Course Outline for A3 Dynamics and Kinematics (8 lectures)

Lecturer: Dr Amy Zavatsky **Michaelmas Term 2004**

Expanded syllabus:

Dynamics: Angular momentum (general definition), rigid body motion with rotation and translation. Kinematics: Velocity and acceleration; motion in rotating frames of reference. Mechanisms; instantaneous centres, calculation of velocity and acceleration (including velocity and acceleration diagrams), simple, compound and epicyclic gear trains (calculation of gear ratios and torques, effective inertia).

Lecture Synopsis:

1. *Dynamics of rigid bodies: plane kinematics*

Translation (rectilinear, curvilinear), rotation about a fixed axis, and general plane motion. Linear and angular velocity and acceleration. Absolute and relative velocity and acceleration. Instantaneous centre of rotation. Rate of change of a vector with respect to a rotating coordinate frame. General motion in a rotating frame of reference.

2. *Dynamics of rigid bodies: plane kinetics*

Force, mass, and acceleration. General equations of motion. Free-body diagram, equivalent force-couple, kinetic diagram. Angular momentum about a fixed point, about the mass centre, about an arbitrary point. Work and energy (translational and rotational), power, impulse and momentum (linear and angular). General problem formulation and review.

3. *Linkages and mechanisms: basic concepts*

Definitions: mechanism, linkage (or kinematic chain), types of motion transformation, open-loop versus closed-loop, lower and higher joints (or pairs), mobility (or degrees-of-freedom). Grübler's equation for mobility. The four-bar linkage (input, output, coupler, transmission angle) and Grashof's law. Kinematic inversion. Introduction to slider-crank mechanism and quick-return mechanism. Analysis versus synthesis.

4. *Linkages and mechanisms: position and velocity*

Relative velocity (link with revolute joint, link with revolute joint and slider). Velocity diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Velocity image. Instantaneous centres and centrodes. Kennedy's theorem for three instantaneous centres. Velocity pole method for finding angular velocities in linkages (four-bar linkage, slider-crank).

5. *Linkages and mechanisms: acceleration*

Relative acceleration (link with revolute joint, link with revolute joint and slider). Acceleration diagrams: four-bar linkage, slider-crank, quick-return mechanism, Geneva mechanism. Acceleration image. Relative acceleration of coincident particles at the point of contact of rolling elements. Equivalent mechanisms.

6. *Linkages and mechanisms: computational analysis*

Kinematic analysis by complex numbers (link with revolute joint, link with revolute joint and slider). Loop closure equations (slider-crank, quick-return mechanism). Solution of the position problem using the Newton-Raphson method. Vector chains to describe motion of an arbitrary point on a mechanism. Introduction to generalised coordinates and constraint equations.

7. *Linkages and mechanisms: forces in linkages*

Review of two-force bodies, three-force bodies, and couples. Static forces (slider-crank, four-bar linkage). D'Alembert's principle (inertia force and concept of dynamic equilibrium) in contrast to direct "F=ma" approach. Dynamic analysis (four-bar linkage, slider-crank). Virtual work method to find forces and torques. Kinetic energy and equivalent inertia.

8. *Gear trains: simple, compound, and epicyclic gear trains*

Friction drives versus gears. Examples of spur gears, rack and pinion, bevel gears, and worm gears. Spur gear terminology. Fundamental law of gearing (circular gears). Gear trains: simple, compound, planetary. Equivalent inertia.

Examples Sheets: 3A3C, 3A3D

Learning Outcomes:

1. Ability to calculate absolute and relative positions, velocities, and accelerations for two-dimensional rigid body motions in fixed, translating, and rotating frames of reference.
2. Ability to calculate for a rigid body moving in two dimensions the angular momentum with respect to a fixed point, to the mass centre of the body, or to any arbitrary point.
3. Ability to solve dynamics problems for two-dimensional rigid body motions including both translation and rotation.
4. Ability to identify the type of motion produced by a mechanism and to calculate its mobility.
5. Ability to find the instantaneous centres for the various components of a mechanism.
6. Ability to calculate the relative positions, velocities and accelerations of the various components of a mechanism using vector methods, diagrammatic methods, and complex numbers / loop closure equations.
7. Ability to calculate the static and dynamic forces within a mechanism and the equivalent inertia for a mechanism.
8. Ability to calculate the gear ratios, torques, and effective inertia for simple, compound, and planetary gear trains.

Gear Trains

Purpose of gears: to couple two shafts together in such a manner that the rotation of one shaft (the *driven shaft* or *output shaft*) will be a function of the rotation of another shaft (the *driving shaft* or *input shaft*).

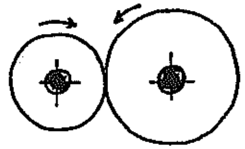
Gears are used to:
- Increase rotation speed but decrease torque
- Keep speed and torque constant
- Reduce rotation speed but increase torque

Note that gear shifting in a gear transmission consists of selecting and engaging different sets of gears.

Most gears and gearboxes are selected as "off-the-shelf" components.

For a given pair of gears, the axis of the input shaft and the axis of the output shaft can be parallel, can intersect at an angle, or can be non-intersecting and nonparallel.

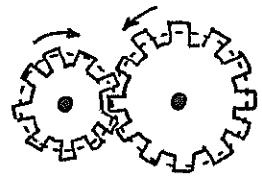
Friction Drive versus Gears



In a friction drive, the power that can be transmitted is limited by the friction that can be developed at the contacting surfaces.

If appreciable torque or power is transmitted from one shaft to another, excess slippage and wear can occur.

Because some slippage will inevitably occur, the angular relationship between the shafts cannot be uniquely maintained in such a drive.



Gears are used to transmit motion from one rotating shaft to another, or from a rotating shaft to a body which translates.

When two gears are meshed with each other, the smaller gear is often called the *pinion*.

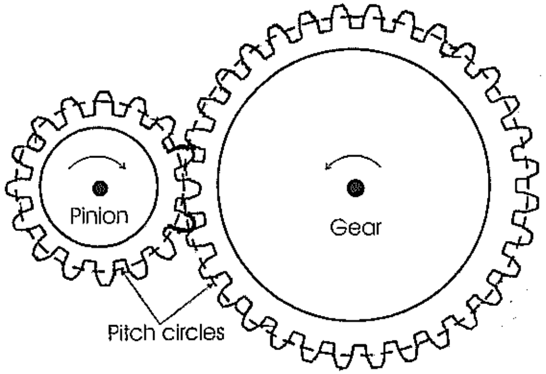
Gears

Spur Gears

These are cut from cylindrical blanks, with the faces of the gear teeth oriented parallel to the shaft on which the gear is mounted.

Conceptually, the pinion and gear are regarded as two cylinders pressed against each other which roll smoothly together. The cylinders are represented by *pitch circles*.

The effective radius of a spur gear, which equals the radius of the conceptual rolling cylinder or pitch circle, is the *pitch radius r*.



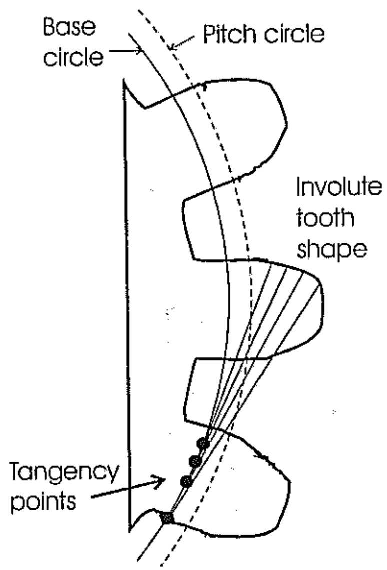
Module = $m = 2r/N$
Diametral pitch = $p = N/2r$
where N = the number of teeth on the gear

In order for a pinion and gear to be compatible, they must have the same value of m (or p) or the gearset will bind as the shafts begin to rotate.

With arbitrarily shaped gear teeth, if the driving gear were to turn at a constant angular velocity, the output gear would rotate with slight variations in its speed, as the teeth engage and disengage.

For spur gears, with *involute* shaped teeth, if one gear turns at constant speed, so will the other.

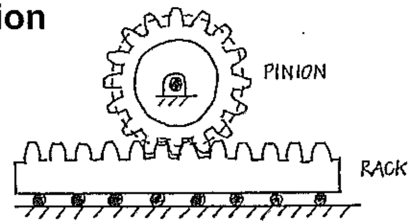
An involute profile can be sketched by drawing a circle (the base circle) slightly smaller than the pitch circle. Wrap a short length of string around the base circle. Hold it at one end and gradually unwrap the other end tangent to the circle. Mark down the curve that the free end of the string follows. This gives the involute profile.



Fundamental law of gearing :

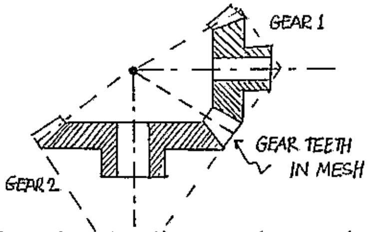
In order for a pair of gears to transmit a constant angular velocity ratio, the shape of their contacting surfaces must be such that the common normal passes through a fixed point (the *pitch point* P) on the line of centres. This stems from Kennedy's theorem that three bodies in relative coplanar motion have their three ICs in one straight line.

Rack and Pinion



- Used to covert the rotational motion of a shaft into the straight-line motion of a slider
- Often used in the mechanism for steering a car's front wheels.

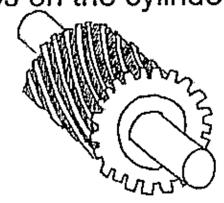
Bevel Gears



- Produced by forming teeth on a truncated cone (as opposed to cylindrical blanks)
- Used where two shafts must be connected at a right angle and where extensions to the shaft centrelines intersect

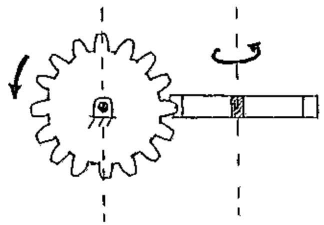
Helical Gears

- Teeth are formed on a cylinder but not parallel to the gear's shaft. They are inclined at an angle so that each tooth wraps on the cylinder in the shape of a shallow helix.



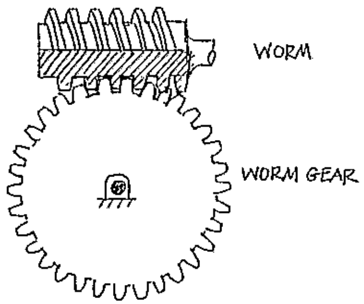
- Produce less noise and vibration than other types of gears. This is because contact starts at one edge of a tooth and proceeds gradually across its width, unlike spur gears where teeth separate and lose contact along the tooth's entire width all at once.

Crossed Helical Gears



- Connect two perpendicular shafts offset from one another so that their centrelines do not intersect

Worm Gearsets



- If the helix angle on a pair of crossed helical gears is large enough, the resulting pair is called a worm and a worm gear.
- The worm has only one tooth that wraps several times around a cylindrical body similar to the thread of a screw.

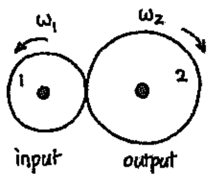
Gear Trains

A gear train is composed of two or more gears in mesh for the purpose of transmitting motion from one shaft to another.

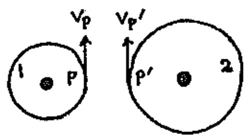
Simple Gear Train

A gear train with only one gear on each shaft.

Velocity



pitch radius R_1, R_2
number of teeth N_1, N_2
angular velocity ω_1, ω_2



$V_p = V_p'$

$\omega_1 R_1 = -\omega_2 R_2$ (ω positive ccw)

$\frac{\omega_2}{\omega_1} = \frac{-R_1}{R_2}$

The pitch p (= $N/2R$) is the same for each gear, so

$$\frac{N_1}{N_2} = \frac{R_1}{R_2} \quad \text{and} \quad \frac{\omega_2}{\omega_1} = -\left(\frac{N_1}{N_2}\right)$$

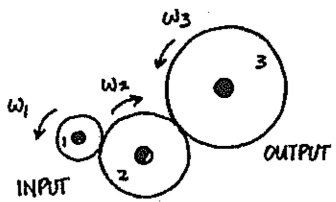
If $N_2 > N_1$, then $\omega_2 < \omega_1$, and the output gear has a slower rate of rotation.

The "velocity ratio" $VR = \frac{\text{output speed}}{\text{input speed}} = \frac{\omega_2}{\omega_1} = \frac{N_1}{N_2}$

The "gear ratio" $G = \frac{N_2}{N_1}$ (down).

NOTE THAT FOR AN INTERNAL GEAR ARRANGEMENT,

$\frac{\omega_2}{\omega_1} = \frac{+N_1}{N_2}$

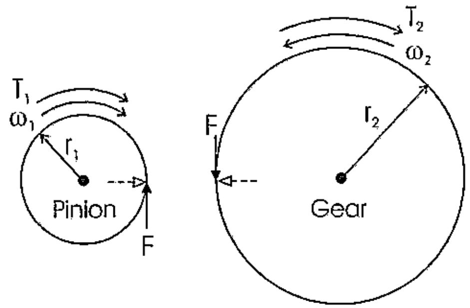


$\frac{\omega_3}{\omega_1} = \frac{\omega_3}{\omega_2} \cdot \frac{\omega_2}{\omega_1} = \left(\frac{-N_2}{N_3}\right)\left(\frac{-N_1}{N_2}\right)$

$\frac{\omega_3}{\omega_1} = \frac{N_1}{N_3}$

GEAR 2 IS AN 'IDLER GEAR' USED TO CHANGE THE DIRECTION OF ROTATION OF GEAR 3 WITHOUT CHANGING ITS ANGULAR VELOCITY.

Torque



The pinion is driven by a motor which applies torque T_1.

The shaft of the gear is connected to a mechanical load (crane, pump, etc), which applies torque T_2.

The driving force F is the means by which torque is transmitted between the pinion and gear. The full meshing force is the resultant of F and another component. The latter tends to separate the pinion and gear but does not serve to transmit torque.

The equations of motion for the gears are:

$$\Sigma M_1 = Fr_1 - T_1 = I_1\dot{\omega}_1 \quad \text{and} \quad \Sigma M_2 = Fr_2 - T_2 = I_2\dot{\omega}_2$$

For constant speed, $\dot{\omega}_1 = \dot{\omega}_2 = 0$. So $\frac{r_1}{r_2} = \frac{T_1}{T_2}$.

The "torque ratio" $TR = \frac{\text{output torque}}{\text{input torque}} = \frac{T_2}{T_1} = \frac{r_2}{r_1} = \frac{N_2}{N_1}$

Note that the torque ratio TR is the inverse of the velocity ratio VR.

If a gear set is designed to increase the speed of its output shaft relative to that of the input shaft ($VR > 1$), then the amount of torque that is transferred will be reduced by an equal factor ($TR < 1$), and vice versa.

A gear set exchanges speed for torque – it is not possible to increase both simultaneously.

Example: car in low gear, rotation speed of engine crankshaft reduced by the transmission in order to increase the torque applied to the drive wheels.

Power

$P_1 = T_1\omega_1$ is the power supplied by the motor to the pinion

$P_2 = T_2\omega_2$ is the power transmitted to the mechanical load by the gear.

$$P_2 = T_2\omega_2 = \left(\frac{N_2}{N_1}\right)T_1\left(\frac{N_1}{N_2}\right)\omega_1 = T_1\omega_1 = P_1$$

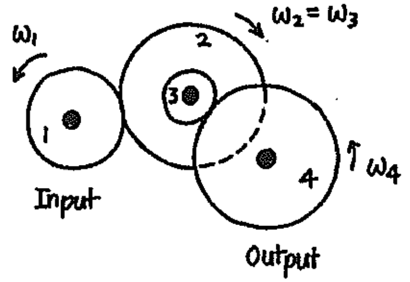
So, power in equals power out.

This is a good approximation for quality gears and bearings and for gear sets where friction is low relative to the overall power. Power reductions are associated with frictional losses.

Compound Gear Train

A gear train in which some shafts carry two gears locked together.

Advantage: a much larger speed reduction from the first to the last shaft can be obtained with small gears. For the same speed reduction in a simple gear train, the last gear would have to be very large. The gear ratio limit in a simple gear train limit is usually 7:1.

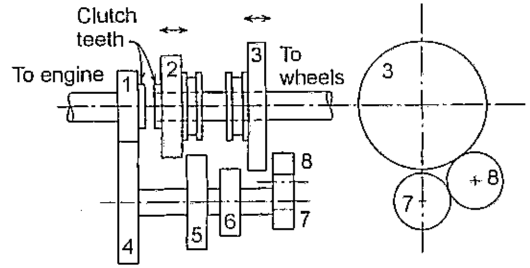


Gears 2 and 3 on same shaft.

$$\frac{\omega_4}{\omega_1} = \frac{\omega_4}{\omega_3} \cdot \frac{\omega_3}{\omega_2} \cdot \frac{\omega_2}{\omega_1} = \left(\frac{-N_3}{N_4}\right)(1)\left(\frac{-N_1}{N_2}\right) = \frac{N_1 N_3}{N_2 N_4}$$

$$\frac{\omega_4}{\omega_1} = \frac{\text{product of number of teeth on driving gears}}{\text{product of number of teeth on driven gears}}$$

A *reverted gear train* has the first and last gears as coaxial. These are used in car transmissions.



'Sliding mesh' gearbox – rarely used today.

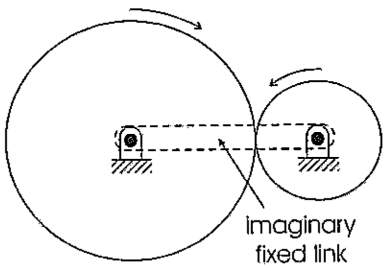
Gear 1 is driven by the engine.
Gears 4, 5, 6, and 7 rotate as a unit.
Gear 8 is an idler gear.

Gears 2 and 3 can slide axially on the splined shaft to the rear wheels.

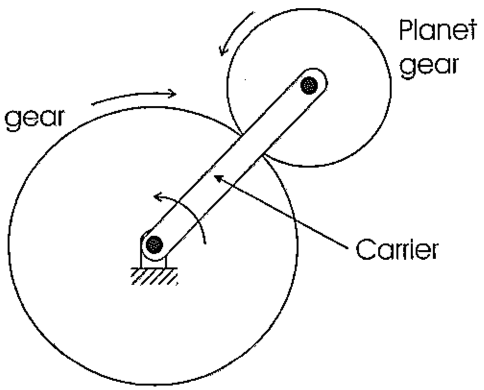
Figure shows "neutral" position in which there is no connection between the shaft to the engine and the shaft to the wheels.

Planetary Gear Trains

In gear sets and in simple and compound gear trains, the centres of the gear shafts do not move.

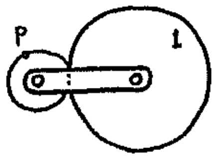


In "planetary" gear trains, the centres of certain gears do move.

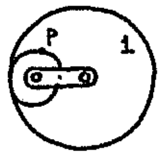


Planetary gear trains are often used as speed reducers and find applications in machine tools, hoists, aircraft propeller reduction drives, automobile differentials, automatic transmissions, aircraft servo-drives, etc.

Another name for a planetary gear train is "epicyclic" gear train.

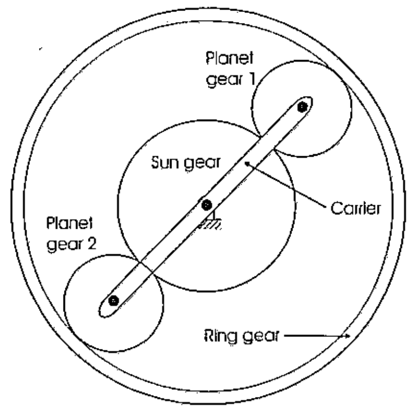


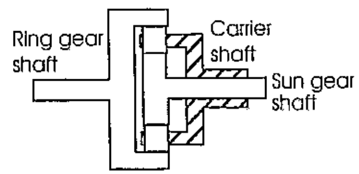
Trace of movement of point P generates an 'epicycloid'



Trace of movement of point P generates a 'hypocycloid'

The sun gear can be connected to a power source or mechanical load, but since the planet gear is moving, its shaft is not easy to connect to another machine or load. To overcome this problem, a ring gear is added. Usually have more than one planet gear to reduce noise, vibration, and forces applied to gear teeth.

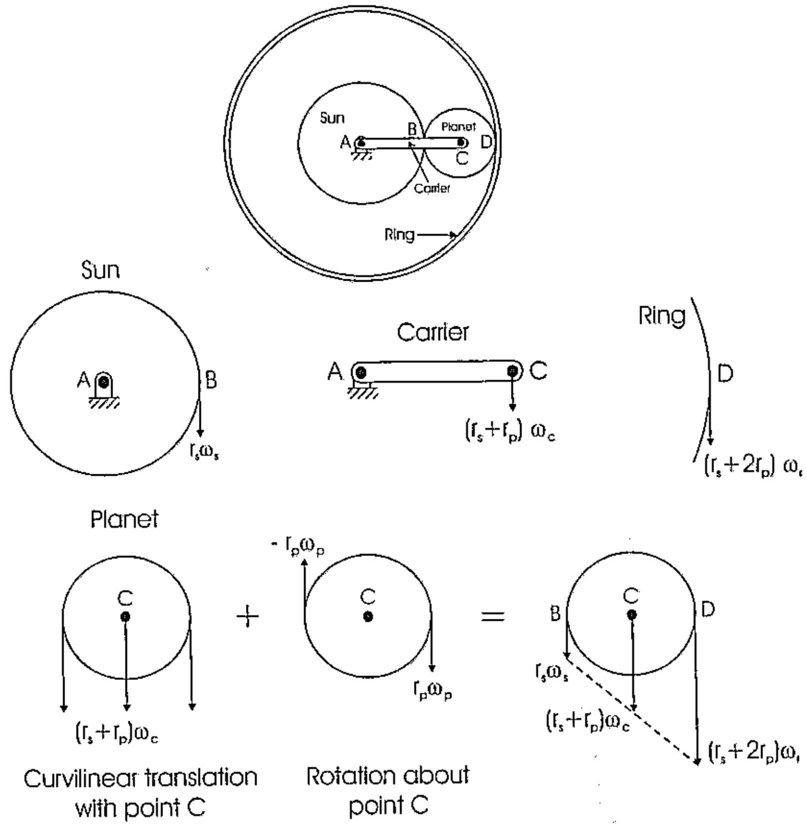




3 connections for input-output:
ring – carrier – sun

- two input – one output
- one input – two output
- one input – one output – one fixed
- two shafts coupled together

Very versatile – can get very large or very small velocity ratios in a gear train with reasonable size.



Contact points on planet gear and sun gear have the same velocities at point B:

$$r_s\omega_s = (r_s + r_p)\omega_c - r_p\omega_p \qquad \omega_p = \frac{(r_s + r_p)\omega_c - r_s\omega_s}{r_p}$$

Contact points on planet gear and ~~sun~~ ring gear have the same velocities at point D:

$$(r_s + 2r_p)\omega_r = (r_s + r_p)\omega_c + r_p\omega_p$$

Substitute for ω_p to get:

$$(r_s + 2r_p)\omega_r = 2(r_s + r_p)\omega_c - r_s\omega_s$$

Define $n = \frac{r_s}{r_p} = \frac{N_s}{N_p}$ and substitute for n to get:

$$(2+n)\omega_r + n\omega_s - 2(1+n)\omega_c = 0$$

(with clockwise rotation positive)

Tabular approach

1. Disconnect any gears from the ground (if there are any fixed to the ground) and fix all gears rigidly to the rotating carrier.

2. Motion with carrier. Rotate the carrier with the rigidly attached gears by a number of revolutions proportional to the angular velocity of the carrier. (If the angular velocity of the carrier is unknown, rotate the arm by x revolutions).

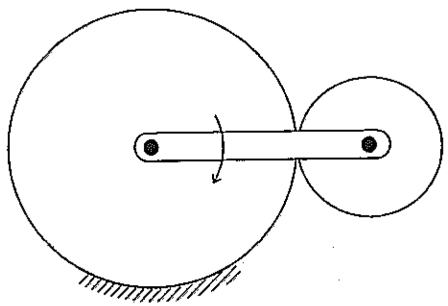
3. Motion relative to the carrier. Now the carrier is assumed to be in its "final" orientation, but some components of the rest of the drive are not. Therefore, unlock the gears from the carrier and while holding the carrier fixed, rotate the rest of the drive back so that

the total rotations (steps 2 and 3) of one or more of the gears match their given rotations.

4. Find the total number of rotations of each gear by algebraically adding its numbers of rotations in steps 2 and 3.

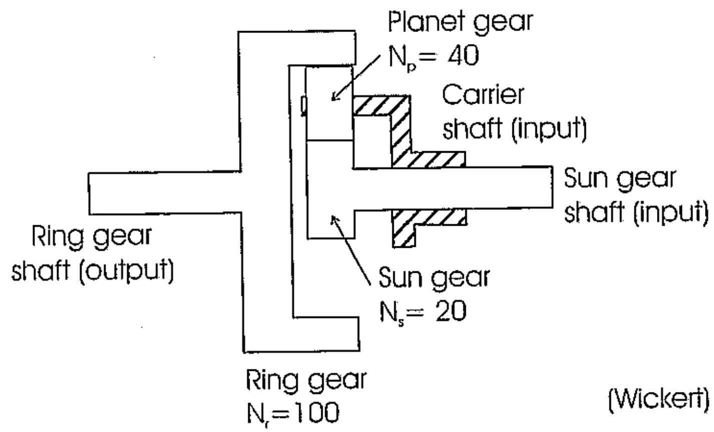
Example of tabular approach

$N_s = 40$
$N_p = 20$
$\omega_c = 20$ rpm
Find ω_p



	Sun	Planet	Carrier
Motion with carrier	+20	+20	+20
Motion relative to carrier	-20	+20 (N_s/N_p)	0
Total motion	0	+20(1+ N_s/N_p)	+20

$\omega_p = 20(1+ N_s/N_p) = 20(1+40/20) = 20(1+2)$
$\omega_p = 60$ rpm clockwise



(Wickert)

When viewed from the right-hand side, the hollow carrier shaft is driven at 3600 rpm clockwise, and the shaft for the sun gear turns at 2400 rpm counterclockwise.

(a) Determine the speed and direction of the ring gear.

$$n = N_s/N_p = 20/40 = 0.5$$
$$\omega_c = 3600 \text{ rpm}, \omega_s = -2400 \text{ rpm}$$

$$(2+n)\omega_r + n\omega_s - 2(1+n)\omega_c = 0$$
$$(2+0.5)\omega_r + 0.5(-2400) - 2(1+0.5)(3600) = 0$$
$$\omega_r = 4800 \text{ rpm (clockwise)}$$

	Sun	Planet	Carrier	Ring
Motion with carrier	+3600	+3600	+3600	+3600
Motion rel to carrier	-6000	-6000($-N_s/N_p$)	0	-6000($-N_s/N_p$)($+N_p/N_r$)
Total motion	**-2400**	+3600 - 6000(-20/40) = 6600 rpm	**+3600**	+3600 + 6000(20/40)(40/100) = **4800 rpm**

(b) The sun gear is driven by an electric motor that is capable of reversing its rotation direction. Repeat the calculation for the case in which the sun gear is instead driven clockwise at 2400 rpm.

Use equations to get $\omega_r = 3840$ rpm (clockwise)

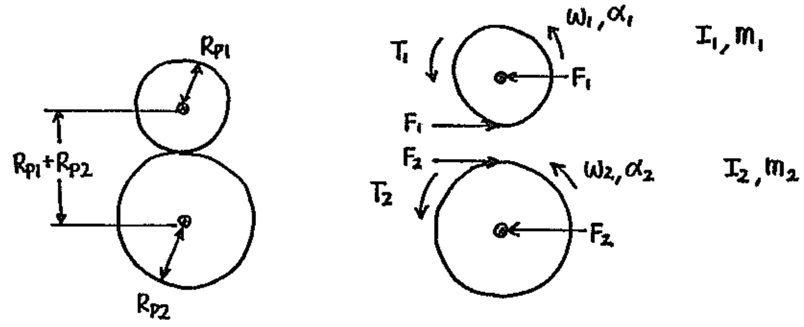
	Sun	Planet	Carrier	Ring
Motion with carrier	+3600	+3600	+3600	+3600
Motion rel to carrier	-1200		0	
Total motion	**+2400**		**+3600**	**= 3840 rpm**

(c) For what speed and direction of the sun gear would the ring gear not rotate?

Use equations with $\omega_r = 0$ to get $\omega_s = 21{,}600$ rpm cw

	Sun	Planet	Carrier	Ring
Motion with carrier	+3600	+3600	+3600	+3600
Motion rel to carrier	$-3600(+N_r/N_p)(-N_p/N_s)$	$-3600(+N_r/N_p)$	0	-3600
Total motion	+3600 + 3600(100/40)(40/20) **= 21,600 rpm**	+3600 - 3600(100/40) = -5400 rpm	**+3600**	**0**

Equivalent Inertia



Assuming No friction present,

$$I_1\alpha_1 = T_1 + F_1 R_{P1} \qquad (*)$$

$$I_2\alpha_2 = T_2 - F_2 R_{P2}$$

$$\frac{\omega_1}{\omega_2} = \frac{\alpha_1}{\alpha_2} = \frac{-R_{P2}}{R_{P1}} = -k$$

$$F_1 + F_2 = 0 \qquad F_2 = -F_1$$

$$I_2\left(-\frac{1}{K}\alpha_1\right) = T_2 + F_1(R_{P1}k)$$

$$\left(\frac{1}{K}\right)^2 I_2\alpha_1 = -\frac{1}{K}T_2 - F_1 R_{P1} \qquad (**)$$

Adding (*) and (**) gives

$$I_1\alpha_1 + I_2\left(\frac{1}{K}\right)^2\alpha_1 = T_1 - \frac{1}{K}T_2$$

$$\underbrace{\left(I_1 + \left(\frac{1}{K}\right)^2 I_2\right)}_{} \alpha_1 = T_1 - \frac{1}{K}T_2$$

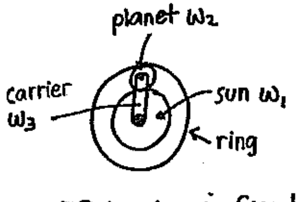
Equivalent inertia of the system referred to the input shaft (1)

The inertia referred to the output shaft is

$I_2 + k^2 I_1$

Finally, if $\alpha_1 = \alpha_2 = 0$, then $kT_1 = T_2$

The effective inertia of an epicyclic gear train can be calculated using energy methods.



If the ring is fixed,

$$\tfrac{1}{2} I_{eq} \omega_1^2 = \tfrac{1}{2} \Sigma \left(I_1 \omega_1^2 + I_3 \omega_3^2 + I_2 \omega_2^2 + m_2 v_2^2 \right)$$

linear velocity of centre of mass of each planet

mass of planet(s)

moment of inertia of all planets

The moment of inertia can be referred to any of the components by replacing ω_1 with the appropriate angular velocity.

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