B entails that a conjunction of determinate truths is determinate

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Dialectic: outline the paradox. In [REF] I argued that B was the culprit. The other main candidate was distributivity, the main moving part in Field's resolution of this puzzle. I argue that B *entails* distributivity, thus if one is to reject distributivity one must already reject B. But if one is to reject B *anyway* one might as well keep distributivity, as without B one already has an adequate solution to the paradox. I shall use the symbol $\Box p$ to be read as informally as 'p and it's no vague whether p', and $\Diamond p$ as it's dual.

Distributivity

D. $\bigwedge_{i<\omega} \Box\phi_i \to \Box \bigwedge_{i<\omega} \phi_i$.

KB

- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- $\phi \to \Box\Diamond\phi$
- if $\vdash \phi$ then $\vdash \Box\phi$

Infinitary conjunction

C1. $\bigwedge_{i<\omega} \phi_i \to \phi_n$ for each $n < \omega$.

C2. $\bigwedge_{i<\omega}(\phi_i \to \psi_i) \to (\bigwedge_{i<\omega} \phi_i \to \bigwedge_{i<\omega} \psi_i)$.

C3. If $\vdash \phi_i$ for each $i < \omega$, $\vdash \bigwedge_{i<\omega} \phi$.

Claim: D is independent of K (and KT) + C1-C3.

Construct a Montague-Scott frame as follows: Let $\mathcal{W} := \mathbb{N}$ and for each world $w \in \mathcal{W}$ let the necessary propositions at w, $N(w)$, be the cofinite subsets of $\mathcal{W}$ (if we are trying to model T as well we let $N(w) := \{X \cup \{w\} \mid X \text{ is cofinite}\}$.) Then $\langle \mathcal{W}, N \rangle$ satisfies:

1. $\mathcal{W} \in N(w)$ for all $w \in \mathcal{W}$
2. If $X, Y \in N(w)$ then $X \cap Y \in N(w)$
3. (For T) If $X \in N(w)$ then $w \in X$.

Thus our frame models K (/KT) including C1-C3. However it does not model D, as can be seen by letting $[\![p_i]\!] := \{n \in \mathbb{N} \mid n > i\}$ (for KT: $\{n \in \mathbb{N} \mid n > i\} \cup \{0\}$, allowing D to fail at 0.) On the other hand any Kripke frame (reflexive Kripke frame) will validate K (KT) along with D.

Although the distributivity of infinite conjunctions over $\Box$ is independent of K, the distributivity of $\Diamond$ over infinite conjunctions, perhaps surprisingly, is not independent in this way and can be show given just some relatively uncontroversial principles governing infinite conjunction.

Lemma 0.1. $\Diamond \bigwedge_{i<\omega} p_i \to \bigwedge_{i<\omega} \Diamond p_i$

Proof. First note that $\Diamond \bigwedge_{i<\omega} p_i \to \Diamond p_j$ for each j, by C1 and the background modal logic of K. Then by C3 and then C2 we can infer $\Diamond \bigwedge_{i<\omega} p_i \to \bigwedge_{i<\omega} \Diamond p_i$. ∎

Theorem 0.2. *Although D is independent of K (and KT) it is not independent of, and is in fact entailed by, KB (and thus KTB.)*

Proof. B directly gives us:

$$\bigwedge_{i<\omega} \Box p_i \to \Box\Diamond \bigwedge_{i<\omega} \Box p_i \tag{1}$$

We may also infer from our lemma that

$$\Box\Diamond \bigwedge_{i<\omega} \Box p_i \to \Box \bigwedge_{i<\omega} \Diamond\Box p_i \tag{2}$$

by applying necessitation and the K principle. Finally we have

$$\Box \bigwedge_{i<\omega} \Diamond\Box p_i \to \Box \bigwedge_{i<\omega} p_i \tag{3}$$

because we have $\Diamond\Box p_i \to p_i$ for each i by B. So by C3 and then C2 we get $\bigwedge_{i<\omega} \Diamond\Box p_i \to \bigwedge_{i<\omega} p_i$, and by necessitation and K that gives (3).

But (1), (2) and (3) give distributivity. ∎

It should be noted that this argument does not appeal to any characteristically classical principles. Indeed this argument can be carried out provided one has the following rules of inference as primitive or derived:

→1. $\phi, \phi \to \psi \vdash \psi$

→2. $\phi \to \psi, \psi \to \chi \vdash \phi \to \chi$

→3. $\phi \to \psi \vdash \neg\psi \to \neg\phi$

and where KB is understood to contain K and the axioms $\phi \to \Box\Diamond\phi$ and $\Diamond\Box\phi \to \phi$.[1]

[1] Without the second version of B the most we could prove without double negation elimination would be things of the form $\Diamond\Box\neg\phi \to \neg\phi$.