

INTRODUCTION TO LOGIC

Lecture 2

Syntax and Semantics of Propositional Logic.

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Logic is the beginning of wisdom.

Thomas Aquinas

Outline

- ➊ Syntax vs Semantics.
- ➋ Syntax of \mathcal{L}_1 .
- ➌ Semantics of \mathcal{L}_1 .
- ➍ Truth-table methods.

Syntax

Syntax is all about **expressions**: words and sentences.

Examples of syntactic claims

- 'Bertrand Russell' is a proper noun.
- 'likes logic' is a verb phrase.
- 'Bertrand Russell likes logic' is a sentence.
- Combining a proper noun and a verb phrase in this way makes a sentence.

Semantics

Semantics is all about **meanings** of expressions.

Examples of semantic claims

- 'Bertrand Russell' refers to a British philosopher.
- 'Bertrand Russell' refers to Bertrand Russell.
- 'likes logic' expresses a property Russell has.
- 'Bertrand Russell likes logic' is true.

Use vs Mention

Note our use of quotes to talk about expressions.

‘Bertrand Russell’ refers to Bertrand Russell.

Mention

- The first occurrence of ‘Bertrand Russell’ is an example of mention.
- This occurrence (with quotes) refers to an expression.

Use

- The second occurrence of ‘Bertrand Russell’ is an example of use.
- This occurrence (without quotes) refers to a man.

Syntax: English vs. \mathcal{L}_1 .

English has **many** different sorts of expression.

Some expressions of English

- (1) **Sentences:** 'Bertrand Russell likes logic', 'Philosophers like conceptual analysis', etc..
- (2) **Connectives:** 'it is not the case that', 'and', etc..
- (3) **Noun phrases:** 'Bertrand Russell', 'Philosophers', etc..
- (4) **Verb phrases:** 'likes logic', 'like conceptual analysis', etc..
- (5) Also: **nouns, verbs, pronouns**, etc., etc., etc..

\mathcal{L}_1 has **just two** sorts of basic expression.

Some basic expressions of \mathcal{L}_1

- (1) **Sentence letters:** e.g. 'P', 'Q'.
- (2) **Connectives:** e.g. ' \neg ', ' \wedge '.

Combining sentences and connectives makes new sentences.

Some complex sentences

- ‘It is not the case that’ and ‘Bertrand Russell likes logic’ make: ‘It is not the case that Bertrand Russell likes logic’.
- ‘ \neg ’ and ‘ P ’ make: ‘ $\neg P$ ’.
- ‘Bertrand Russell likes logic’ and ‘and’ and ‘Philosophers like conceptual analysis’ make: ‘Bertrand Russell likes logic and philosophers like conceptual analysis’.
- ‘ P ’, ‘ \wedge ’ and ‘ Q ’ make: ‘ $(P \wedge Q)$ ’.

Logic convention: no quotes around \mathcal{L}_1 -expressions.

- P , \wedge and Q make: $(P \wedge Q)$.

Connectives

Here's the full list of \mathcal{L}_1 -connectives.

name	in English	symbol
conjunction	and	\wedge
disjunction	or	\vee
negation	it is not the case that	\neg
arrow	if ... then	\rightarrow
double arrow	if and only if	\leftrightarrow

The syntax of \mathcal{L}_1

Here's the official definition of \mathcal{L}_1 -sentence.

Definition

- (i) All sentence letters are sentences of \mathcal{L}_1 :
 - $P, Q, R, P_1, Q_1, R_1, P_2, Q_2, R_2, P_3, \dots$
- (ii) If ϕ and ψ are sentences of \mathcal{L}_1 , then so are:
 - $\neg\phi$
 - $(\phi \wedge \psi)$
 - $(\phi \vee \psi)$
 - $(\phi \rightarrow \psi)$
 - $(\phi \leftrightarrow \psi)$
- (iii) Nothing else is a sentence of \mathcal{L}_1 .

Greek letters: ϕ ('PHI') and ψ ('PSI'): not part of \mathcal{L}_1 .

How to build a sentence of \mathcal{L}_1

Example

The following is a sentence of \mathcal{L}_1 :

$$\neg\neg(((P \wedge Q) \rightarrow (P \vee \neg R_{45})) \leftrightarrow \neg((P_3 \vee R) \vee R))$$

Definition of \mathcal{L}_1 -sentences (repeated from previous page)

- (i) All sentence letters are sentences of \mathcal{L}_1 .
- (ii) If ϕ and ψ are sentences of \mathcal{L}_1 , then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$ are sentences of \mathcal{L}_1 .
- (iii) Nothing else is a sentence of \mathcal{L}_1 .

Object vs. Metalanguage

I mentioned that ϕ and ψ are **not** part of \mathcal{L}_1 .

- $\neg P$ is a \mathcal{L}_1 -sentence.
- $\neg\phi$ describes many \mathcal{L}_1 -sentences (**but is not one itself**).
e.g. $\neg P$, $\neg(Q \vee R)$, $\neg(P \leftrightarrow (Q \vee R))$, ...

ϕ and ψ are part of the metalanguage, not the object one.

Object language

The object language is the one we're theorising **about**.

- The object language is \mathcal{L}_1 .

Metalanguage

The metalanguage is the one we're theorising **in**.

- The metalanguage is (augmented) English.

ϕ and ψ are used as variables in the metalanguage:
in order to generalise about sentences of the object language.

Bracketing conventions

There are conventions for dropping brackets in \mathcal{L}_1 .
Some are similar to rules used for $+$ and \times in arithmetic.

Example in arithmetic

- $4 + 5 \times 3$ does not abbreviate $(4 + 5) \times 3$.
- \times ‘binds more strongly’ than $+$.
 $4 + 5 \times 3$ abbreviates $4 + (5 \times 3)$.

Examples in \mathcal{L}_1

- \wedge and \vee bind more strongly than \rightarrow and \leftrightarrow .
 $(P \rightarrow Q \wedge R)$ abbreviates $(P \rightarrow (Q \wedge R))$.
- One may drop outer brackets.
 $P \wedge (Q \rightarrow \neg P_4)$ abbreviates $(P \wedge (Q \rightarrow \neg P_4))$.
- One may drop brackets on strings of \wedge s or \vee s.
 $(P \wedge Q \wedge R)$ abbreviates $((P \wedge Q) \wedge R)$.

Semantics

Recall the characterisation of validity from week 1.

Characterisation

An argument is **logically valid** if and only if there is no interpretation of subject-specific expressions under which:

- (i) the premisses are all true, and
- (ii) the conclusion is false.

We'll adapt this characterisation to \mathcal{L}_1 .

- Logical expressions: $\neg, \wedge, \vee, \rightarrow$ and \leftrightarrow .
- Subject specific expressions: P, Q, R, \dots
- Interpretation: **\mathcal{L}_1 -structure**.

\mathcal{L}_1 -structures

We interpret sentence letters by assigning them truth-values: either T for True or F for False.

Definition

An \mathcal{L}_1 -**structure** is an assignment of exactly one truth-value (**T** or **F**) to every sentence letter of \mathcal{L}_1 .

Examples

We can think of an \mathcal{L}_1 -structure as an infinite list that provides a value T or F for every sentence letter.

	P	Q	R	P_1	Q_1	R_1	P_2	Q_2	R_2	\dots
$\mathcal{A}:$	T	F	F	F	T	F	T	T	F	\dots
$\mathcal{B}:$	F	F	F	F	F	F	F	F	F	\dots

We use \mathcal{A} , \mathcal{B} , etc. to stand for \mathcal{L}_1 -structures.

Truth-values of complex sentences 1/3

\mathcal{L}_1 -structures **only** directly specify truth-values for P, Q, R, \dots

- The logical connectives have fixed meanings.
- These determine the truth-values of complex sentences.
- Notation: $|\phi|_{\mathcal{A}}$ is the truth-value of ϕ under \mathcal{A} .

Truth-conditions for \neg

The meaning of \neg is summarised in its **truth table**.

ϕ	$\neg\phi$
T	F
F	T

In words: $|\neg\phi|_{\mathcal{A}} = \text{T}$ if and only if $|\phi|_{\mathcal{A}} = \text{F}$.

Worked example 1

$|\phi|_{\mathcal{A}}$ is the truth-value of ϕ under \mathcal{A} .

ϕ	$\neg\phi$
T	F
F	T

Compute the following truth-values.

Let the structure \mathcal{A} be partially specified as follows.

P	Q	R	P_1	Q_1	R_1	P_2	Q_2	R_2	\dots
T	F	F	F	T	F	T	T	F	\dots

Compute:

$ P _{\mathcal{A}} = \text{T}$	$ Q _{\mathcal{A}} = \text{F}$	$ R_1 _{\mathcal{A}} = \text{F}$
$ \neg P _{\mathcal{A}} = \text{F}$	$ \neg Q _{\mathcal{A}} = \text{T}$	$ \neg R_1 _{\mathcal{A}} = \text{T}$
$ \neg\neg P _{\mathcal{A}} = \text{T}$	$ \neg\neg Q _{\mathcal{A}} = \text{F}$	$ \neg\neg R_1 _{\mathcal{A}} = \text{F}$

Truth-values of complex sentences 2/3

Truth-conditions for \wedge and \vee

The meanings of \wedge and \vee are given by the truth tables:

ϕ	ψ	$(\phi \wedge \psi)$	ϕ	ψ	$(\phi \vee \psi)$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	F	T	T
F	F	F	F	F	F

$|(\phi \wedge \psi)|_{\mathcal{A}} = \text{T}$ if and only if $|\phi|_{\mathcal{A}} = \text{T}$ and $|\psi|_{\mathcal{A}} = \text{T}$.

$|(\phi \vee \psi)|_{\mathcal{A}} = \text{T}$ if and only if $|\phi|_{\mathcal{A}} = \text{T}$ or $|\psi|_{\mathcal{A}} = \text{T}$ (or both).

Truth-values of complex sentences 3/3

Truth-conditions for \rightarrow and \leftrightarrow

The meanings of \rightarrow and \leftrightarrow are given by the truth tables:

ϕ	ψ	$(\phi \rightarrow \psi)$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	ψ	$(\phi \leftrightarrow \psi)$
T	T	T
T	F	F
F	T	F
F	F	T

$|(\phi \rightarrow \psi)|_{\mathcal{A}} = \text{T}$ if and only if $|\phi|_{\mathcal{A}} = \text{F}$ or $|\psi|_{\mathcal{A}} = \text{T}$.

$|(\phi \leftrightarrow \psi)|_{\mathcal{A}} = \text{T}$ if and only if $|\phi|_{\mathcal{A}} = |\psi|_{\mathcal{A}}$.

Worked example 2

Let $|P|_{\mathcal{B}} = \text{T}$ and $|Q|_{\mathcal{B}} = \text{F}$.

Compute $|\neg(P \rightarrow Q) \rightarrow (P \wedge Q)|_{\mathcal{B}}$

What is the truth value of $\neg(P \rightarrow Q) \rightarrow (P \wedge Q)$ under \mathcal{B} ?

- ❶ $|(P \rightarrow Q)|_{\mathcal{B}} = \text{F}$ and $|(P \wedge Q)|_{\mathcal{B}} = \text{F}$
- ❷ $|\neg(P \rightarrow Q)|_{\mathcal{B}} = \text{T}$
- ❸ $|\neg(P \rightarrow Q) \rightarrow (P \wedge Q)|_{\mathcal{B}} = \text{F}$

ϕ	$\neg\phi$	ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \rightarrow \psi)$
T	F	T	T	T	T
F	T	T	F	F	F
		F	T	F	T
		F	F	F	T

For actual calculations it's usually better to use tables.

Suppose $|P|_{\mathcal{B}} = \text{T}$ and $|Q|_{\mathcal{B}} = \text{F}$.

Compute $|\neg(P \rightarrow Q) \rightarrow (P \wedge Q)|_{\mathcal{B}}$

P	Q	$\neg(P \rightarrow Q) \rightarrow (P \wedge Q)$							
T	F	T	T	F	F	F	T	F	F

ϕ	$\neg\phi$	ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \rightarrow \psi)$
T	F	T	T	T	T
T	F	T	F	F	F
F	T	F	T	F	T
F	T	F	F	F	T

Using the same technique we can fill out the full truth table for $\neg(P \rightarrow Q) \rightarrow (P \wedge Q)$

P	Q	$\neg(P \rightarrow Q) \rightarrow (P \wedge Q)$							
T	T	F	T	T	T	<u>T</u>	T	T	T
T	F	T	T	F	F	<u>F</u>	T	F	F
F	T	F	F	T	T	<u>T</u>	F	F	T
F	F	F	F	T	F	<u>T</u>	F	F	F

The main column (underlined) gives the truth-value of the whole sentence.

ϕ	$\neg\phi$	ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \rightarrow \psi)$
T	F	T	T	T	T
F	T	T	F	F	F
		F	T	F	T
		F	F	F	T

Validity

Let Γ be a set of sentences of \mathcal{L}_1 and ϕ a sentence of \mathcal{L}_1 .

Definition

The argument with all sentences in Γ as premisses and ϕ as conclusion is **valid** if and only if there is no \mathcal{L}_1 -structure under which:

- (i) all sentences in Γ are true; and
- (ii) ϕ is false.

Notation: when this argument is valid we write $\Gamma \models \phi$.

$\{P \rightarrow \neg Q, Q\} \models \neg P$ means that the argument whose premises are $P \rightarrow \neg Q$ and Q , and whose conclusion is $\neg P$ is valid.

Also written: $P \rightarrow \neg Q, Q \models \neg P$

Worked example 3

We can use truth-tables to show that \mathcal{L}_1 -arguments are valid.

Example

Show that $\{P \rightarrow \neg Q, Q\} \models \neg P$.

	P	Q	$P \rightarrow \neg Q$			Q	$\neg P$
►	T	T	T	<u>F</u>	F	T	<u>F</u> T
►	T	F	T	<u>T</u>	T	F	<u>F</u> T
►	F	T	F	<u>T</u>	F	T	<u>T</u> F
►	F	F	F	<u>T</u>	T	F	<u>T</u> F

Rows correspond to interpretations.

One needs to check that there is no row in which all the premisses are assigned T and the conclusion is assigned F.

Other logical notions

Definition

A sentence ϕ of \mathcal{L}_1 is **logically true** (a **tautology**) iff:

- ϕ is true under all \mathcal{L}_1 -structures.

e.g. $P \vee \neg P$, and $P \rightarrow P$ are tautologies.

Truth tables of tautologies

Every row in the main column is a T.

P	$P \vee \neg P$				$P \rightarrow P$		
T	T	<u>T</u>	F	T	T	<u>T</u>	T
F	F	<u>T</u>	T	F	F	<u>T</u>	F

Definition

A sentence ϕ of \mathcal{L}_1 is a **contradiction** iff:

- ϕ is not true under any \mathcal{L}_1 -structure.

e.g. $P \wedge \neg P$, and $\neg(P \rightarrow P)$ are contradictions.

Truth tables of contradictions

Every row in the main column is an F.

P	$P \wedge \neg P$				$\neg(P \rightarrow P)$			
T	T	<u>F</u>	F	T	<u>F</u>	T	T	T
F	F	<u>F</u>	T	F	<u>F</u>	F	T	F

Definition

Sentences ϕ and ψ are **logically equivalent** iff:

- ϕ and ψ are true in exactly the same \mathcal{L}_1 -structures.
- P and $\neg\neg P$ are logically equivalent.
- $P \wedge Q$ and $\neg(\neg P \vee \neg Q)$ are logically equivalent.

Truth tables of logical equivalents

The truth-values in the main columns agree.

P	Q	$P \wedge Q$	$\neg(\neg P \vee \neg Q)$
T	T	<u>T</u> T	<u>T</u> F T F F T
T	F	T <u>F</u> F	<u>F</u> F T T T F
F	T	F <u>F</u> T	<u>F</u> T F T F T
F	F	F <u>F</u> F	<u>F</u> T F T T F

Worked example 4

Example

Show that the sentence $(P \rightarrow (\neg Q \wedge R)) \vee P$ is a tautology.

Method 1: Full truth table

- Write out the truth table for $(P \rightarrow (\neg Q \wedge R)) \vee P$.
- Check there's a T in every row of the main column.

P	Q	R	$(P \rightarrow (\neg Q \wedge R)) \vee P$						
T	T	T	T	F	F	T	F	T	<u>T</u>
T	T	F	T	F	F	T	F	F	<u>T</u>
T	F	T	T	T	T	F	T	T	<u>T</u>
T	F	F	T	F	T	F	F	F	<u>T</u>
F	T	T	F	T	F	T	F	T	<u>T</u>
F	T	F	F	T	F	T	F	F	<u>T</u>
F	F	T	F	T	T	F	T	T	<u>T</u>
F	F	F	F	T	T	F	F	F	<u>T</u>

Worked example 4 (cont.)

Show that the sentence $(P \rightarrow (\neg Q \wedge R)) \vee P$ is a tautology.

Method 2: Backwards truth table.

- Put an F in the main column.
- Work backwards to show this leads to a contradiction.

P	Q	R	$(P \rightarrow (\neg Q \wedge R)) \vee P$
			$T_3? \quad F_1 \qquad \qquad \qquad F \quad F_2$

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ϕ	$\neg\phi$	ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \vee \psi)$	$(\phi \rightarrow \psi)$
T	F	T	T	T	T	T
T	F	T	F	F	T	F
F	T	F	T	F	T	T
		F	F	F	F	T

Worked example 5

Example

Show that $P \leftrightarrow \neg Q \models \neg(P \leftrightarrow Q)$

Method 1: Full truth table

- Write out the full truth table.
- Check there's no row in which the main column of the premiss is T and the main column of the conclusion is F.

Worked example 5 (cont.)

Show that $P \leftrightarrow \neg Q \models \neg(P \leftrightarrow Q)$

Method 2: Backwards truth table

- Put a T in the main column of the premiss and an F in the main column of the conclusion.
- Work backwards to obtain a contradiction. x

P	Q	$P \leftrightarrow \neg Q$				$\neg(P \leftrightarrow Q)$			
		T_4	T	T_5	?	F	T_2	T	T_3
		F_4	T	F_5	?	F	F_2	T	F_3

ϕ	$\neg\phi$	ϕ	ψ	$(\phi \leftrightarrow \psi)$
T	F	T	T	T
T	F	T	F	F
F	T	F	T	F
F	T	F	F	T

<http://logicmanual.philosophy.ox.ac.uk>