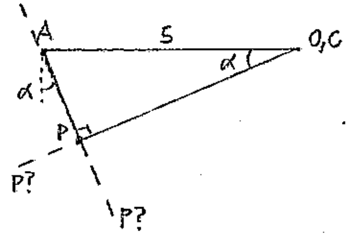
$a_{P/C}$ radial $= 3.46 + 35.3 = 38.76$ m/s ← Compare $\ddot{x}_P$ on page ③.

$\dot{\omega}_{AP}(AP) = \dot{\omega}_{CP}(CP) = 0$ Compare $\ddot{y}_P$ on page ③.

⑤ CRANK ANGLE = 90°



Velocity Diagram:



$V_A = (\omega_{OA})(OA) = 5$ m/s ←

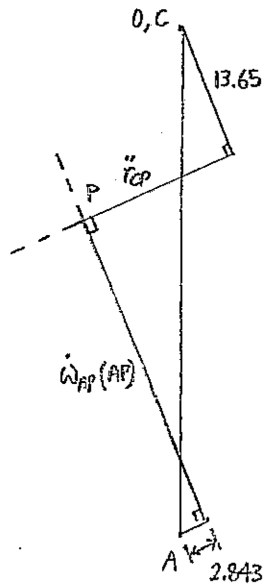
$V_{P/A} = \omega_{AP}\, r_{P/A} = 5 \sin\alpha = 1.923$ m/s ↘

and $\omega_{AP} = 1.479$ rad/s

$V_{P/C} = 5\cos\alpha = 4.615$ m/s ↙

Compare $\dot{r}_3$ on page ② and V_P on page ③.

Acceleration Diagram:



$a_A = 50$ m/s² ↓

$a_{P/A}$ radial $= (\omega_{AP})^2(AP) = 2.843$ m/s² ↗

$a_{P/A}$ tangential $= \dot{\omega}_{AP}(AP)$

$a_{P/C}$ radial $= (\omega_{CP})^2\cancel{(CP)}^{0} + \ddot{r}_{CP}$

$a_{P/C}$ tangential $= 2\omega_{CP}\dot{r}_{CP} + \dot{\omega}_{CP}\cancel{(CP)}^{0}$

$= 2\omega_{AP}\dot{r}_{CP} = 13.65$ m/s²

$\dot{\omega}_{AP}(AP) = 32.50$ m/s² and $\dot{\omega}_{AP} = 25$ rad/s²

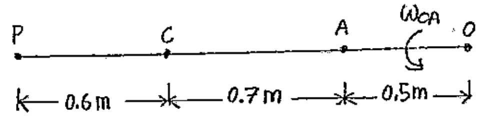
$\ddot{r}_{CP} = 16.38$ m/s²

$a_P = 21.32$ m/s²

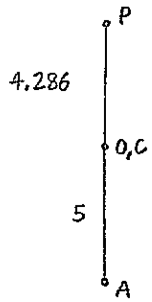
Compare $\ddot{\theta}_3$ and $\ddot{r}_3$ on page ②.

Compare a_P on page ③

(6) CRANK ANGLE = 180°



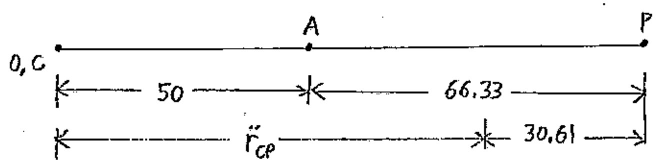
Velocity Diagram:



Compare $\dot{r}_3$ on page (2)

Compare $\dot{y}_P$ on page (3)

Acceleration Diagram:



Compare $\ddot{r}_3$ on page (2)

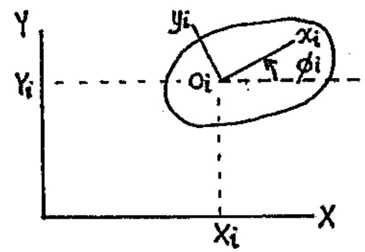
Compare $\ddot{x}_P$ on page (3)

Generalised co-ordinates and constraint equations

Analyse kinematics directly from "equations of constraint". Differentiating these leads to equations for velocity and acceleration.

First, the configuration of the mechanism (position and orientation) must be specified uniquely. For this, we use a system of generalised co-ordinates which vary with time (since the system is in motion).

The generalised Cartesian co-ordinates for a single planar rigid body (3 dof) are:



$$q_i = [X_i, Y_i, \phi_i]^T$$

If there are N_b rigid bodies, then the vector of generalised co-ordinates is:

$$q = [q_1^T\, q_2^T\, q_3^T \ldots q_{N_b}^T]^T$$

In a mechanism, the bodies are coupled together (through lower and higher pairs). We can write "equations of kinematic constraint" that imply the geometry of each joint. The total number of these equations is N_c.

The resulting number of degrees of freedom of the system is then simply $N = 3N_b - N_c$

To determine the motion of the system, we must define either N additional "driving constraints" (kinematic analysis) or a set of forces that act on the system (dynamic analysis).

In general terms, the equations of kinematic constraint can be written as:

$$\underline{\Phi}^k(\underline{q}) = 0$$

The equations corresponding to the driving constraints can be written as:

$$\underline{\Phi}^d(\underline{q}, t) = 0$$

Note that the latter are functions of time. Assume that the former are not explicitly functions of time.

The constraint equations can be assembled into a column vector having $3N_b$ elements.

$$\underline{\Phi}(\underline{q},t) = \begin{bmatrix} \underline{\Phi}^k(\underline{q}) \\ \underline{\Phi}^d(\underline{q},t) \end{bmatrix} = 0$$

This equation can be differentiated (with $\underline{q}$ as a function of time) and solved to give the generalised velocities.

$$\dot{\underline{\Phi}} = \underline{\Phi}_q \dot{\underline{q}} + \underline{\Phi}_t = 0$$

$$\dot{\underline{q}} = -\underline{\Phi}_q^{-1} \underline{\Phi}_t$$

where the subscript represents partial differentiation, with

$$\underline{\Phi}_q = \left[\frac{\partial \Phi_i}{\partial q_j} \right]$$

a square matrix called the Jacobian matrix.

To obtain accelerations, we must differentiate again. It is convenient to define the ($3N_b$ x 1) column vector:

$$\underline{\Psi} = \underline{\Phi}_q \dot{\underline{q}}$$

So,

$$\dot{\Phi} = \Psi + \Phi_t = 0$$

Differentiating this, again using the chain rule:

$$\ddot{\Phi} = \Psi_q \dot{q} + \Psi_t + \Phi_{tq}\dot{q} + \Phi_{tt} = 0$$

Substituting back in for Ψ:

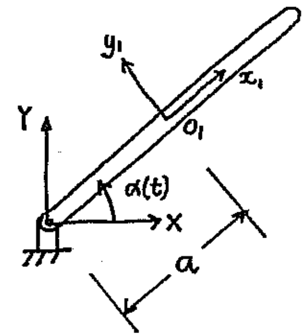
$$\ddot{\Phi} = (\Phi_q \dot{q})_q \dot{q} + \left[\Phi_{qt}\dot{q} + \Phi_q \ddot{q}\right] + \Phi_{tq}\dot{q} + \Phi_{tt} = 0$$

$$\Phi_{qt} = \Phi_{tq}$$

$$\ddot{\Phi} = \Phi_q \ddot{q} + (\Phi_q \dot{q})_q \dot{q} + 2\Phi_{tq}\dot{q} + \Phi_{tt} = 0$$

$$\ddot{q} = -\Phi_q^{-1}\left\{(\Phi_q \dot{q})_q \dot{q} + 2\Phi_{tq}\dot{q} + \Phi_{tt}\right\}$$

Example



$$\dot{\alpha} = \omega = \text{constant}$$

$$q = [X_1\ Y_1\ \alpha]^T$$

$$\Phi^k(q) = \begin{bmatrix} X_1 - a\cos\alpha \\ Y_1 - a\sin\alpha \end{bmatrix} = 0 \qquad \Phi^d(q,t) = \alpha - \omega t = 0$$

$$\Phi = \begin{bmatrix} X_1 - a\cos\alpha \\ Y_1 - a\sin\alpha \\ \alpha - \omega t \end{bmatrix}$$

$$\Phi_q \dot{q} + \Phi_t = 0$$

$$\begin{bmatrix} 1 & 0 & a\sin\alpha \\ 0 & 1 & -a\cos\alpha \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{q} = -\begin{bmatrix} 1 & 0 & -a\sin\alpha \\ 0 & 1 & a\cos\alpha \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -\omega \end{bmatrix}$$

$$\dot{X}_1 = -a\omega\sin\alpha$$
$$\dot{Y}_1 = a\omega\cos\alpha$$
$$\dot{\alpha} = \omega$$

For accelerations,

$$\ddot{q} = -\Phi_q^{-1}\left\{(\Phi_q \dot{q})_q \dot{q} + 2\Phi_{tq}\dot{q} + \Phi_{tt}\right\}$$

Φ_q^{-1} was given on the previous page.

$$\Psi = \Phi_q \dot{q} = \begin{bmatrix} \dot{X}_1 + a\dot{\alpha}\sin\alpha \\ \dot{Y}_1 - a\dot{\alpha}\cos\alpha \\ \dot{\alpha} \end{bmatrix}$$ from previous page

$$(\Phi_q \dot{q})_q = \Psi_q = \begin{bmatrix} 0 & 0 & a\dot{\alpha}\cos\alpha \\ 0 & 0 & a\dot{\alpha}\sin\alpha \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_{qt} = 0 \quad , \quad \Phi_{tt} = 0$$

$$\ddot{q} = -\begin{bmatrix} 1 & 0 & -a\sin\alpha \\ 0 & 1 & a\cos\alpha \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & a\dot{\alpha}\cos\alpha \\ 0 & 0 & a\dot{\alpha}\sin\alpha \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ \dot{\alpha} \end{bmatrix}$$

$$\ddot{q} = -\begin{bmatrix} 0 & 0 & a\dot{\alpha}\cos\alpha \\ 0 & 0 & a\dot{\alpha}\sin\alpha \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -a\omega\sin\alpha \\ a\omega\cos\alpha \\ \omega \end{bmatrix}$$

$$\ddot{q} = \begin{bmatrix} -a\omega^2\cos\alpha \\ -a\omega^2\sin\alpha \\ 0 \end{bmatrix} \qquad \begin{aligned} \ddot{X}_1 &= -a\omega^2\cos\alpha \\ \ddot{Y}_1 &= -a\omega^2\sin\alpha \\ \ddot{\alpha} &= 0 \end{aligned}$$

If $\Phi^d(q,t) = \alpha - \omega t \underbrace{- \varepsilon X_1^2}_{\text{Feedback}} = 0, \quad \Phi_{qt} \neq 0$.

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