Rethinking Newton's *Principia*

Simon Saunders[1]

1. Introduction

Few writings in physics have occasioned as much philosophical interest as Newton's *Principia,* but still puzzles remain. One is an alleged inconsistency in Newton's stated principles in application to a universe that is, quite possibly like the actual universe, infinite and approximately homogeneous in space.[2] Another is that the motions identified by Newton's methods as privileged ('true', according to Newton) could not, realistically, be considered even as motions relative to an inertial frame[3] – or not if the universe is sufficiently large.

The two problems are evidently related. Indeed, if the material universe is in fact infinite and if Newton's theory really is inconsistent in that application we should expect that *no* true motions in Newton's sense could have been identified. But that raises a further puzzle in its own right. For Newton surely did pick out definite motions as somehow preferred; his theory as laid down in the *Principia* (what I shall call Newton's theory of gravity or NTG) was empirically successful; it was genuinely informative; how can an inconsistent theory be empirically informative?

Of course there are other cases of alleged inconsistencies in empirically successful theories. Relativistic quantum electrodynamics, with its famous divergences, is another example. It is tempting to place NTG in the same category, and to view the inconsistency problem as an instance of a more general puzzle (of how inconsistencies can in practise be 'contained'). That has been the tenor of several recent discussions of the subject.[4]

That is not the strategy taken here. My objective is to give a consistent reading of Newton's *Principia* – discarding the Scholium to the Definitions – under which it applies equally to an infinite as to a finite mass distribution, and hence to the actual universe. On this reading the nearest thing to an inertial frame is a non-rotating frame with arbitrary (and possibly time-dependent) linear acceleration. The privileged frame (the centre of mass of the solar system), wrongly identified by Newton as inertial, is exemplary: it is a local freely-falling non-rotating frame. The latter, moreover, is just the sort of frame that is privileged in Einstein's general theory of relativity (GTR),[5] and in Cartan's non-relativistic theory of gravity (so-called Newton-Cartan theory or NCT). These theories have more in common than is normally thought.

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[1] Faculty of Philosophy, University of Oxford, 6 May 2012.

[2] See e.g. Norton [1993], Vickers [2009]. For the early history of this problem, see Norton [1999].

[3] This question was first raised by Kant. I shall come back to Kant's views shortly.

[4] An exception is Malament [1995], who showed how the problem can be solved (effectively by going over to Newton-Cartan theory; see below).

[5] See Knox [2009] for a defence of the claim that such frames play the same functional role in GTR that inertial frames in practise played in NTG.

There is more. As John Norton has recently explained (Norton [1995]), acceleration in NCT in an infinite universe is *relative*, not absolute. On the reading of *Principia* I shall propose the same is true, but in a rather more explicit sense. It is that accelerations, as all other motions, are relational in roughly the sense of the absolute-relational debate in Newton's time. This reading extends to any theory satisfying Newton's laws, yielding a form of relationalism that applies to the entire scope of classical particle mechanics and Newtonian gravity.[6] In this respect it is unlike relationalism in the Machian sense, as developed by Julian Barbour.[7] The latter has the following feature: only systems which (in NTG terms) have total energy and angular momentum zero can be defined. No such constraint – and no such prediction – follows from relationalism in our sense.

What about relationalism in Norton's sense, in NCT? To explain this we will need to use the modern apparatus of differentiable geometry, and the concept of a connection Γ – a rule for defining differentiation of vectors by continuous transport along smooth curves. Thus, suppose there is a manifold M equipped with global rectilinear coordinates $x^\alpha = \langle x^0, x^j \rangle$, $j = 1, 2, 3$ (so M has the topology of a Euclidean space). The coordinates x^α are eventually to be identified as inertial coordinates. In terms of these, let the connection Γ have as its only non-zero components the quantities:

$$\Gamma^j_{00} = \frac{\partial \phi}{\partial x^j} \tag{1}$$

where ϕ (the gravitational potential) satisfies:

$$\nabla^2 \phi = -4\pi G \rho \tag{2}$$

with G is the universal gravitational constant $G = 6.67 \times 10^{-11}\ m^3 \sec^{-2} kg^{-1}$ and ρ the mass density. The equation of motion for a point particle with curve $x : \mathbb{R} \rightarrow M$, parameterized by real numbers λ, is then the same as in GTR; it is the geodesic equation:

$$\frac{d^2 x^\alpha}{d\lambda^2} = \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}. \tag{3}$$

Geodesics are curves whose tangent vector $\frac{dx^\beta}{d\lambda}$ at each point is the parallel transport of the tangent vector at any other point. Curves that depart from these represent the motions of accelerated bodies.

One can free the theory from dependence on a special coordinate system by introducing spatial and temporal metrics, satisfying suitable compatability conditions, and by introducing a Riemann tensor as defined by the connection. The theory can then be written in a generally covariant form as in GTR. The result is Newton-Cartan theory. It is more general than the theory based on (1)

[6] I also believe that it extends to non-relativistic field theory, but I will not try to defend that claim here. See Hood [1970], Rosen [1972] for the construction of such a theory, specifically in non-relativistic quantum mechanics.

[7] See Barbour [1999] for a masterly and self-contained introduction.

through (3). When the total mass is finite (an 'island universe') the connection has a unique decomposition into a gravitational and inertial part:

$$\Gamma = \Gamma_{grav} + \Gamma_{inertial} \tag{4}$$

such that there exist global rectilinear coordinates in which Γ_{grav} satisfies (1) and $\Gamma_{inertial}$ vanishes. Acceleration, as defined by departures from straight-line motions with respect to $\Gamma_{inertial}$, is then uniquely defined. More prosaically, in the coordinates in which $\Gamma_{inertial}$ vanishes, non-accelerating motions are given by linear equations in x^0 and x^j. In this context (and in this sense) acceleration is absolute. But Γ can still be defined even in the case of an infinite and homogeneous mass distribution. In that case the decomposition (4) is no longer unique, there are no privileged inertial frames, and there is no absolute meaning to acceleration. Acceleration is only defined relative to a decomposition of the connection into an inertial and a gravitational part, and in the case of an infinite mass distribution, no such decomposition is preferred.

When I say NTG can be read as a relational theory, I mean something rather different. In the first instance we are speaking of a particle theory, not a field theory. The mathematics is also elementary – it is a reading of Principia. And my claim is that motions are relative whether or not the matter distribution is infinite, in roughly the way relationalists like Huygens and Leibniz thought it was. In the case of a finite number of particles, it is true, the system as a whole can be taken to define a frame of reference, on the usual reading of Principia an inertial frame of reference. Kant indeed took this as the only true inertial frame obtainable by Newton's methods, with the solar system only the first step (yielding only a first approximation) in an iterative procedure that in principle can only be completed when the entire universe is taken into account. Leibniz and Mach thought the same. I am taking the opposite strategy to Kant's. I am suggesting a reading in which this limit is not necessary to the structure of the theory and plays no role in its application, and in which the concept of inertial motion cannot even be defined.

As emphasized by Michael Friedman [1992] and Robert DiSalle [2006], the key to applying NTG to only a part of the actual matter distribution – to beginning the iterative procedure envisaged by Kant as concluding only with the entire universe – is Corollary vi to the Laws. On my reading of Principia it is put to wider use: Cor.vi frees NTG from the need to give any operational significance to the notion of inertial frame. And it poses a further question. Newton thought absolute space and time and absolute velocities were needed to make sense of his Definitions and Laws, but the latter admitted more symmetries; they included boosts among inertial frames as proved in Cor.v (the relativity principle). As a result Newton's absolute space and time and absolute velocities were dispensed with, and replaced by Galilean space-time and relative velocities, with accelerations as the only absolute quantities. But the Definitions and Laws, as applied to the motions of bodies 'among themselves', have still another symmetry, namely the group of time-dependent boosts among non-rotating frames, as demonstrated by Cor.vi. What should then replace Galilean space-time and absolute accelerations?

The passage from Newtonian to Galilean space-time was consistent with all the propositions of Bk.1 and Bk.2 of Principia, but it did violence to the Scholium to the Definitions. It is a conservative extension of NTG all the same. I shall begin with this. In §3 I shall give an argument from cosmology – a near neighbour to the one I began with above – to show we cannot in practise determine any inertial motions, not in an infinite nor in a finite universe (so long as it is sufficiently large), by Newton's principles – so knowing those motions cannot possibly have played any role in the actual application of NTG to planetary motions. The section following is on Cor.vi, and the explanation of the irrelevance of inertial motions to Newton's model of the solar system even in the favourable case of an island universe. In §6 and 7 the ladder is kicked away. The last sections are on relationalism and the definition of the 'right' space-time for *Principia*.

2. Newton's absolute space

Newton worked hard, in the Scholium to the Definitions, to explain why a theory of motion required an absolute, immaterial backdrop – absolute space – that was not subject to motions of its parts or to any change. He saw plainly the inadequacy of an appeal to the stellar background or any other observable body as an absolute standard of rest: that could define only a 'relative space', a 'movable dimension or measure' of motion – quantities that may have nothing to do with true motions. No observable body was immovable. No idealization was safe.

The avowed aim of *Principia*, as stated in the Scholium to the Definitions, was to show how these true motions were to be determined, including absolute velocities. The Laws that immediately followed spoke of straight-line motions, and motions deviating from these – absolute accelerations. Non-accelerating motions were 'straight' by reference to the sequence of positions of bodies in absolute space at equal times, with equal distances traversed in equal (absolute) times (in 4-dimensional terms, straight lines in Newtonian space-time). These locations in space, and space itself, were part of the furniture of the universe even if they were insensible. The relative motions of bodies, subject to forces satisfying the Laws, were supposed to eventually yield the true motions. But by wide assent what Newton's methods actually gave was motions referred to Galilean rather than Newtonian space-time, in which no velocity is preferred.

Relationalists like Huygens and Leibniz went further. According to Leibniz, space was 'nothing at all without bodies, but the possibility of placing them'; and 'instants, consider'd without the things, are nothing at all; ... they consist only in the successive order of things' (Alexander [1956 p.26-27]). His most damaging criticism was that according to Newton's principles, supposedly determinate physical properties and relations were undetermined by any possible measurement, however indirect – not just absolute positions but the velocities of particles too. They could be made whatever you like – there could be no 'sufficient reason' as to why they are thus and so – nor could you ever know that they are thus and so. Neither the speed nor the direction of the motion of the material universe as a whole, with respect to absolute space, could have any empirical meaning; hence neither could the absolute speed or absolute direction of motion of any body within it.

The critique applies equally to Galilean (or for that matter to Minkowski) space-time, if taken in the substantivalist sense (I shall come back to this sense in a moment), but it is particularly telling in the context of Newton's absolute space and time. There the problem can be stated independent of Leibniz's principles and of substantivalism. In space-time terms, Newton's absolute space picks out a preferred velocity – absolute rest – but the laws of motion do not. The symmetry group of Newtonian space-time is thus smaller than the symmetry group of the laws. The theory itself says there are these absolute quantities, whose values can never be obtained, for from the laws it follows (the principle of relativity):[8]

COROLLARY V

> The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion

Given that the motions of bodies among themselves – if necessary, all bodies – are all that any possible observation can ever have to go on, absolute velocities can never be determined.

Newton was clearly aware of the difficulty. For textual evidence, consider: the third edition of *Principia* contains only three hypotheses, one in Book 2, two in Book 3, the first and third of a technical nature. The second reads

HYPOTHESIS I

> That the centre of the system of the world is immovable

It was followed with the remark:

> This is acknowledged by all, while some contend that the earth, others that the sun, is fixed in that centre. Let us see what may from hence follow.

The proposition which followed and its proof resolved Leibniz's under-determination problem – in a manner of speaking:

PROPOSITION XI. THEOREM XI

[8] I am taking the laws to include the requirement that mass is a scalar and that force-functions do not depend on absolute positions or velocities, for only then is Cor.v a corollary (see Barbour [1989 p.31-2], Brown [2005 p.37-8]). From §5 on I shall, for simplicity, assume force functions do not depend on relative velocities either.

> That the common centre of gravity of the earth, the sun, and all the planets is immovable.
> For (by corollary iv of the Laws) that centre either is at rest, or moves uniformly forwards in a right line; but if that centre moved, the centre of the world would move also, against the Hypothesis.

From the point of view of Galilean space-time, having argued, by Cor.iv, that the centre of mass of the solar system moves inertially, Newton took it as a matter of *convention* that it be considered at rest. I shall come back to this use of Cor.iv shortly.

Newton's appeal to absolute space was excoriated by Mach. It was considered by Hans Reichenbach and other members of the Vienna circle as purely metaphysical (one might have thought logical empiricists would look on Newton's conventionalist maneuvering with more sympathy). But philosophers of science have been more forgiving of Newton since: Howard Stein, in particular, did much to rehabilitate him.

How does Galilean space-time stand up to Leibniz's critique? A general approach to the treatment of exact symmetries is this: insist that physically real quantities, and by extension, physically real properties and relations built out of those quantities, as specified by a physical theory, be *invariant* under the symmetries of that theory.[9] In the Galilean case that lets in relative angles and distances, (magnitudes of) relative velocities, and (magnitudes of) absolute accelerations; it disallows absolute directions and absolute speeds. The symmetry group of the space-time is bigger, and now includes the symmetries of the relativity principle; and the theory (in terms of invariants) no longer speaks of absolute positions or velocities, but only of (magnitudes of) absolute accelerations. We then have all the quantities ordinarily thought to be measurable in NTG, and gets rid of the ones forever underdetermined by any possible observation.

Commitment to invariant quantities does not in itself rule on the reality of points of space-time, for such points can still be relationally discerned by invariant quantities (again: relative distances, angles, and intervals in time). But if space-time points are to bear spatiotemporal relations to material particles, as substantivalists assume, the difficulty recurs: such relations will be changed by symmetry transformations, actively construed – by relative drags between material bodies and space-time – hence they can never be known. The same difficulty arises for Minkowski space and for general relativistic space-times with symmetries.[10]

So much is familiar ground. We are about to move on to new territory, and to the question of what Newton's laws really allow us to determine. But to that end we will need some of the machinery of affine-space theory, and it will be helpful to first set it up for the familiar space-times just encountered.[11]

[9] I have defended this reading of exact symmetries in physics in several places; see, e.g. Saunders [2003a,b], [2006], [2012].

[10] I do not myself think this difficulty is insurmountable, however. See my [2003a,b], and especially [2012 §12].

[11] We could use the more usual perspective of a differentiable manifold, but besides keeping

An affine space has much of the structure of the structure of a vector space. It is a set in which pairs of points define vectors, in turn used to define the straight lines talked of in the Laws – the straight line motions, real or imagined, of bodies moving inertially. Quite generally, let $\mathcal{X}$ be a set, let $\mathbb{V}^n$ be an n-dimensional vector space, and let $+$ on $\mathcal{X}$ be an action of $\mathbb{V}^n$ on $\mathcal{X}$ which is:

- transitive, i.e. any $a, b \in \mathcal{X}$, there is a vector $\overrightarrow{c} \in \mathbb{V}^n$ such that $a + \overrightarrow{c} = b$.
- free, i.e. if for some $a \in \mathcal{X}$, $a + \overrightarrow{c} = a + \overrightarrow{d}$, then $\overrightarrow{c} = \overrightarrow{d}$.

If free, the vector in $\mathbb{V}^n$ (which exists by transitivity) is unique, denote $a - b$.

The structure $\langle \mathcal{X}, \mathbb{V}^n, + \rangle$ is an *(n-dimensional) affine space*, denote $\mathcal{A}^n$. A notion of parallelism for pairs of points can now be defined: a, b is *parallel* to a', b'' if $a - b = a' - b'$. A transformation $g : \mathcal{A}^n \to \mathcal{A}^n$ is *affine* if it preserves parallels, i.e. if $a - b = a' - b'$, then $g(a) - g(b) = g(a') - g(b')$.

If $\mathbb{V}^n$ is equipped with the Euclidean scalar product with norm $|.|$, then a relative distance function h is defined on $\mathcal{A}^n \times \mathcal{A}^n$ by $h(a, b) = |a - b|_n$. Since $\mathcal{A}^n$ is affine, h is non-degenerate and satisfies the triangle inequality: it is a *metric* on $\mathcal{A}^n$. The resulting space $\langle \mathcal{X}, \mathbb{V}^n, +, h \rangle$ is *n-dimensional Euclidean space.*

It is now clear how to proceed. Space-time should be a $4-$dimensional affine space (to define straight-line inertial motions), but if it is to respect the relativity principle its Euclidean structure should be limited to simultaneous $3-$dimensional surfaces with no preferred velocity (so no decomposition of vectors in $\mathbb{V}^4$ into a spatial and temporal part). Define *neo-Newtonian* space-time as the structure $\langle \mathcal{X}, \mathbb{V}^4, +, \mathbb{W}, h, t \rangle$, where

- $\mathbb{V}^4$ is a 4-dimensional real vector space
- $+$ is a free, transitive action of $\mathbb{V}^4$ on $\mathcal{X}$
- $\mathbb{W}$ is a 3-dimensional subspace of $\mathbb{V}^4$
- h is a Euclidean inner product on $\mathbb{W}$, defining a norm $|.|_3$
- t is a Euclidean inner product on the quotient space $\mathbb{V}^4/W$, defining a norm $|.|_t$.

Since it is a 4-dimensional affine space the notion of straight line, and straight-line parallelism, is well-defined. The lines which (as vectors) lie in W connect simultaneous points in $\mathcal{X}$, i.e. points a, b such that $a - b \in W$; and for such points h defines a metric $h(a, b) = |a - b|_n$ on points in $\mathcal{X}$. We could, indeed, have begun with a simultaneity relation on $\mathcal{X}$ and defined W in its terms.[12]

The affine transformations that preserve S, t and h are the *Galilean transformations* (in which I include mirroring and time reversal). Defining a rectilinear

the mathematics as simple as possible, I am concerned to use methods as close as possible as those of *Principia* (an 'affine-space plus' theory in Stachel's [1993] terminology).

[12] The method followed by Stein [1967].

coordinate system $\overrightarrow{x}, t$ on $\mathcal{X}$ so as to assign equal times to simultaneous points, they are:

$$\begin{aligned} \overrightarrow{x} &\rightarrow R \cdot \overrightarrow{x} + \overrightarrow{u}t + \overrightarrow{d} \\ t &\rightarrow t + s, t \rightarrow -t \end{aligned} \tag{5}$$

where R is a matrix of determinant ± 1, $\overrightarrow{u}$, $\overrightarrow{d}$ are 3-dimensional Euclidean vectors, and s is a real number. Excluding the inversions, these transformations are parameterized by 10 real numbers, the (proper, orthochronous, inhomogeneous) Galilean group. In this way we turn our affine-space plus into a differentiable manifold (Galilean space-time). The measurable quantities – the invariants of these transformations – are time intervals, instantaneous relative distances and their derivatives, and (norms of) relative velocities and absolute accelerations, and derivatives.

What then is Newtonian space-time? It is the structure $\langle \mathcal{X}, \mathbb{V}^3 \oplus \mathbb{V}^1, +, h, t\rangle$; that is, it is neo-Newtonian space-time with the additional structure that any vector in $\mathbb{V}^4$ is uniquely decomposed into the sum of a vector in $\mathbb{V}^3$ (with the Euclidean metric h) and a vector in $\mathbb{V}^1$ (with the temporal metric t). Then any two points a,b in $\mathcal{X}$ have a unique spatial separation, namely $h(a, b) = |P(a-b)|_3$, where P is the projector from $\mathbb{V}^3 \oplus \mathbb{V}^1$ to $\mathbb{V}^3$.

Invariance of the decomposition $\mathbb{V}^4 = \mathbb{V}^3 \oplus \mathbb{V}^1$ narrows the space-time symmetries to the 7-dimensional subgroup consisting of rotations in space and translations in space and time. The relativity principle may be a symmetry of the laws, but it is not a symmetry of the space-time. Only with neo-Newtonian (or Galilean) space-time are the two correctly aligned. Or are they?

3. An inconvenient truth[13]

By the late 19th century, analytical methods took centre stage, and increasingly group theory. The concept of inertial coordinates was the crucial one, and of inertial frame. The latter term was first coined by Ludwig Lange in 1885, taking up a suggestion by Carl Neumann in his habilitation address of 1869. Neumann's work, published as 'On the principles of the Galilean-Newtonian theory' in 1870, began the debates over the foundations of NTG by Mach and others that cleared the way for Einstein's discoveries.

Neumann's monograph was followed soon after by Ernst Mach's *History and Root of the Principle of the Conservation of Energy*; both argued for the need for operational definitions in mechanics. The following year the mathematician Peter Tait offered the following construction (Tait [1983]): let there be three force-free bodies, no two of which are at relative rest, that are non-collinear; let the sequence of equal increments in the relative distance of any two serve as a clock; then their relative distances and angles with the third, if inertial, must satisfy certain constraints over time. In fact the first three instantaneous

[13] The first part of this section draws heavily on Barbour [1989 Ch.12, 1999].

relative configurations – 'snapshots' – are unconstrained (any will do, inertial or no); only with the fourth is there a constraint, amounting to a test of whether the three bodies are really inertial.

Tait's construction was followed by several others, among them one by Lange. But its weakness as an operational definition of inertial motion was also clear, for it depended on the availability of force-free ('fundamental') bodies. Which, precisely, were these? Unlike every other force, gravitational force cannot be screened off. The proposal, like Newton's appeal to absolute space, was a purely conceptual device: it had nothing to do with anything that could in practise be measured. As Bertrand Russell complained:

> If motion means motion relative to fundamental bodies (and if not, their introduction is no gain from a logical point of view), then the law of gravitation becomes strictly meaningless if taken to be universal – a view which seems impossible to defend. The theory requires that there should be matter not subject to any forces, and this is denied by the law of gravitation. (Russell [1903].)

The situation is different if we use Newton's methods. Rather than force-free, he used gravitating bodies, principally the earth, sun and moon. Because of the complexity of this system he made use of an iterative procedure, but in essence the method was an operational one. Thus to a first approximation he used the area law (Kepler's second law), itself derived from his principles for two bodies subject only to centripetal forces, to define a clock (with equal areas swept out in equal times). Time defined in this way, taking into account perturbations introduced by a third (and further bodies, notably Jupiter), is *ephemeris time*. It is time defined so as to make the observed relative angles subtended by the planets at the earth satisfy Newton's equations of motion. In the early 1950s it had acquired sufficient precision to replace sidereal time as the standard of time. It was replaced in turn by atomic clock time in 1972.

Does that secure the status of Prop.xi, but replacing 'at rest' by 'inertial'? No, for the same problem returns: the solar system as a whole is no more free from gravitational forces than the 'fundamental bodies' of which Russell spoke. The sun and planets are acted on by bodies beyond the solar system: in principle, arbitrarily many.[14] Newton's reasoning in deriving Prop.xi, even granting Hyp.i, was fallacious. Here is Cor.iv to the Laws, on which the proof turns:

COROLLARY IV

[14] Recognition of this point was slow. The question of the proper motions of stars among themselves was widely discussed by the mid 19th century, as we learn from Mary Somerville's *On the Connexion of the Physical Sciences* [1840] - but not much before, and not even then in terms of their action on the solar system. (I am grateful to George Smith on this point). Nor was the universality of gravity taken for granted: Somerville cited the motions of binary stars as conforming to Newton's laws as evidence that the stars, and not just bodies in the solar system, truly gravitate.

> The common centre of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common centre of gravity of all bodies acting upon each other (excluding external actions and impediments) is either at rest, or moves uniformly in a right line.

The proviso 'excluding external actions' robs it of any application to local gravitating bodies, like the planets and sun, which cannot be screened from the rest of the universe.

Are the accelerations thus produced at least likely to be small, so that Prop.xi (given Hyp.i) holds to a good approximation? The acceleration due to the stars in our local galaxy is certainly tiny – no more than 10^{-10} msec^{-2} – but include ever-larger regions of space filled with stars and their influence builds. It is the same as with Olber's paradox: the influence of remote stars falls off as the inverse square, but the number of stars at a given distance, if uniform, increases as the square. The result is an acceleration that scales linearly with the diameter of the universe, given an approximately uniform mass distribution.

Any elementary observation that escaped Newton should be spelt out in full.[15] Suppose, for simplicity, matter is uniformly distributed with density ρ over an enormously large sphere, and suppose the solar system is located within the sphere at distance R from its centre. The gravitational forces due to the matter at distance greater than R cancels, leaving only that of the matter in the interior. This mass, denote M, is as a function of R:

$$M(R) = \frac{4}{3}\pi R^3 \rho$$

The gravitational action of the sphere of mass M on the solar system (by hypothesis located on the boundary of the sphere) is the same as if all the mass were concentrated at the centre, and hence produces an acceleration a (using the proportionality of gravitational to inertial mass):

$$a = \frac{M(R)G}{R^2} = \frac{4}{3}\pi G \rho R. \tag{6}$$

The result is not materially effected on removing the simplifying symmetries (so long as the mass density is roughly uniform). For sufficiently large R, the acceleration of the solar system will as large as you like – or do *not* like.

But there is always the centre of mass frame for the universe as a whole, however large, so long as the total mass is finite. To this Cor.iv applies exactly and without ambiguity. Relative to this frame, whatever it is, if stellar densities

[15] It also eluded Stein. Having correctly observed 'Newton's theory affords a way of assigning kinematical states up to a Galilean transformation, on condition that one has succeeded in accounting completely for the relative motions by a system of action-reaction pairs, and on the further condition that there is no reason to suspect the system in question to be subject to an outside influence imparting equal accelerations to all its members', Stein [1977] continued: 'Newton had the good luck to find such a system: namely, the solar system; and the skill to effect its thorough dynamical explication.' It would be by good luck indeed were this to be true.

are similar to those of our local Hubble volume, the acceleration of the earth due to the rest of the universe swamps that due to the sun for a universe of size 10^{38} m and greater. As it happens, that is much larger than the *visible* universe (by about twelve orders of magnitude) – but there is every reason to think the actual universe is much larger, if not in fact infinite.

What if the actual universe *is* infinite? In that case there is a real difficulty. It was first recognized by Hugo Seelinger in 1894 and was much discussed subsequently. By the above method the acceleration of the solar system due to the rest of the universe can be given any number whatsoever. The problem was considered so severe that some concluded the inverse square law itself should be modified.[16] Shortly after, Einstein made the same suggestion, but for a very different reason. He too assumed the universe was approximately homogeneous, but in terms of his newly discovered theory of GTR; and in that context he saw it could not possibly be static. The solution was to modify the field equations, thus introducing his 'cosmological constant'. But now exactly the same reasoning applied to NTG, which Einstein used as a warming-up exercise. The modification to NTG eliminated the divergence of Eq.(6) as $R \to \infty$.[17]

Pursuit of these questions led directly to big-bang cosmologies on the one hand (because Einstein's solution was ineffectual), and to Newton-Cartan theory on the other (because Einstein's geometric methods were so successful). We saw in §1 how the problem of acceleration is resolved in that setting. But let us not take our eye off the ball. There is and never was any evidence to show the universe is less than 10^{38} m in size. Even in the favourable case, supposing we live in an island universe, by Newton's principles we never had good reason to think the solar system or any other concretely-defined frame of reference was even approximately inertial. That is, the central concept of NTG and classical mechanics, the concept of an inertial frame of reference, was never in practice operationally defined, not even approximately.

4. Corollary VI

The conclusion just reached has an air of paradox. It seems self-defeating if true: how is it NTG and classical mechanical theories *were*, in fact, successfully applied?

In only one place in *Principia* did Newton raise the question of the influence of the stars on the solar system: in Cor.ii to Prop.iv of Book 3. The proposition itself was that the aphelions and nodes of the orbits of the planets are fixed (the aphelion of an orbit is its furthest point from the sun; its nodes are the points of intersection of the plane of an orbit with the ecliptic). The result was proved in Book 1 (at Prop.xi) in the approximation in which the mutual actions

[16] This was the inconsistency discussed by Norton [1993]. See Norton [1999] for the history of these developments.

[17] Eq.(2) is replaced by:

$$\nabla^2\phi + \Lambda\phi = -4\pi G\rho.$$

This is equivalent to replacing the inverse square law by $\frac{1}{r^2}\exp(-\Lambda/r)$. Here Λ is the cosmological constant.

of the planets and comets among themselves are neglected, but Newton stated it again in Book 3 to draw two corollaries. The first is that 'the fixed stars are immovable, seeing they keep the same position to the aphelions and nodes of the planets' – meaning, the fixed stars are non-rotating with respect to the aphelions and nodes, and therefore absolutely non-rotating. The second reads:

> And since these stars are liable to no sensible parallax from the annual motion of the earth, they can have no force, because of their immense distance, to produce any sensible effect in our system. Not to mention that the fixed stars, everywhere promiscuously dispersed in the heavens, by their contrary attractions destroy their mutual actions, by Prop.lxx, Book 1.

On the first point, we must infer Newton never made the calculation Eq.(6); he simply failed to see that the fall-off as the inverse square of the gravitational force with distance is cancelled by the quadratic increase in the number of stars. But on the second the error is more subtle. Prop. lxx of Book 1 proved that the gravitational force vanished everywhere in the interior of a uniform shell of matter; Newton's reasoning, then, would seem to be that the stars are approximately uniformly distributed ('promiscuously dispersed'), so their action may be considered as that due to a concentric system of such shells with the sun at their centre proceeding to infinity. Given all this, their influence is shown to be zero.

We used Prop. lxx ourselves, in the case of an island universe, but in that case the assumption that the solar system is at the centre of the universe is clearly unwarranted. In the infinite case the assumption does not even make sense. Since homogeneous and infinite, no point is at the centre. If one applies the same construction anyway, one can build a system of spheres about any point whatsoever, so points at completely arbitrary distances from the solar system. Whereupon by the argument of Eq.(6) the acceleration can be anything and we are back to Seelinger's problem.

We are also back to Einstein's problem. Such a universe cannot possibly be static. Newton recognized rather better than did Einstein that here a balancing act was impossible – that nothing short of divine providence could ensure a static universe.[18] That sounds like a reductio, but it is clear from correspondence with

[18] Newton wrote:

> That there should be a central particle, so accurately placed in the middle as to be always equally attracted on all sides, and thereby continue without motion, seems to me a supposition fully as hard as to make the sharpest needle stand upright on its point upon a looking glass. For if the very mathematical centre of the central particle be not accurately in the very mathematical centre of the attractive power of the whole mass, the particle will not be attracted equally on all sides. And much harder is it to suppose all the particles in an infinite space should be so accurately poised one among another, as to stand still in a perfect equilibrium. For I reckon this as hard as to make, not one needle only, but an infinite number of them (so many as there are particles in infinite space) stand accurately poised upon their points. Yet I grant it possible, at least by a divine power; and if they were once to be placed, I agree with you that they would

Bishop Bentley that Newton welcomed the conclusion. Like Bentley, Newton looked to science for arguments for the existence of God.

But our concern is with the definition of inertial frames, and on this point Newton had more to say. The remark is to be found not in the *Principia* itself but in an informal draft of Book 3, published posthumously in English translation as *The System of the World* in 1728:

> It may be alleged that the sun and planets are impelled by some other force equally and in the direction of parallel lines, but by such a force (by Cor.vi of the Laws of Motion) no change would happen in the situation of the planets one to another, nor any sensible effect follow: but our business is with the causes of sensible effects. Let us, therefore, neglect every such force as imaginary and precarious, and of no use in the phenomena of the heavens... (Cajori [1938 Sec.8].)

Corollary vi to the Laws is the statement:

COROLLARY VI

> If bodies moved in any manner among themselves, are urged in the direction of parallel lines by equal accelerative forces[19], they will all continue to move among themselves, after the same manner as if thy had not been urged by those forces.

The proof was brief and there was no comment on it in the Scholium that followed. It was used to prove Prop.iii of section II (and Prop.lvii and lxiv in later sections of Book 1); no other reference to it is to be found in *Principia*. But Prop.iii was important to the argument for the inverse square law of force for gravity in Book 3. The proposition is:

PROPOSITION III

> Every body [A], that by a radius drawn to the centre of another body [B], however moved, describes areas about that centre proportional to the time, is urged by a force compounded of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

That is, given the premise, the force by which body A is moved is compounded of a centripetal force, and of whatever force is needed to make it accelerate at the same rate as is B. But according to the Phenomena of Book 3 the moons of

> continue in that posture without motion forever, unless put into new motion by the same power. (Letter Jan. 17, Bentley [1838 p.208].)

Einstein's solution (by means of a cosmological constant) required just such a balancing act; small wonder he later called it 'the greatest blunder of my life' (on this point see Bianchi and Roveli [2010]).

[19] Meaning, forces that produce equal accelerations (see Section 5).

jupiter *do* describe equal areas in equal times, so they must have those forces acting on them, so in particular (apart from the force that makes them accelerate at the same rate as jupiter) the moons experience a centripetal force towards jupiter. This is the first part of Newton's deduction from phenomena of the force of gravity as centripetal and falling off as the inverse square.

But *how* is it that all the moons of jupiter are accelerated the same as jupiter by the sun? The answer, of course, lies in Galileo's equivalence principle – the principle that neglecting friction, all bodies are accelerated the same by a gravitational field, whatever their composition. In Newton's terms that meant the constancy of the ratio of gravitational mass μ to inertial mass m; and it is to this that Newton immediately turned after establishing the force law of gravity, at Prop.vi of Book 3.[20] Newton argued for the universality of this ratio at length, on purely empirical grounds, with reference to experiments with pendulums that he had himself conducted.[21]

If, instead, the gravitational force is taken as given, the key to applying the Laws to the jupiter system is not so much Cor.vi but the combination of Cor.vi with Galileo's principle, which may be stated as:

COROLLARY VI*

> If bodies moved in any manner among themselves, are urged in the direction of parallel lines by equal gravitational forces due to outside bodies, they will all continue to move among themselves, after the same manner as if they had not all been urged by that force.

In this by 'equal gravitational force' I mean equal force per unit gravitational mass, denote $\overrightarrow{g}$, so that acting on a body of inertial mass m and gravitational mass μ the force is $\mu\overrightarrow{g}$. The resulting acceleration is

$$\frac{d^2\overrightarrow{x}}{dt^2} = \frac{\mu}{m}\overrightarrow{g}.$$

From the constancy of μ/m for any material body, whatever its constitution, and given that $\overrightarrow{g}$ is uniform, the condition of Cor.vi is met. Since the gravitational force of the sun is approximately uniform over the dimensions of the jupiter system, we can conclude that the motions of jupiter and its moons 'among themselves' will be the same as if the centre of mass system were moving inertially. By Cor.v, these are the same as if the centre of mass were at rest, whereupon Newton's laws apply.

It needs only a small rearrangement of this reasoning to conclude:

EQUIVALENCE PRINCIPLE

[20] Although he had intimated it much earlier, at Def.vii of Bk.1.

[21] An argument from induction, in Newton's sense, meeting the requirements of his Rules of Reasoning in Philosophy (in particular the third).

> If bodies moved in any manner among themselves are described in relation to an accelerating but non-rotating frame of reference, they will all move in relation to that frame as if acted on by uniform gravitational forces, producing the opposite acceleration.

With that we have a version of Einstein's equivalence principle – and the germ of an explanation for the universal proportionality of gravitational to inertial mass. However, we are now remote not so much from Newtonian concepts, as from his purpose in Book 3 of *Principia* – which was to *deduce* the inverse square law from the observed motions.

Cor.vi in conjunction with Newton's dynamical determinations of a state of non-rotation and of equal time intervals[22] solves the problem of inertial frames for gravitational physics. The reason that NTG works in application to the solar system, without ever giving an operational meaning to the notion of inertial frame, is that none is needed: all that matters is that motions are referred to a freely-falling non-rotating frame. That, to an excellent approximation, can be defined in terms of the dynamical behaviour of the solar system – using Prop.xiv of Book 3 – regardless of the distribution of matter in the rest of the universe. All that is needed is that the stars and galaxies are sufficiently distant so as to give rise to gravitational fields approximately uniform over the dimensions of the planetary orbits; whether or not they are changing in time – and, indeed, whether or not the universe is infinite – is irrelevant.

But now notice Cor.vi* together with Cor.vi *also* solves the problem of inertial frames for non-gravitational forces as well – so long as they are described by Newton's laws. The difference, on going over to non-gravitational physics, is that there is no analog of the constancy of gravitational to inertial mass; Cor.vi* cannot be used to justify the neglect of long-range forces other than gravity. But precisely because charge-to-mass ratios of every other force *do* vary, the existence of such forces would show up in the relative motions of bodies, whether referred to an inertial frame or – and this is the crucial point, by Cor.vi – to an accelerating but non-rotating frame. So long as the latter (as a material system - say, a system of gyroscopes in orbit around the earth) can be screened off from any non-gravitational forces that might produce a torque, and so long as Newton's laws apply, a dynamical analysis will yield a criterion of non-rotation, and thereby define a non-rotating frame. In principle such coordinates can be extended arbitrarily and used to describe any bodies, anywhere in the universe.

What happens if the bodies comprising the freely-falling frame *cannot* be screened off from non-gravitational forces due to distant bodies producing a torque? If there were such forces then there would be a difficulty. It may be, in such a world, that nothing short of the centre of mass frame of the entire universe would do – supposing it is a finite universe. In a world like that it may well be that physics as we know it would not be possible. But that world, thankfully, is not ours. The only long-range force other than gravity is electromagnetism,

[22] At Prop.1 Bk.1, deriving Kepler's second law (see Barbour [1989 pp.546-56] for an illuminating discussion).

which (except in extreme cases) can always be screened off. Nor are there large remote sources of this kind.

But if all of this is true, is there really any need of 'real' inertial frames, that are anyway operationally unavailable? Why not make do with local, freely-falling, non-rotating frames, as we do in GTR? If need be, the concept could be idealized as the limiting case of such a frame. The idealization is safe, as in practise such frames can be concretely realized (gyroscopes in orbit) to enormous accuracy.[23]

That question carries with it another: what has any of this really to do with gravity? Given that any rigid motion can be decomposed into a linear acceleration and a rotation, all that really matters is that the frame of reference be non-rotating. Free-fall, and the appeal to gravitational physics and screening from all other forces, is simply the easiest way of materializing a non-rotating frame. Cor.vi is more fundamental than the equivalence principle.

5. Kicking away the ladder

Of course our reasoning up to this point has been based on the laws (Axioms, or Laws of Motion, Bk.1), which suppose, by wide consent, that motions can be referred to an inertial frame. Presented analytically, as from the time of Lagrange, it is assumed there exist rectilinear inertial coordinates $\langle \overrightarrow{x}, t\rangle$ and the equations written in terms of these. For N particles subject to forces F that satisfy the third law (and that for simplicity are independent of relative velocities), we have the system of equations:

$$m_j \frac{d^2 \overrightarrow{x}_j}{dt^2} = \sum_{k \neq j = 1}^{N} F(\overrightarrow{x}_j - \overrightarrow{x}_k, t);\ j = 1, ..., N. \tag{7}$$

The symmetry group of this system of equations (assuming the m_j's transform as scalars) is the Galilean group (5). It would seem, then, that inertial coordinates, in the usual Galilean sense, are needed to set up the theory ab initio. It may be that in practise we an refer all the motions to local freely-falling frames, and the motions of the particles among themselves will all be the same as if referred to an inertial frame, but this inertial frame has to be defined, at least conceptually, to even write down the equations.

But the key concept to our reading of NTG as a relational theory, that of 'motions among themselves', plays a role here as elsewhere. It is most naturally expressed in terms of *difference* equations for particle pairs, in terms of the relative distance vectors

$$\overrightarrow{r}_{jk} = \overrightarrow{x}_j - \overrightarrow{x}_k. \tag{8}$$

From the N equations (7), define the $N(N-1)/2$ equations

[23] The Gravity B Probe, recently used to test for dragging of inertial frames – hence as a test of GTR – is an example.

$$\frac{d^2\overrightarrow{r}_{jk}}{dt^2} = \frac{1}{m_j}\sum_{l\neq j} F(\overrightarrow{r}_{jl}, t) - \frac{1}{m_k}\sum_{l\neq k} F(\overrightarrow{r}_{kl}, t). \tag{9}$$

Of these only $N-1$ are linearly independent. From such a set, together with the equation

$$\sum_k m_k \frac{d^2\overrightarrow{x}_k}{dt^2} = \sum_k \sum_{j\neq k} F(\overrightarrow{r}_{jk}, t) = 0 \tag{10}$$

the N equations (7) can be derived, and vice versa. Eq.(10) follows from Newton's third law: it states that the total momentum is conserved. It is invariant under Galilean boosts

$$\overrightarrow{x}_k \to \overrightarrow{x}_k + \overrightarrow{u}t$$

(by Cor.v), as are the $N(N-1)$ equations (9). But (8) and (9) are invariant under the much wider class of symmetry transformations:

$$\overrightarrow{x}_k \to \overrightarrow{x}_k + \overrightarrow{f}(t) \tag{11}$$

where $\overrightarrow{f}$ is any twice-differentiable vector-valued function of the time.[24] The imposition of Eq.(10) is what distinguishes 'true' inertial frames from our local freely-falling frames. If we dispense with (10), and allow that the system of equations (8), (9) define the entire theory – say a new theory, different from NTG – the symmetries of the equations will be those of Cor.vi, not just Cor.v.

There is, however, something odd about this way of putting it. For wasn't Cor.vi *already* a consequence of the Laws, just as was Cor.v? Why think Eq.(7) is the correct expression of Newton's laws, rather than (8) and (9)? The latter are clearly better suited to define the motions of a system of bodies 'among themselves'. If it was a check on the correct equations for the expression of Newton's laws that they exhibit the symmetries of the laws (Cor.v), leading to (7), why isn't it a check that they have the symmetries of Cor.vi, namely (11), leading to (8) and (9) and not (7)?

The answer, presumably, will turn on the extent to which we can view the *Principia* as at bottom a theory of the relative motions of particle pairs. As a first step one would like to see if the Definitions and Laws can be rewritten in a way that obviously translates into Eq.(8) and (9), rather than (7), *and* that preserves intact the methods of proof used in the first two books of *Principia*.

An obvious way is simply to relativize them – to simply suppose that 'absolute space' refers to any rigid non-rotating frame, in the same way that passing from Newtonian to neo-Newtonian space-time one takes 'absolute space' to refer to any inertial frame. No matter which one is chosen, the subsequent geometrical constructions (the main proof procedure used in Principia) go through unchanged. But this is to quantify over these sorts of reference frames, and (arguably) to space-time points, whether or not they are occupied by matter. Proceeding in this way, space-time may have less structure than Newton thought, but must still be presupposed.

[24] Eq.(11) together with the second of (5) were called the 'Newtonian group' by Ehlers [1973].

On a genuine eliminativism, only reference to material bodies is permissible, and the laws should be couched in this way. For example, and keeping as close as possible to Newton's language[25], the laws might be restated as:

First Law* Every body continues in its state of relative rest, or uniform relative motion in a right line, with respect to another, unless compelled to change that state by a difference in motive forces impressed upon them.

Second Law* The change in relative motion is proportional to the difference in motive forces impressed upon them, and is made in the direction of the right line in which the difference in motive forces is impressed.

Third Law To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

(The third law is unchanged.)

As stated these laws clearly support (9) and the second equality of Eq.(10). But it is not obvious whether the laws, so weakened, can be applied using Newton's methods in *Principia* - or can have the same consequences. Some changes are inevitable. Take for example the law of conservation of momentum (Cor.iv); from the modified laws above it follows not that the centre of mass of all bodies acting upon each is in uniform motion, excluding external actions and impediments, but that it is at rest or in uniform motion relative to another (that is free from external actions and impediments, or for which the relative external force vanishes). However the first part of Cor.iv stands unchanged.

But what does 'uniform, straight line motion' mean, if inertial motions (in the usual sense of 'inertial') are undefined? Just this: relative straight-line motions are those for which the relative distances (lengths of lines joining pairs of bodies at each instant of time) and relative angles between lines joining instantaneous bodies at various times one time, change in a characteristic way. What way exactly? – the way they change under the usual representation of force-free bodies, as moving inertially. (I shall come back to the question of how we measure such quantities shortly.)

In any case, whatever its relation to *Principia*, the system of equations (8) and (9) is of independent interest – especially if, as seems to be promised, it yields exactly the same relative motions as does (7). Can we kick away the ladder? – can we dispense with Eq.(10)? *Formally* there can be no obstacle: we are already assured that any solution to the N equations (7) will yield $N(N-1)$ difference vectors $\overrightarrow{r}_{jk}$ satisfying (9). Conversely, any solution to the $N(N-1)$ equations (9) satisfying (8) (of which only $N-1$ functions $\overrightarrow{r}_{jk}$ are linearly independent) defines a solution to the N equations (7). Or put it like this: the set of difference equations, Eq.(9), for antisymmetric F, has a gauge freedom –

[25] First law: 'Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it'. Second law: 'The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed'..

an overall acceleration, constant in space, as an arbitrary function of time. Up to this choice of gauge, they are in 1:1 correspondence with solutions to (7).[26] As Newton said: 'Let us, therefore, neglect every such force as imaginary and precarious, and of no use in the phenomena of the heavens'.

6. Is it relationalism?

I have urged that a key concept of *Principia* is that of 'motions among themselves', and made this out in terms of difference equations. Is a theory based on Eqs.(8), (9) a distinctively relational theory? It *appears* so. It is certainly a theory in which no meaningful notion of the velocity or acceleration of a single body is available: there are only relative distances and velocities of simultaneous bodies and their time derivatives, of two or more bodies among themselves.

We should be clear on just what these quantities are. As in Galilean-covariant theories, they include rates of change of the norms of the $\overrightarrow{r}_{jk}$'s, the quantities:

$$\frac{d}{dt}\left|\overrightarrow{x}_j - \overrightarrow{x}_k\right|, \frac{d^2}{dt^2}\left|\overrightarrow{x}_j - \overrightarrow{x}_k\right|. \tag{12}$$

These require comparisons of spatial lengths at different times. Considering three or more bodies, one also has rates of change of relative angles, of the form

$$\frac{d}{dt}\left|(\overrightarrow{x}_j - \overrightarrow{x}_k)\cdot(\overrightarrow{x}_k - \overrightarrow{x}_l)\right|. \tag{13}$$

Quantities of the form (12), (13) were always acceptable to relationalists.

The traditional difficulty has lain rather with rotations − say, two bodies rotating about their common centre. If the motion is purely rotational, then all quantities of the form (12), (13) are zero. Nevertheless the bodies have non-zero relative velocities, and, referred to inertial frames, non-zero accelerations. How do we describe their motion on our relationalist reading of *Principia*? The answer is that the notion of relative velocity is independent of the principle of inertia, so carries over unchanged, and the acceleration goes over to relative accelerations of the rotating bodies, with no need to refer it to an inertial frame. The two bodies have a relative velocity because the direction of the line connecting them changes in time; and they have a relative acceleration because the direction of the relative velocity changes in time. The space-time structure needed to make sense of rotations is that a spatial direction at one time can be compared with a spatial direction at another time, with no need of the concept of straight-line motion or departure from straight-line motion.

So much is obvious if we write the derivative as the limit of a difference equation:

$$\frac{d}{dt}\overrightarrow{r}_{jk}(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\overrightarrow{r}_{jk}(t+\Delta t) - \overrightarrow{r}_{jk}(t)\right]. \tag{14}$$

[26] As a result Noether's first theorem no longer applies, and the total momentum is not a conserved quantity - as has already been observed.

The mistake was to think this can be expressed as

$$\begin{aligned}\frac{d}{dt}(\overrightarrow{x}_j - \overrightarrow{x}_k)(t) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[(\overrightarrow{x}_j(t+\Delta t) - \overrightarrow{x}_k(t+\Delta t)) - (\overrightarrow{x}_j(t) - \overrightarrow{x}_k(t))\right] \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[(\overrightarrow{x}_j(t+\Delta t) - \overrightarrow{x}_j(t)\right] - \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[(\overrightarrow{x}_k(t+\Delta t) - \overrightarrow{x}_k(t)\right]\end{aligned}$$

i.e. as the difference between two velocities. Only the relative velocity can be defined, not velocities of which it is the difference.

Of course Machians go further, and reject quantities like (14) as well. The comparison of spatial directions at different times is omitted according to Barbour [1999], along with comparisons of 4−dimensional vectors in space-time. The solution to a dynamical problem in Barbour's terms is fully determined given two neighboring relative configurations of particles, without information on their relative orientation. Our form of relationalism is from a Machian point of view a half-way house.

Along with Machians,one may well ask: how are these angles between lines in space at different times determined? Clearly not in the way angles between lines in space normally are – the two lines cannot be brought into comparison at a single time. The answer is, as always, dynamically. The comparison (for a pure rotation) is in the first instance made between the relative velocities of the two bodies at neighbouring times – their relative acceleration. *This* quantity is given by a centripetal force law in terms of masses and relative distances of bodies alone. Since (for a pure rotation) their relative distance is constant in time, so is the angular velocity and the norm of their relative velocity at each time. The rate of change of relative velocity then determines the relative velocity (and the angle between the lines joining the two bodies at two times) uniquely. All these determinations (or rather their norms) are invariant under boosts to arbitrary non-rotating frames, Eq.(11), as also, for time-independent force-laws, under the symmetries (5).

Was it considerations like this that led Huygens to reconsider his earlier view, in reluctant agreement with Newton, that cases of pure rotation could only be understood as motions in relation to absolute space? At the end of his life he wrote instead:

> Circular motion is relative motion along parallel lines, where the direction is continually changed and the distance is kept constant through a bond. Circular motion in one body is the relative motion of the parts, while the distance remains constant owing to the bond. (Huygens [1888-1950, Vol.21, p.507].)

Passages like these were quoted by Stein as evidence of the depth of Huygens' thinking (Stein [1977]), but on this point there is hardly a consensus:

> [T]o analyze rotation in terms of an objective, or absolute, notion of velocity difference rather than objective, or absolute velocity is to possess exactly the insight Newton lacked, but it is also to reject

> the full-blown relational conception of motion, something that was beyond the ken of Huygens' philosophical dogmas. (Earman [1989 p.71].)[27]

Earman seems to equate Huygen's insight with the shift from Newtonian to neo-Newtonian space-time, but the point goes deeper: only a comparison of velocities at the same time is needed, not at different times. We read Huygens as dispensing with the principle of inertia. Earman thought no theory like this was available in Newton's day.

That claim, I have shown, is debatable. And whether or not a theory based on (8) and (9) is a 'full-blown' relationalism, it is surely a theory in which:

> True and simple motion of any one whole body can in no way be conceived – what it is – and does not differ from rest of that body. (Huygens [1888-1950, fragment 8, vol.16].)

7. Newton-Huygens spacetime

The limiting process used in Sec.6 shows what is needed in terms of manifold structures: a connection as a rule for comparing spatial directions at different times – a rule for the parallel transport of *directions in space* (compatible with the temporal and spatial metrics), not *space-time*. Such a construction (with the connection an 'absolute' structure) was given by Earman; he called it 'Maxwell space-time'.[28] Here I conclude with a construction that goes the other way. Galilean space-time is the right manifold corresponding to neo-Newtonian space-time, an affine-space; what does Maxwell space-time correspond to? It should be a weaker structure – call it 'Newton-Huygens space-time' – but be sufficient for the constructions used in *Principia*. It might also provide a cleaner route to Newton-Cartan spacetime then the conventional one.[29]

Its outline should already be clear. We are looking for a structure in which affine notions and not just metrical ones are restricted to surfaces of constant time. Define Newton-Huygens spacetime as the structure $\langle \mathcal{X}, \mathbb{V}^3, \mathbb{V}^1, +_3, +_1 \rangle$, where

- $\mathbb{V}^3$and $\mathbb{V}^1$ are real vector spaces of dimensions 3 and 1 respectively, with Euclidean inner products $\langle .,.\rangle_3$, $\langle .,.\rangle_1$, respectively.
- $+_3$ is a free (but not transitive) action of $\mathbb{V}^3$ on $\mathcal{X}$, i.e. for any $a \in \mathcal{X}$, $\overrightarrow{b} \in \mathbb{V}^3$, $a +_3 \overrightarrow{b} \in \mathcal{X}$

[27] Barbour [1989 p.675] is likewise critical of Huygens on this point.

[28] Earman [1989 §2.3, §4.7]. He still thought it inadequate to relationalism, however, as flouting his relational requirement R1, that space-time 'cannot have structures that support absolute quantities of motion'. (But whether rotation is absolute or relative is part of what is in contention.)

[29] I owe this observation to David Wallace.

- $+_1$ is a free and transitive action of $\mathbb{V}^1$ on the cosets of $\mathcal{X}$ under the action of $\mathbb{V}^3$.

The action of $\mathbb{V}^3$ on $\mathcal{X}$ defines an equivalence relation on $\mathcal{X}$, 'simultaneous with'; the cosets of $\mathcal{X}$ under $\mathbb{V}^3$ are time-slices. By construction, $+_3$ acts transitively on any time-slice. Hence, since $+_3$ is free, for any simultaneous $a, b \in \mathcal{X}$, there is a unique $\overrightarrow{c} \in \mathbb{V}^3$ such that $a +_3 \overrightarrow{c} = b$, denote $b - a \in \mathbb{V}^3$. Now consider two pairs of simultaneous points in $\mathcal{X}$. If a is simultaneous with b, and a' is simultaneous with b' (possibly at a different time), the quantity:

$$\cos\theta = \frac{\langle (a-b), (a'-b') \rangle_3}{|a-b|_3 |a'-b'|_3}$$

is well-defined. θ is the angle between the directions in space $a - b$ and $a' - b'$. But there is no notion of straight line connecting non-simultaneous points; the principle of inertia does not apply.

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