Donaldson Invariants and Moduli of Yang-Mills Instantons

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Invariants are gotten by cooking up a number (homology group, derived category, etc.) using auxiliary data (a metric, a polarisation, etc.) and showing independence of that initial choice.

Donaldson and Seiberg-Witten invariants count in moduli spaces of solutions of a pde (shown to be independent of conformal class of the metric).

Example (baby example)

\[ f : (M, g) \rightarrow \mathbb{R}, \text{ the number of solutions of} \]

\[ \text{grad}_g f = 0 \]

is independent of \( g \) (Poincaré-Hopf).
Recall (Setup) $(X, g)$ smooth oriented Riemannian 4-manifold and a principal $G$-bundle $\pi : P \to X$ over it.

Consider the Yang-Mills functional

$$YM(A) = \int_M |F_A|^2 d\mu.$$ 

The corresponding Euler-Lagrange equations are

$$d_A^* F_A = 0.$$ 

Using the gauge group $\mathcal{G}$ one generates lots of solutions from one. But one can fix a gauge

$$d_A^* a = 0.$$
Recall (ASD Equations)

In dimension 4, the Hodge star splits the 2-forms $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$. This splits the curvature

$$F_A = F_A^+ + F_A^-.$$ 

Have

$$\kappa(P) = \|F_A^+\|^2 - \|F_A^-\|^2 \leq YM(A) = \|F_A^+\|^2 + \|F_A^-\|^2,$$

where $\kappa(P)$ is a topological invariant (e.g. for SU(2)-bundles, $\kappa(P) = -8\pi^2 c_2(P)$). This comes from Chern-Weil theory.

**Key Idea.** Anti-self-dual connections, e.g. ones with $F_A^+ = 0$, minimise the Yang-Mills functional among connections on $P$. 

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Atiyah-Drinfeld-Hitchin-Manin constructions describes all finite energy instantons over $\mathbb{R}^4$ and $S^4$ in terms of algebraic data: a $k$-dimensional complex vector space $\mathcal{H}$, self adjoint linear operators $T_i : \mathcal{H} \to \mathcal{H}$, $n$-dim. vector space $E_\infty$, and a map $P : E_\infty \to \mathcal{H} \otimes S_+$ satisfying the *ADHM equations* [ADHM].

**Example**

Consider $SU(2)$ bundle $P \to \mathbb{R}^4$ with $c_2(E) = 1$. Then the ADHM matrices $T_i$ are $1 \times 1$ and $P \in \text{Hom}(E_\infty, S^+)$ is rank 1. So the moduli space of YM instantons is $\mathbb{R}^4 \times \mathbb{R}^+$.  

**The Point.** Instantons occur in moduli spaces.
The Ambient Space

We begin with the space of all connections, modulo gauge.

Let $G$ be $\text{SO}(3)$ or $\text{SU}(2)$. Let $P \to X$ be a principal $G$-bundle over a smooth oriented Riemannian 4-manifold $X$.

A infinite-dimensional affine space of connections on $P$

$\mathcal{G}$ infinite-dimensional group of automorphisms of $P$, it acts on connections via

$$u \cdot A := A - (\nabla_A u)u^{-1}.$$ 

**Definition**

The space

$$\mathcal{B} := \mathcal{A}/\mathcal{G}$$

(with quotient topology) is the space of *connections modulo gauge* (on $P$).
\( \mathcal{B} \) carries an \( L^2 \)-metric

\[ \| A - B \|_{L^2} := \left( \int_M |A - B|^2 d\mu \right)^{1/2} \]

descends to a distance function

\[ d([A], [B]) := \inf_{u \in \mathcal{G}} \| A - u \cdot B \| \]

(clear, except for \( d([A], [B]) = 0 \Rightarrow [A] = [B] \); uses compactness of \( \mathcal{G} \))
Local Models of the Orbit Space $\mathcal{B}$

**Definition**

Fix $A \in \mathcal{A}$ and $\epsilon > 0$. Define

$$T_{A,\epsilon} = \{ a \in \Omega^1(g_E) | d_A^* a = 0, \| a \|_{L^2_{\ell-1}} < \epsilon \}$$

By Uhlenbeck’s gauge fixing theorem and the implicit function theorem, $A + T_{A,\epsilon}$ meets every nearby orbit.

**Definition (/Lemma)**

Write $\Gamma_A = \text{Iso}_G A$. It is isomorphic $C_G(H_A)$, where $H_A$ denotes the holonomy group of the connection $A$.

**Theorem**

For small $\epsilon > 0$ the projection $\mathcal{A} \to \mathcal{B}$ induces a homeomorphism from $T_{A,\epsilon}/\Gamma_A$ to a neighborhood of $[A]$ in $\mathcal{B}$. 
Write $\mathcal{A}^*$ for the subspace of $\mathcal{A}$ with minimal $G$-isotropy groups

$$\mathcal{A}^* = \{ A \in \mathcal{A} | \Gamma_A = C(G) \}.$$ 

These are called *irreducible connections*.

**Theorem**

The quotient $\mathcal{B}^* = \mathcal{A}^*/G$ is a smooth Banach manifold (the moduli space of irreducible connections).
Local Models of the ASD Moduli Space

Let $M_P(g)$ be the moduli space of ASD connections on $P \to (X, g)$. Define a map $\psi : T_{A, \epsilon} \to \Omega^+(g_E)$ by

$$\psi(a) = F^+(A + a)$$

and write

$$Z(\psi) := \psi^{-1}(0) \subset T_{A, \epsilon}$$

**Theorem**

*The previous homeomorphism induces a homeomorphism from $Z(\psi)/\Gamma_A$ to a neighborhood of $[A]$ in $M_P(g)$.*

Have a bundle $\mathcal{E} := A^* \times_{G/\mathbb{C}(G)} \Omega^+(g_E) \to B^*$. Taking the ASD portion of curvature yields a section $\Psi_g : B^* \to \mathcal{E}$ s.t.

$$\psi^{-1}(0) = M^*_P(g) \subset B^*$$

(the *moduli space of irreducible ASD connections*).
Deformation-Obstruction Theory

Have a linear elliptic operator

\[ C^\infty(\text{ad}(P) \otimes \Lambda^1 T^* X) \xrightarrow{d^+_A \oplus d^*_A} C^\infty(\text{ad}(P) \otimes (\Lambda^2_+ T^* X) \oplus (\Lambda^0 T^* X)) \]

its kernel is deformations of the ASD condition along gauge fixing.

\[ \ker (d^+_A \oplus d^*_A) = T_A M \]

and

\[ \text{coker} (d^+_A \oplus d^*_A) = O_A M. \]

Remark. It will turn out that for generic \(g\), \(O_A M = 0\). This is a nice deformation-obstruction theory.
The deformation-obstruction theory of the ASD moduli problem is governed by an elliptic complex

\[ \Omega_X^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega_X^1(\mathfrak{g}_E) \xrightarrow{d_A^+} \Omega_X^+(\mathfrak{g}_E) \]

\((d_A^+ \circ d_A = 0\) by ASD condition).

Note \( H^1_A \cong \ker \delta_A \) and \( H^2_A \cong \text{coker} \delta_A \).

The **virtual dimension** of the moduli problem is

\[ 8c_2(E) - 3(1 - b_1(X) + b_+(X)) \]

for SU(2) and

\[ s = -2p_1(E) - 3(1 - b_1(X) + b_+(X)) \]

for SO(3).
Analogy with other gauge theories

Flat connections (Casson theory)

$$\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{d_A} \Omega^2 \xrightarrow{d_A} \Omega^3$$

Holomorphic structures (Donaldson-Thomas theory)

$$\Omega^{0,0} \xrightarrow{\bar{\partial}_A} \Omega^{0,1} \xrightarrow{\bar{\partial}_A} \Omega^{0,2} \xrightarrow{\bar{\partial}_A} \Omega^{0,3}$$

$G_2$-instantons (Donaldson-Segal programme, ongoing)

$$\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{\psi \wedge d_A} \Omega^6 \xrightarrow{d_A} \Omega^7$$

Spin(7)-instantons (unsure of current state)

$$\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{\pi_7 \circ d_A} \Omega^2$$
Theorem

If $A$ is an ASD connection over a neighborhood of $X$, a neighborhood of $[A]$ in $M$ is modelled on a quotient $f^{-1}(0)/\Gamma_A$, where $f : \ker (d_A^+ \oplus d_A^*) \to \text{coker} (d_A^+ \oplus d_A^*)$ is a $\Gamma_A$-equivariant map.

Definition

An ASD connection is called regular if $H^2_A = 0$ (unobstructed).

Definition

An ASD moduli space is called regular if all of its points are regular.

If the moduli space is regular, 0 is a regular value, and Sard-Smale is applicable.
We need some bits of Fredholm theory.

**Theorem (Sard-Smale)**

Let $F : P \to Q$ be a smooth Fredholm operator between paracompact Banach manifolds. Then, the set of regular values of $F$ is everywhere dense in $Q$. Moreover, if $P$ is connected then the fibre $F^{-1}(y) \subset P$ over any regular value $y \in Q$ is a smooth submanifold of dimension equal to the Fredholm index of $F$.

**Theorem (Fredholm Transversality)**

Let $F : P \to Q$ be a smooth Fredholm operator between paracompact Banach manifolds and let $S$ be a finite-dimensional manifold equipped with a smooth map $h : S \to Q$. Then there is a $C^\infty$ arbitrarily close map $h' : S \to Q$, to $h$, such that $h'$ is transverse to $F$. 
Write $\mathcal{C}$ for the space of conformal classes of Riemannian metrics on $X$. We obtain a parameterized bundle $\overline{\mathcal{E}} \rightarrow \mathcal{B}^* \times \mathcal{C}$ along with section $\overline{\Psi} : \mathcal{C} \times \mathcal{B}^* \rightarrow \overline{\mathcal{E}}$ s.t. $\overline{\Psi}\{g\} \times \mathcal{B}^* = \Psi g$.

**Theorem (Uhlenbeck-Freed)**

Let $\pi : P \rightarrow X$ be a principal $G$-bundle over a smooth oriented 4-manifold $X$. Then the zero set of $\overline{\Psi}$ in $\mathcal{B}^* \times \mathcal{C}$ is regular.

**Corollary**

There is a dense subset $\mathcal{C}' \subset \mathcal{C}$ such that for $[g] \in \mathcal{C}'$ the moduli space $M^*_P(g)$ is a regular (as the zero set of the section $\Psi g$ determined by $[g]$).
Compactness

**Definition**

An *ideal* ASD connection over $X$ is a pair $([A], (x_1, \ldots, x_\ell))$. A sequence of gauge equivalence classes $[A_\alpha]$ converges weakly to a limiting ideal connection $([A], (x_1, \ldots, x_\ell))$ if

1. $\int_X f|F(A_\alpha)|^2 d\mu \to \int_X f|F(A)|^2 d\mu + 8\pi^2 \sum_{r=1}^{\ell} f(x_r)$ (the action densities converge as measures)

2. there are bundle maps $\rho_\alpha : P_{\ell}|_{X \{x_1, \ldots, x_\ell\}} \to P_{k}|_{X \{x_1, \ldots, x_\ell\}}$ s.t. $\rho^*_\alpha(A_\alpha)$ converges (in $C^\infty$ on compact subsets of the punctured manifold) to $A$

Write $\overline{M}$ for the set of all ideal connections (with a metrizable topology induced from the notion of weak convergence).

**Theorem**

*Any infinite sequence in $\overline{M}$ has a weakly convergent subsequence with a limit point in $\overline{M}$.***
One can make sense of orientation data on all of $\mathcal{B}$. At each connection $A$ (instanton or not) write elliptic operator

$$L_A : \text{ad}(E) \otimes T^*X \to \text{ad}(E) \otimes (\Lambda^2_+ \oplus \Lambda^0)$$

At each point form

$$\Lambda^{\text{top}}(\ker L_A) \otimes (\Lambda^{\text{top}}(\text{coker } L_A))^{-1}$$

as the fibre of a real line bundle $L \to \mathcal{B}$ (orientation line bundle).

This induces an orientation on $M \subset \mathcal{B}$. 
First suppose a zero-dimensional moduli space $M_{\text{ASD}}(g)$. Wish to show a count (with correct multiplicities) is independent of $g$.

**Example (Toy Example.)**

Take $\pi : V \to B$ finite-rank vector bundle and $s$ section of $\pi$ which is transverse to the zero section. Extract a homology class

$$[Z(s)] \in H_d(B),$$

$d := \dim B - \text{rk } V$. This is the Euler class of $V$.

For each choice of metric $g$, define an integer

$$q(g) = \sum_{A \in \mathcal{M}_P(g)} \epsilon(A),$$

where the sign $\epsilon$ is determined by our orientation data $L$. 
Theorem

For any two generic metrics $g_0$ and $g_1$ on $X$,

$$q(g_0) = q(g_1).$$

Following the masters, we return to our toy example before proving this theorem.

Let $C$ be a finite-dimensional manifold parameterising sections $s_c$. Have a parameterized zero set $Z \subset B \times C$.

The number of zeros of some $s_c$ is the number of elements of the fibre $\pi : Z \to C$ over $c \in C$—the degree of a proper map.
Theorem

The degree of \( \pi : Z \to C \) is invariant of the choice in \( c \in C \) in the target.

Proof.

Take a path \( \gamma : I \to C \) transverse to \( \text{im}(\pi) \). Define

\[
W = \{(z, t) | \pi(z) = \gamma(t) \subset Z \times [0, 1]\},
\]

which is a smooth one-dimensional manifold with boundary. The boundary component at 0 is \( \pi^{-1}(c_0) \) and the boundary component is \( \pi^{-1}(c_1) \). And the total oriented boundary of a compact one-dimensional manifold is zero.
Proof.

Take a path \( \gamma : [0, 1] \rightarrow \mathcal{C} \) in the space of conformal classes of metrics which joins \( g_0 \) and \( g_1 \). Using Fredholm transversality, perturb it to be transverse to the Fredholm map \( \mathcal{M}^* \rightarrow \mathcal{C} \). For generic \( \gamma \), the space

\[
\mathcal{M} = \{(A, t) | A \in M(\gamma(t))\} \subset \mathcal{B}^* \times [0, 1]
\]

is an oriented one-manifold with boundary components \( M(g_0) \) and \( M(g_1) \). Uhlenbeck’s compactness theorem completes the proof [We2].
The dimension of $M_{ASD}$ is not, in general, zero. Suppose $b^+(X) > 1$ (avoid reducible solutions). Take $M_{ASD}$ even dimensional

$$\dim M_{ASD} = 2d = 8c_2(E) - 3(b^+(X) + 1).$$

Let $[\Sigma_1], \ldots, [\Sigma_d]$ be classes in $H_2(X, \mathbb{Z})$. Atiyah-Bott [] shown that the cohomology ring $B^*_X$ is a polynomial algebra generated in degrees two and four, where the two-dimensional generators are gotten from a certain map

$$\mu : H_2(X) \rightarrow H^2(B^*_X).$$

In this way, in positive dimension, we can get integers

$$q = \langle \mu(\Sigma_1) \cup \ldots \cup \mu(\Sigma_d), [M] \rangle$$
This gives a symmetric multilinear function

\[ q : H_2(X) \times \ldots \times H_2(X) \to \mathbb{Z} \]

such that

- \( \bar{q}([\Sigma_1], \ldots, [\Sigma_d]) = -q([\Sigma_1], \ldots, [\Sigma_d]) \) (reversing orientation)
- if \( f : X \to Y \) is an orientation preserving diffeomorphism then

\[ q(f([\Sigma_1]), \ldots, f([\Sigma_d])) = q([\Sigma_1], \ldots, [\Sigma_d]) \]

Proving these more general invariants exists and satisfy the above properties is analytically more difficult than the aforementioned simple (dim. zero) invariants (done in [DK, Sect. 9.2.]).
When $b_+^2 = 1$, the Donaldson invariants are not independent of the metric.

The paths between $g_0$ and $g_1$ cannot always avoid reducible solutions—which occur as codimension one walls in $\mathcal{C}$.

Within chambers, the Donaldson invariants are constant. When crossing a wall, they change by an explicit formula

$$\Phi^{X,g_1} - \Phi^{X,g_2} = \delta^X_{g_1,g_2},$$

where the $\delta^X_{g_1,g_2}$ can be computed explicitly using modular forms [Gott96].

