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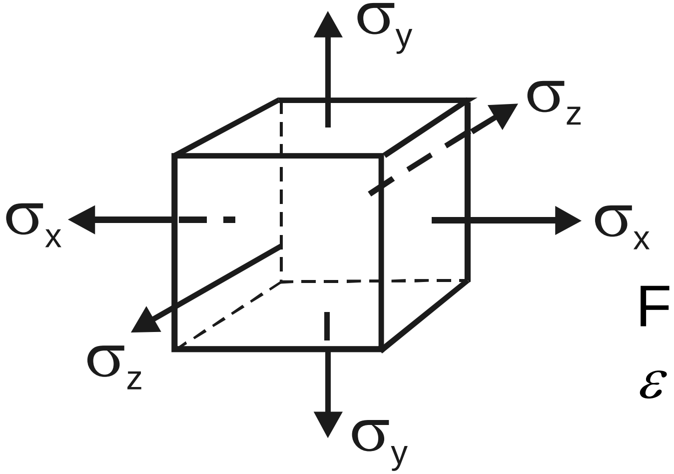
HT08

Lecture 7

Further Development of Theory and Applications

Hooke's law for plane stress.
Relationship between the elastic constants.
Volume change and bulk modulus.
Spherical and cylindrical pressure vessels.

Generalized Hooke's Law



Apply σ_x, get ε_x, $\varepsilon_y = -\nu\varepsilon_x$, $\varepsilon_z = -\nu\varepsilon_x$
Apply σ_y, get ε_y, $\varepsilon_x = -\nu\varepsilon_y$, $\varepsilon_z = -\nu\varepsilon_y$
Apply σ_z, get ε_z, $\varepsilon_x = -\nu\varepsilon_z$, $\varepsilon_y = -\nu\varepsilon_z$

For an isotropic linearly elastic material,
$\varepsilon = \sigma / E$ in the x, y, and z directions.

Use superposition to get ε_x:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\varepsilon_y - \nu\varepsilon_z$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\left(\frac{\sigma_y}{E}\right) - \nu\left(\frac{\sigma_z}{E}\right)$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\,\sigma_y - \nu\,\sigma_z\right)$$ etc.

The resulting equations are:

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\,\sigma_y - \nu\,\sigma_z\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\,\sigma_x - \nu\,\sigma_z\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\,\sigma_x - \nu\,\sigma_y\right)$$

Hooke's Law for Plane Stress

For plane stress, substitute $\sigma_z = 0$ into the generalized Hooke's Law equations to get:

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\,\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\,\sigma_x\right)$$

HLT, page 108

$$\varepsilon_z = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right)$$

Remember also the shear strain: $\gamma_{xy} = \frac{\tau_{xy}}{G}$

These equations can be re-written in terms of stresses:

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\,\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\,\varepsilon_x\right)$$

HLT, page 108

$$\sigma_z = 0$$

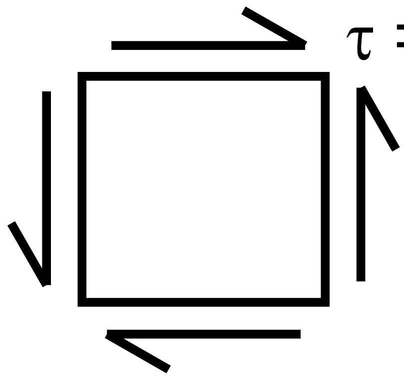
$$\tau_{xy} = G\gamma_{xy}$$

These equations contain three material constants: *E*, *G*, and ν. We can show that these constants are related by the equation:

$$G = \frac{E}{2(1+\nu)}$$

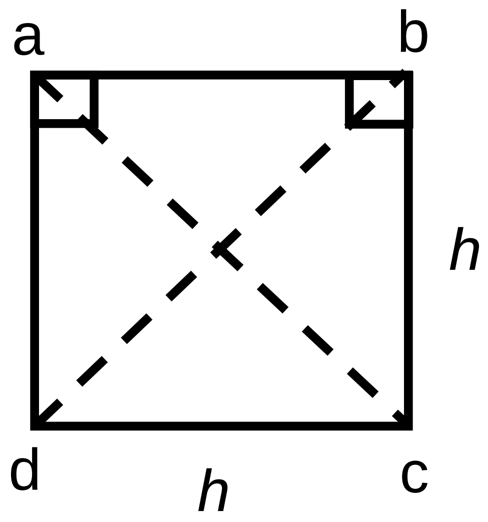
HLT, page 110

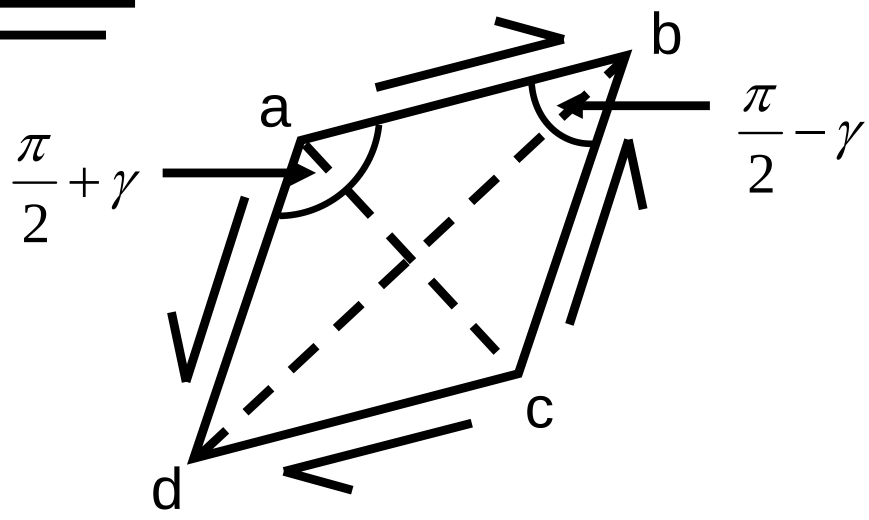
Consider a plane stress element in “pure shear” and relate the **shear** strains and stresses to the strains and stresses along the θ = 45° direction.



$\tau = G\gamma$

Start with strains.





Element before shear is applied.

$$L_{bd} = \sqrt{2}\,h$$

Element after shear applied.

bd lengthens, ac shortens

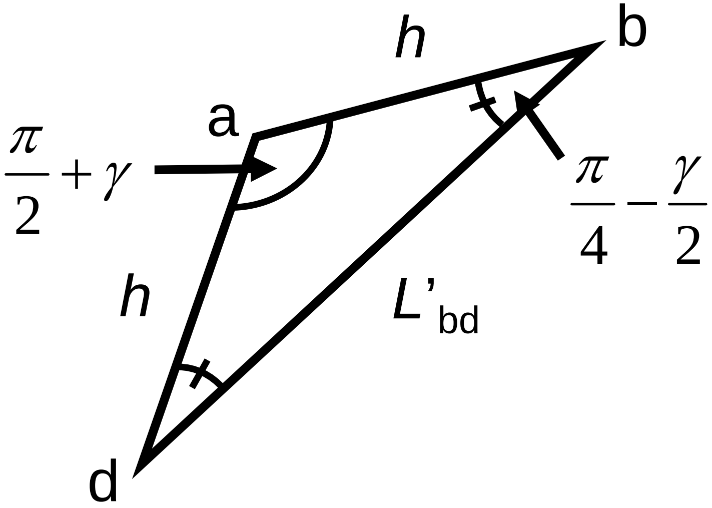
Angle changes related to γ

$$L_{bd} = \sqrt{2}\, h \qquad L'_{bd} = ?$$

$$\varepsilon_{bd} = \frac{\Delta L_{bd}}{L_{bd}} \qquad \Delta L_{bd} = \varepsilon_{bd} L_{bd}$$

$$L'_{bd} = L_{bd} + \Delta L_{bd} = L_{bd}(1+\varepsilon_{bd})$$

$$L'_{bd} = \sqrt{2}\, h\,(1+\varepsilon_{bd})$$



Using geometry, the normal strain ε_{bd} can be related to the shear strain γ.

Using the cosine rule:

$$(L'_{bd})^2 = h^2 + h^2 - 2h(h)\cos(\pi/2+\gamma)$$

$$\left[\sqrt{2}\, h\,(1+\varepsilon_{bd})\right]^2 = 2h^2\left[1-\cos(\pi/2+\gamma)\right]$$

$$(1+\varepsilon_{bd})^2 = 1-\cos(\pi/2+\gamma)$$

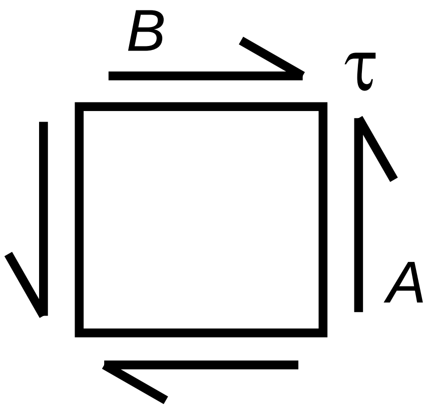
$$1+2\varepsilon_{bd} + (\varepsilon_{bd})^2 = 1+\sin\gamma$$

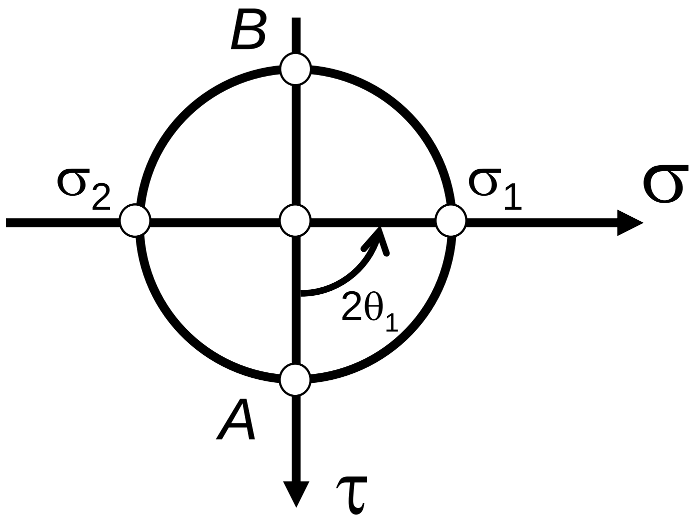
Since ε_{bd} and γ are small,

$$(\varepsilon_{bd})^2 \approx 0 \qquad \sin\gamma \approx \gamma$$

$$\boxed{\varepsilon_{bd} = \gamma/2}$$

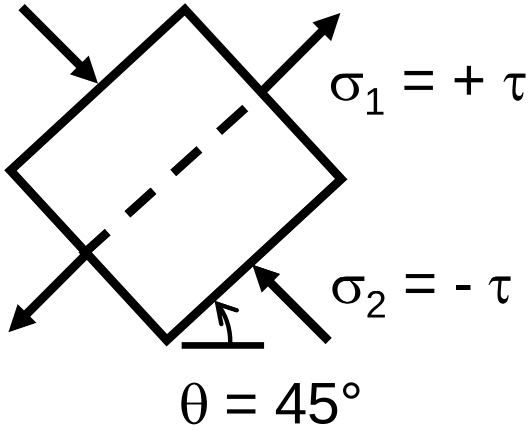
Now use Mohr's circle and Hooke's law to relate strains to stresses.
Find the stress along the $\theta = 45°$ direction :





$\sigma_1 = +\tau$
$\sigma_2 = -\tau$

$2\theta_1 = 90°$
$\theta_1 = 45°$



The strain in the σ_1 direction is:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E}$$

$$\varepsilon_1 = \frac{\tau}{E} - \nu \frac{(-\tau)}{E}$$

$$\boxed{\varepsilon_1 = \frac{\tau}{E}\left(1+\nu\right) = \varepsilon_{bd}}$$

$$\varepsilon_{bd} = \gamma / 2 \qquad \varepsilon_{bd} = \frac{\tau}{E}(1+\nu)$$

$$\frac{\gamma}{2} = \frac{\tau}{E}(1+\nu) = \frac{G\gamma}{E}(1+\nu)$$

$$G = \frac{E}{2(1+\nu)}$$

Data from HLT, page 41

Mild Steel: E = 210 GPa, ν = 0.27-0.30, G = 81 GPa
Aluminium 2024: E = 72 GPa, ν = 0.33, G = 28 GPa

Using the equation just derived:

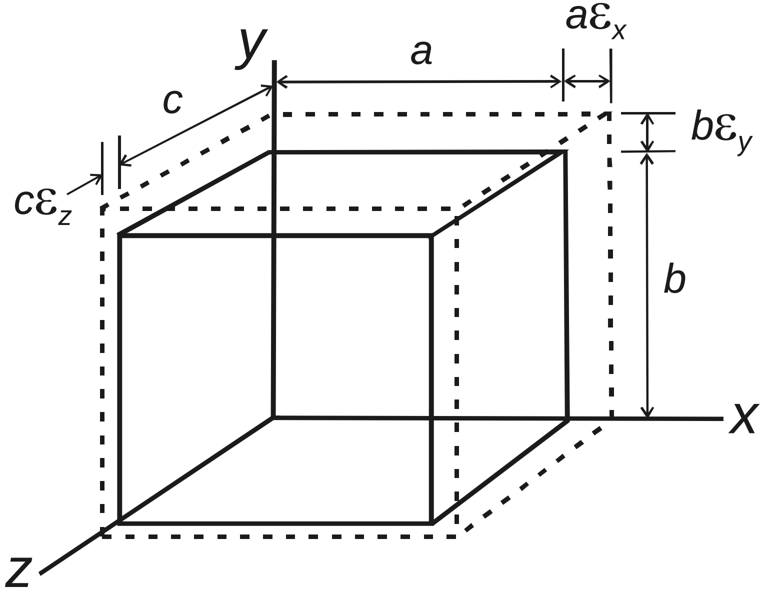
Mild Steel: G = 210 / [2(1+0.27)] = 83 GPa
G = 210 / [2(1+0.30)] = 81 GPa

Aluminium 2024: G = 72 / [2(1+0.33)] = 27 GPa

Volume Change and Bulk Modulus

Normal stresses produce changes in **volume**, whereas **shear** stresses produce changes in **shape**.

Normal stress causes a change in length *dL* of each face. Since $\varepsilon = dL / L$, each $dL = \varepsilon L$.



$$V_o = abc$$ Original volume V_o

New volume V_1

$$V_1 = (a + a\varepsilon_x)(b + b\varepsilon_y)(c + c\varepsilon_z)$$

$$V_1 = abc\,(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)$$

$$V_1 = V_o(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)$$

$$V_1 = V_o(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)$$

This equation is valid for both large and small strains.
If strains are assumed to be small, the product terms all tend to 0.

Assuming that the strains are small,

$$V_1 = V_o(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)$$

The change in volume ΔV is then:

$$\Delta V = V_1 - V_o$$

$$\Delta V = V_o(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z) - V_o$$

$$\Delta V = V_o(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

The **unit** volume change ("dilatation","volumetric strain") is defined as:

$$e = \frac{\Delta V}{V_o} = \frac{V_o(\varepsilon_x + \varepsilon_y + \varepsilon_z)}{V_o}$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Consider next two situations: spherical stress and uniaxial stress.

Spherical stress is defined as $\sigma_x = \sigma_y = \sigma_z = \sigma_o$.

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right) = \frac{1}{E}\left(\sigma_o - \nu\sigma_o - \nu\sigma_o\right)$$ (Hooke's Law)

$$\varepsilon_x = \frac{\sigma_o}{E}\left(1 - 2\nu\right) = \varepsilon_y = \varepsilon_z = \varepsilon_o$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$ (unit volume change)

$$e = 3\varepsilon_o = \frac{3\sigma_o(1-2\nu)}{E}$$

Define the **bulk modulus of elasticity**: $K = \dfrac{E}{3(1-2\nu)}$ HLT, page 110

$$e = \frac{\sigma_o}{K}$$

$$K = \frac{\sigma_o}{e} = \frac{\text{spherical stress}}{\text{volumetric strain}} = \frac{\sigma_o}{3\varepsilon_o}$$

These equations also hold for **hydrostatic stress** ($\sigma_x = \sigma_y = \sigma_z = -\sigma_o$)

Data from HLT, page 41

Mild Steel: E = 210 GPa, ν = 0.27-0.30, K = 160-170 GPa
Aluminium 2024: E = 72 GPa, ν = 0.33, K = 75 GPa

Using the equation for K just derived:

Mild Steel: K = 210 / { 3 [1-2(0.27)] } = 152 GPa
K = 210 / { 3 [1-2(0.30)] } = 175 GPa

Aluminium: K = 72 / { 3 [1-2(0.33)] } = 71 GPa

$$K = \frac{E}{3(1-2\nu)}$$

If $\nu = 0$, $K = E / 3$.

If $\nu = 1/3$, $K = E$.

If $\nu = 1/2$, $K \to \infty$.
This corresponds to a rigid material having no change in volume (that is, the material is incompressible).

Next, consider the unit volume change for **uniaxial stress**, $\sigma_y = \sigma_z = 0$.

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Substituting $\sigma_y = \sigma_z = 0$ into Hooke's Law gives:

$$\varepsilon_y = \frac{-\nu\sigma_x}{E}$$

$$\varepsilon_z = \frac{-\nu\sigma_x}{E}$$

The volumetric strain is then:

$$e = \frac{\sigma_x}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_x}{E}$$

$$e = \frac{\sigma_x}{E}(1 - 2\nu)$$

Note here that the maximum possible value of Poisson's ratio for common materials is 0.5, because a larger value means that the volume would decrease when the material is in tension, which is contrary to ordinary physical behaviour.

Example (based on Gere, 6th ed, p 537, 7.6-10)

A solid steel sphere (E = 210 GPa, ν = 0.3) is subjected to hydrostatic pressure p such that its volume is reduced by 0.4%.

Calculate:

(a) the bulk modulus of elasticity K for the steel

(b) the pressure p

(c) the strain energy stored in the sphere if its diameter d = 150 mm.

Solution:

(a) The bulk modulus of elasticity is found using the equation for K derived earlier.

$$K = \frac{E}{3(1-2\nu)} = \frac{210\times10^9}{3(1-2(0.3))} = 175\times10^9 \text{ Pa} = 175 \text{ GPa}$$

(b) $$e = \frac{\Delta V}{V_o} = 0.4\% = 0.004$$

$$\sigma_o = K\,e = (175 \times 10^9)(0.004) = 700 \times 10^6 \text{ Pa} = 700 \text{ MPa}$$

σ_o is the same as the pressure *p.*

(c) The strain energy density *u* (strain energy per unit volume) is given by the area under the $\sigma_o = K\,e$ curve (which is linear).

$$u = \frac{1}{2}\sigma_o\,e = \frac{\sigma_o^2}{2K} = \frac{(700 \times 10^6)^2}{2(175 \times 10^9)} = 1.40 \times 10^6 \ \frac{\text{J}}{\text{m}^3}$$

The volume *V* of the sphere is

$$V = \frac{4}{3}\pi\,r^3 = \frac{4}{3}\pi\,(75 \times 10^{-3})^3 = 1.77 \times 10^{-3} \text{ m}^3$$

The strain energy density *U* is then simply

$$U = uV = (1.40 \times 10^6)(1.77 \times 10^{-3}) \text{ J} = 2478 \text{ J}$$

Pressure Vessels

• Closed structures containing liquids or gases under pressure. Examples are tanks, pipes, pressurized cabins in aircraft, etc.

• Pressure vessels are considered to be **thin-walled** when the ratio of radius r to wall thickness t is greater than 10.

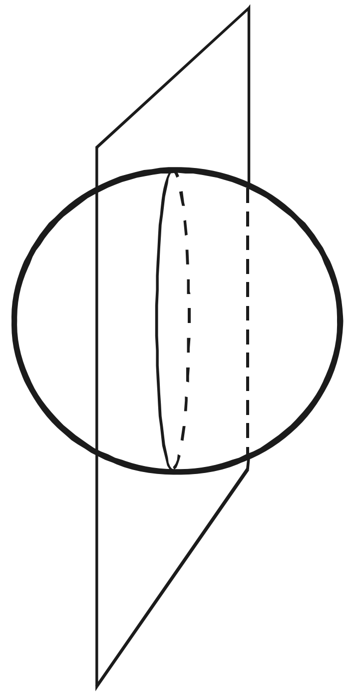
• Assume that the internal pressure p_{in} exceeds the pressure p_{out} acting on the outside of the vessel (usually atmospheric pressure). If $p_{out} > p_{in}$, the vessel could collapse inward due to buckling.

• We are interested in the stresses and strains that develop in the walls of pressure vessels.

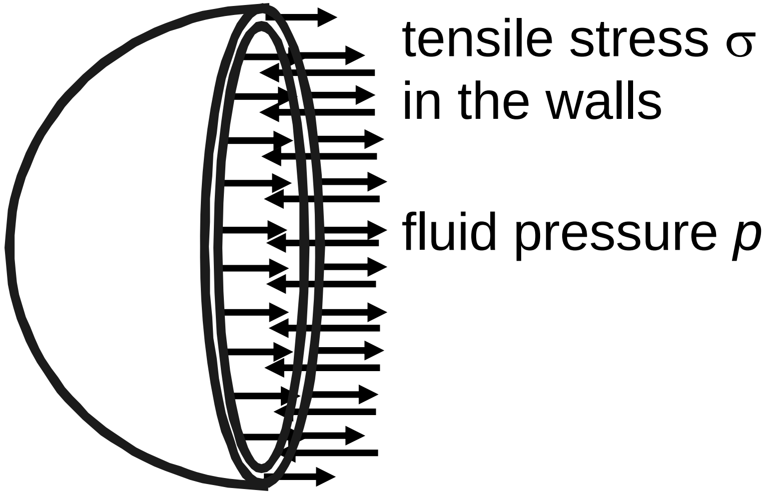
• We will derive equations based on the "net" or "gauge" pressure p, where $p = p_{in} – p_{out}$.

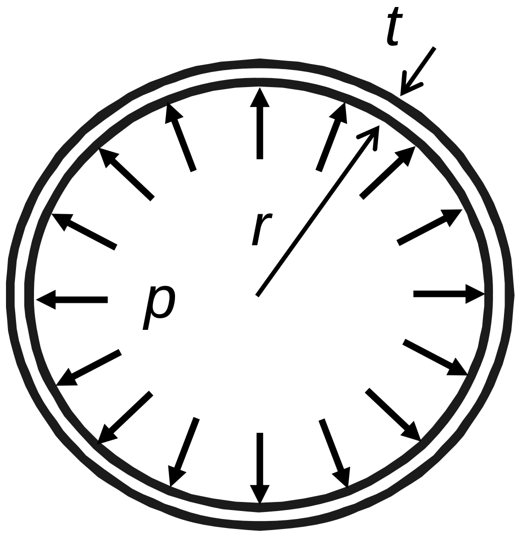
Spherical Pressure Vessels

To determine the stresses in the (thin) walls of a spherical pressure vessel with inner radius *r* and wall thickness *t*, first cut through the sphere on a vertical diametral plane.



Next, isolate half of the sphere and its fluid contents as a single free body.





Fluid force in horizontal direction
$P = p\,(\pi r^2)$

Tensile force in horizontal direction
$T = \sigma\,(2\pi r_m t)$ where $r_m = r + t/2$

For equilibrium, forces in the horizontal direction must balance.

$$\Sigma F = \sigma(2\pi\, r_m\, t) - p(\pi\, r^2) = 0$$

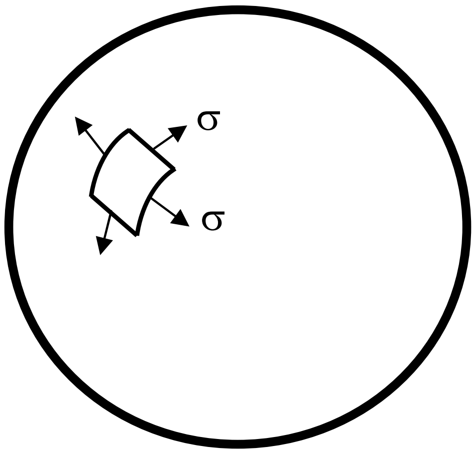
$$\sigma = \frac{pr^2}{2\, r_m t}$$

For thin-walled vessels, $r \cong r_m$ and the equation becomes

$$\boxed{\sigma = \frac{pr}{2t}}$$

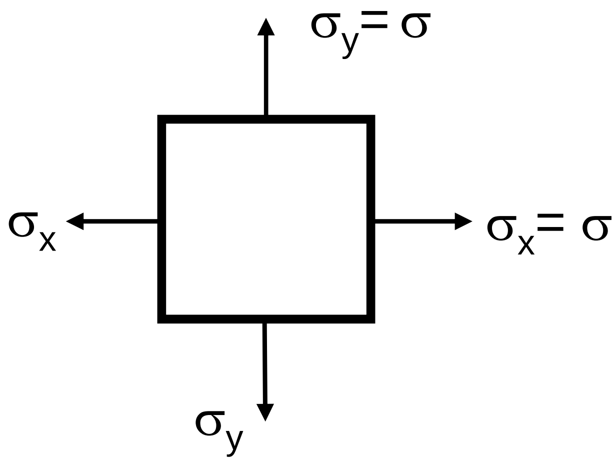
* Note that using r instead of r_m actually gives a result closer to the theoretically "exact" result.

Since the same equation for tensile stresses would result from any slice through the centre of the sphere, we conclude that the wall of a spherical pressure vessel is subjected to uniform tensile stress in all directions.



Sometimes called "membrane stresses"

Stresses at the Outer Surface



$\sigma_x = \sigma_y = \sigma = pr / 2t$
$\sigma_z = 0$
$\tau_{xy} = 0$

When new stress elements on the sphere are obtained from rotating this element about the *z* axis, the normal stresses remain the same and there are no shear stresses. So, every plane tangent to the sphere is a principal plane, and every direction a principal direction.

The principal stresses are $\sigma_1 = \sigma_2 = pr / 2t$, $\sigma_3 = 0$.

The maximum (out-of-plane) shear stress is $\tau_{max} = (\sigma_1 - \sigma_3) / 2 = pr / 4t$

Draw the three Mohr's circles to convince yourself of this.

Comments on the theory

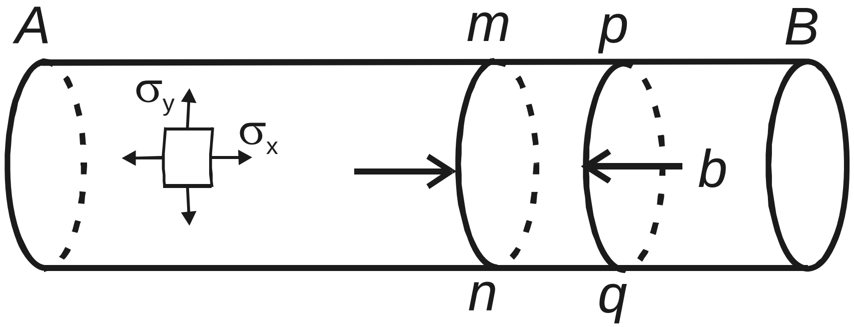
1. The wall thickness must be small in comparison to the other dimensions. The ratio r/t should be 10 or more.

2. The internal pressure must exceed the external pressure to avoid inward buckling.

3. The analysis is based only on the effects of (net) internal pressure. The effects of external loads, reactions, the weight of the contents, and the weight of the structure are not included.

4. The formulae derived are valid throughout the (thin) wall of the vessel, except near points of stress concentration.

Cylindrical Pressure Vessels



The thin-walled cylindrical tank is subjected to a net internal pressure *p*.

Longitudinal Stress

tensile stress σ_L in the walls

fluid pressure *p*

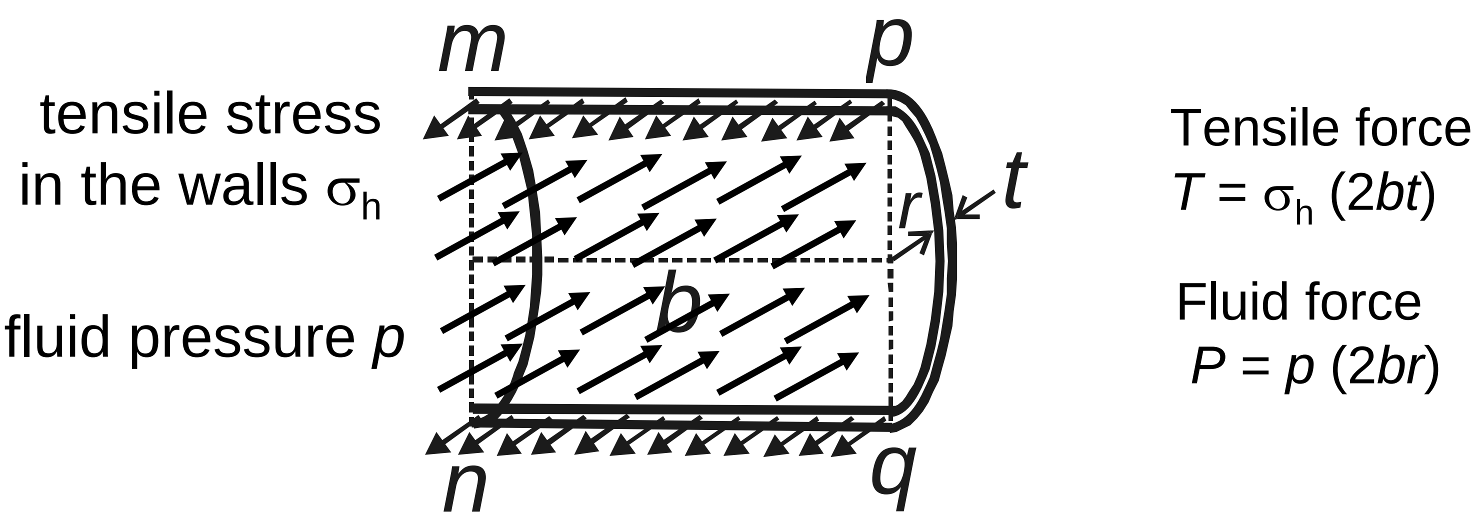
Fluid force in horizontal direction
$P = p\,(\pi r^{2})$

Tensile force in horizontal direction
$T = \sigma_L\,(2\pi r\, t)$

For equilibrium, forces in the horizontal direction must balance.

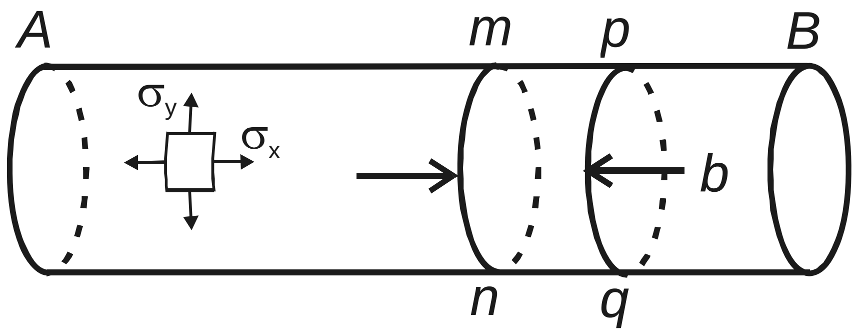
The longitudinal stress is then: $\sigma_L = \dfrac{pr}{2t}$

Circumferential (or Hoop) Stress



For equilibrium, forces in the horizontal direction must balance.

The circumferential (or hoop) stress is then: $\sigma_h = \dfrac{pr}{t}$



The normal stresses σ_x and σ_y are principal stresses since no shear stresses are acting.

Circumferential (Hoop) Direction

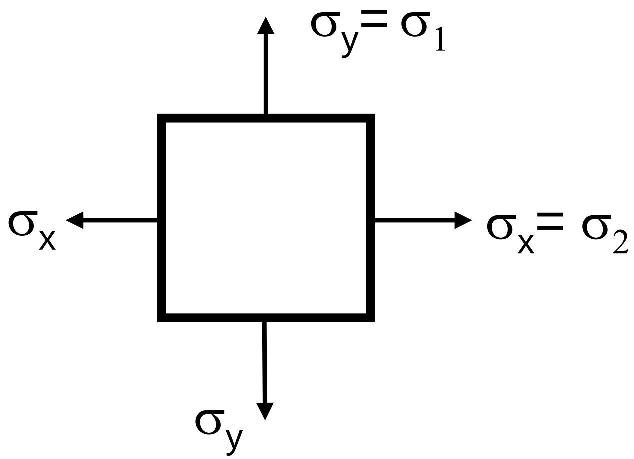
$$\sigma_y = \sigma_1 = \boxed{\sigma_h = \frac{pr}{t}}$$

Longitudinal Direction

$$\sigma_x = \sigma_2 = \boxed{\sigma_L = \frac{pr}{2t}}$$

An obvious discontinuity exists at the ends of the cylinder, where the ends (usually plates or hemispheres) are attached, because the geometry of the structure changes abruptly.

Stresses at the Outer Surface



$\sigma_x = \sigma_2 = pr / 2t$
$\sigma_y = \sigma_1 = pr / t$
$\sigma_z = \sigma_3 = 0$
$\tau_{xy} = 0$

The maximum (in-plane) shear stress is

$$\tau_{\max(1,2)} = \frac{\sigma_1 - \sigma_2}{2} = \frac{(pr/t) - (pr/2t)}{2} = \frac{pr}{4t}$$

The maximum (out-of-plane) shear stresses are

$$\tau_{\max(1,3)} = \frac{\sigma_1 - \sigma_3}{2} = \frac{(pr/t) - 0}{2} = \frac{pr}{2t}$$ overall max shear stress

$$\tau_{\max(2,3)} = \frac{\sigma_2 - \sigma_3}{2} = \frac{(pr/2t) - 0}{2} = \frac{pr}{4t}$$

Example

A spherical pressure vessel having 450 mm inside diameter and 6 mm wall thickness is to be constructed by welding together two aluminium hemispheres. From tests, it is found that the ultimate and yield stresses in tension at the weld are 165 MPa and 110 MPa, respectively. The tank must have a factor of safety of 2.1 with respect to the ultimate stress and 1.5 with respect to the yield stress. What is the maximum permissible pressure in the tank? (Gere and Timoshenko, 3rd ed, p 413)

$r = d/2 = 0.450/2 = 0.225$ m $\quad\quad t = 0.006$ m

$r / t = 37.5$ (>10, so thin-walled assumption okay)

The allowable stress based on the ultimate stress is:

$\sigma_{allow} = \sigma_{ult} / n = 165/2.1 = 78.6$ MPa

The allowable stress based on the yield stress is:

$\sigma_{allow} = \sigma_{y} / n = 110/1.5 = 73.3$ MPa ☆

The latter is lower, so it is the most critical and governs the design.

Tension at the weld for the spherical vessel is $\sigma = pr / 2t$

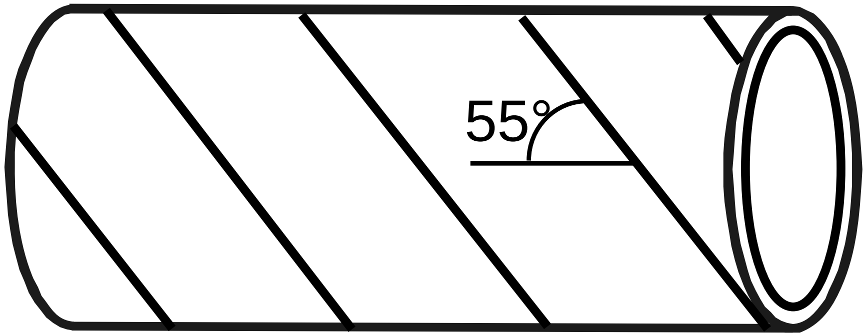
$p = 2\,t\,\sigma_{allow} / r$ = 2 (0.006)(73.3 x 10^6) / 0.225
p = 3.91 x 10^6 Pa = 3.91 MPa

So, the maximum allowable pressure is 3.9 MPa.
(Note that for safety reasons we have rounded down here, not up.)

Example

A cylindrical pressure vessel is constructed with a helical weld that makes an angle of 55° with the longitudinal axis. The tank has inside radius 1.8 m and wall thickness 8 mm. The maximum internal pressure is 600 kPa.

Find the circumferential and longitundal stresses, the absolute maximum shear stress, and the normal and shear stresses acting perpendicular and parallel to the weld. (Gere and Timoshenko, 3rd ed, p 414)



$r\ /\ t$ = 1.8/0.008 = 225 > 10
(so thin-walled assumption okay)

Circumferential and longitudinal stresses

$\sigma_x = \sigma_L = pr\ /\ 2t = (600 \times 10^3)(1.8)/2(0.008) = 67.5$ MPa

$\sigma_y = \sigma_h = pr\ /\ t = 135$ MPa

Absolute maximum shear stress

$\sigma_1 = \sigma_h$, $\sigma_2 = \sigma_L$, $\sigma_3 = 0$ (at the outer surface)

$\tau_{max} = (\sigma_1 - \sigma_3)/2 = 67.5$ MPa

Need stresses perpendicular and parallel to the weld. Consider the stress element below and use either the transformation equations or Mohr's circle with $\theta = 35°$ (why not 55°?) to find $\sigma_{x1} = 89.7$ MPa, $\sigma_{y1} = 112.8$ MPa, $\tau_{x1y1} = 31.7$ MPa.