Recall argument 2 from week 1.

**Argument 2**

<table>
<thead>
<tr>
<th>Validity</th>
<th>Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Valid</td>
<td>(1) Zeno is a tortoise.</td>
</tr>
<tr>
<td></td>
<td>(2) All tortoises are toothless.</td>
</tr>
<tr>
<td></td>
<td>Therefore, (C) Zeno is toothless.</td>
</tr>
</tbody>
</table>

Is the argument propositionally valid?

<table>
<thead>
<tr>
<th>Propositional Formalisation</th>
<th>Not Valid</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $T$</td>
<td></td>
</tr>
<tr>
<td>(2) $A$</td>
<td></td>
</tr>
<tr>
<td>(C) $L$</td>
<td></td>
</tr>
</tbody>
</table>

Dictionary: $T$: Zeno is a tortoise.  
$A$: All tortoises are toothless.  
$L$: Zeno is toothless.

But $T, A \not\models L$.

**Subject-Predicate form.**

To capture the validity of Argument 2 we need a richer formal language:

One that can capture some of the internal structure of sentences like ‘Zeno is a tortoise’ and ‘All tortoises are toothless’.

The language we use is the **language of predicate logic**: $\mathcal{L}_2$.
Designators/Constants

In English, designators are singular noun phrases that (purport to) refer to a single thing.

**English designators**

In $L_2$, the equivalent of designators are constants

**$L_2$ constants**
- $a, b, c$, etc.

In $L_2$, we simply stipulate what the predicates are.

**$L_2$-predicates**
- $P^1, Q^1, R^1, P_1^1, Q_{36}^1$, etc (1-place, unary) Arity: 1
- $P^2, P_1^2, R_{43}^2$, etc. (2-place, binary) Arity: 2
- $P^3, P_1^3, R_{43}^3$, etc (3-place, ternary) Arity: 3

**Arity**

The superscript (or ‘arity index’) tells us what the arity is.

Combining $n$ occurrences of constants with an $n$-ary predicate makes an $L_2$-sentence.

**Some $L_2$-sentences**

Examples: $P^1a, R^2ab, R^2aa, P^4_{23}abca$

Predicates

In English, a predicate is the result of deleting one or more designators from a sentence.

**English predicates**
- is a tortoise Arity: 1
- is bigger than Arity: 2
- opened with Arity: 3

**Arity**

A predicate’s arity is the number of occurrences of designators it takes to make a whole sentence again.

Predicate formalisation

We can now formalise the first premise

**Formalise: Zeno is a tortoise**

Formalisation: $P^1a$

Dictionary: $a$: Zeno. $P^1$: ...is a tortoise.

**More examples**

<table>
<thead>
<tr>
<th>Formalisation</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice paid Beatrice.</td>
<td>$P^2_{ab}$</td>
</tr>
<tr>
<td>Beatrice paid Alice.</td>
<td>$P^2_{ba}$</td>
</tr>
<tr>
<td>Alice was paid by Beatrice.</td>
<td>$P^2_{ba}$</td>
</tr>
<tr>
<td>Paraphrase: Beatrice paid Alice.</td>
<td></td>
</tr>
<tr>
<td>Alice paid Beatrice or was paid by Beatrice</td>
<td>$P^2_{ab} \lor P^2_{ba}$</td>
</tr>
<tr>
<td>Paraphrase: Alice paid Beatrice or Beatrice paid Alice</td>
<td></td>
</tr>
</tbody>
</table>

Dictionary: $a$: Alice. $b$: Beatrice. $P^2$: ...paid ....
Argument 2 revisited, again

What about the second premiss?

All tortoises are toothless

This cannot be formalised as a designator-predicate sentence. To formalise this sentence we additionally need to deal with quantifiers and pronouns.

Everything is such that it has mass

Quantifier     Pronoun       Predicate

In English, pronouns are expressions used instead of nouns

**English pronouns**

‘she’, ‘him’, ‘it’

Distinguish two uses of pronouns: **lazy** and **quantificational**.

**Lazy use**

Tom likes Miranda Fitzwilliam-Carter and she likes him.

The pronouns each refer to a particular person

**Quantificational use**

If a girl likes Tom, Tom likes her.

The pronoun cannot be replaced by the noun to which it refers back (without changing the meaning). The pronoun ‘her’ does not refer to a single person: it is used to make a general claim.

In English, quantifiers are noun phrases typically used to make general claims

**English quantifiers**

(1) ‘Everything is such that’, ‘Something is such that’
(2) ‘Every philosopher’, ‘All tortoises’, ‘No people’
(3) ‘Most people’, ‘Over half of all tortoises’

The $L_2$-quantifiers correspond to the quantifiers in (1).

**$L_2$-quantifiers**

(1) $\forall x, \exists x$; also: $\exists y, \exists z_{34}$, etc.

$L_2$ can formalise the quantifiers in (2) via paraphrase. $L_2$ cannot formalise the quantifiers in (3).
Formalising general sentences

Formalise:

Everything is such that \( \text{it has mass} \)

\[ \forall x \left( Mx \right) \]

\( \forall x \) corresponds to ‘Everything is such that’
\( x \) corresponds to ‘it’

Let \( M \) correspond to ‘has mass’
Then \( Mx \) corresponds to ‘it has mass’

Formalisation: \( \forall x Mx \)
Dictionary: \( M: \ldots \) has mass.

Note: it’s fine to omit the arity index when formalising.

More examples

<table>
<thead>
<tr>
<th>Formalisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall x Mx )</td>
</tr>
<tr>
<td>( \exists x Mx )</td>
</tr>
<tr>
<td>( \exists x (Px \land Mx) )</td>
</tr>
<tr>
<td>( \forall x (Px \rightarrow Mx) )</td>
</tr>
</tbody>
</table>

Dictionary: \( M: \ldots \) has mass. \( P \ldots \) is a person.

Paraphrases.

1. Everything is such that: it has mass.
   Every \( x \) is such that: \( x \) has mass.

2. Some \( x \) is such that: \( x \) has mass.

3. Something that is a person has mass.
   Something is such that (it is a person and it has mass).
   Some \( x \) is such that (\( x \) is a person and \( x \) has mass).

4. Everything that is a person has mass.
   Every \( x \) is such that (if \( x \) is a person, then \( x \) has mass).

Common variants

<table>
<thead>
<tr>
<th>( L_2 )</th>
<th>standard connective</th>
<th>some other formulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall x )</td>
<td>everything</td>
<td>every, all, any, a</td>
</tr>
<tr>
<td>( \exists x )</td>
<td>something</td>
<td>some, at least one, any, a</td>
</tr>
</tbody>
</table>

Examples

<table>
<thead>
<tr>
<th>Formalisations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists x (Dx \land B_1 x) )</td>
</tr>
<tr>
<td>( \forall x (Dx \rightarrow B_2 x) )</td>
</tr>
</tbody>
</table>

Dictionary. \( D: \ldots \) is a dog. \( B_1: \ldots \) barked. \( B_2: \ldots \) barks

Paraphrase (of the most natural readings)

1. Some dog barked
   Some \( x \) is such that (\( x \) is a dog and \( x \) barked).

2. Every dog barks
   Every \( x \) is such that (if \( x \) is dog, then \( x \) barks).
Quantifiers at the end of sentences

**Examples**

<table>
<thead>
<tr>
<th>No.</th>
<th>Sentence</th>
<th>Formalisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Everyone loves Zuleika</td>
<td>$\forall x (P x \rightarrow L x a)$</td>
</tr>
<tr>
<td>2</td>
<td>Zuleika loves everyone</td>
<td>$\forall x (P x \rightarrow L a x)$</td>
</tr>
</tbody>
</table>

**Dictionary:**

- $a$: Zuleika
- $P$: ...is a person.
- $L$: ...loves ...

**Paraphrases**

<table>
<thead>
<tr>
<th>No.</th>
<th>Paraphrase (1)</th>
<th>Paraphrase (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Everyone is such that (if (x) is a person, then (x) loves Zuleika)</td>
<td>Everyone is such that (if (x) is a person, then (x) is loved by Zuleika)</td>
</tr>
<tr>
<td>2</td>
<td>Everyone is such that (if (x) is a person, then Zuleika loves (x))</td>
<td>Some (x) is such that (if (y) is a person, then (y) is loved by (x))</td>
</tr>
</tbody>
</table>

The syntax of \(\mathcal{L}_2\)

Here’s the official syntax of \(\mathcal{L}_2\):

**Definition (Predicate letters)**

- \(P^k, Q^k, R^k\) are predicate letters, where \(k\) and \(n\) are either missing (no symbol) or a numeral ‘1’, ‘2’, ‘3’, ...

- \(P, Q, R, P_1, Q_1, R_1, P_2, \ldots\) Sentence letters
- \(P^1, Q^1, R^1, P_1^1, Q_1^1, R_1^1, P_2^1, \ldots\) Unary predicates
- \(P^2, Q^2, R^2, P_1^2, Q_1^2, R_1^2, P_2^2, \ldots\) Binary predicates

**Definition**

The following are variables: \(x, y, z, x_1, y_1, z_1, x_2, \ldots\)

The following are constants: \(a, b, c, a_1, b_1, c_1, a_2, \ldots\)

We use the same trick to deal with multiple quantifiers.

**Example**

- Someone loves everyone: $\exists x (P x \land \forall y (P y \rightarrow L x y))$
- Dictionary: $P$: \ldots is a person. $L$: \ldots loves \ldots

**Paraphrases**

- Some \(x\) is such that \((x\) is a person and \(x\) loves everyone\)
- Some \(x\) is such that \((x\) is a person and everyone is loved by \(x\)\)
- Some \(x\) is such that \((x\) is a person and everything is such that: if \(y\) is a person, then \(y\) is loved by \(x\)\)
- Some \(x\) is such that \((x\) is a person and every \(y\) is such that: if \(y\) is a person, then \(x\) loves \(y\)\)

**Definition (atomic formulae of \(\mathcal{L}_2\))**

If \(Z\) is a predicate letter of arity \(n\) and each of \(t_1, \ldots, t_n\) is a variable or a constant, then \(Z t_1 \ldots t_n\) is an atomic formula of \(\mathcal{L}_2\).

**Definition (formulae of \(\mathcal{L}_2\))**

1. All atomic formulae of \(\mathcal{L}_2\) are formulae of \(\mathcal{L}_2\).
2. If \(\phi\) and \(\psi\) are formulae of \(\mathcal{L}_2\), then $\neg \phi$, $(\phi \land \psi)$, $(\phi \lor \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$ are formulae of \(\mathcal{L}_2\).
3. If \(v\) is a variable and \(\phi\) is a formula, then $\forall v \phi$ and $\exists v \phi$ are formulae of \(\mathcal{L}_2\).
4.2 The Sentences of $L_2$

Worked example: which of these are $L_2$-formulae?

(i) $P a$
(ii) $Q^3 a x y$
(iii) $P$
(iv) $\forall v \phi$

(No bracketing conventions, etc. have been applied.)

4.3 Free and Bound Occurrences of Variables

Formulae vs Sentences

An $L_2$ formula $\phi$ is a sentence if every occurrence of each variable $v$ in $\phi$ is ‘bound’ by a quantifier $\exists v$ or $\forall v$.

Examples

- $P^1 x$ is not a sentence. No quantifier binds the variable.
- $\forall x P^1 x$ is a sentence: $x$ is bound by $\forall x$.
- $\forall y P^1 x$ is not a sentence: $x$ is not bound by $\forall y$
- $(\forall x P^1 x \lor P^1 x)$ is not a sentence: only the first occurrence is bound by $\forall x$

Bound variables

- An occurrence of a variable $v$ in $\phi$ is bound if it occurs within the scope of $\forall v$ or $\exists v$; otherwise it is free.
- The scope of an occurrence of a quantifier in $\phi$ is the smallest subformula of $\phi$ that contains this occurrence.

Underline the free variables in the following:

$\forall x (\forall y (P^1 x \to P^1 y) \to R^2 x y)$

Notational conventions

These aren’t part of the official syntax of $L_2$.
But it is permitted to apply the following conventions to make it easier to read $L_2$-formulae.
(We’ve been applying them already.)

- The bracketing conventions from $L_1$ may be applied to drop brackets.
- Arity indices may be dropped.
  e.g $P x y$ is used to abbreviate $P^2 x y$
- Letters other than the official $P, Q, R$ and $a, b, c$ may be used in formalisations.
- But you should stick to letters early in the alphabet for constants and letters late in the alphabet (start with: $x, y, z$) for variables.
What of argument 2?

<table>
<thead>
<tr>
<th>Argument 2</th>
<th>Valid</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Zeno is a tortoise.</td>
<td></td>
</tr>
<tr>
<td>(2) All tortoises are toothless.</td>
<td></td>
</tr>
<tr>
<td>Therefore, (C) Zeno is toothless.</td>
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</table>

Formalisation

(1) $T_a$
(2) $\forall x (T_x \rightarrow L_x)$
(C) $L_a$

Dictionary: $a$: Zeno. $T$:...is a tortoise. $L$: ...is toothless

What is it for this $L_2$-argument to be valid? (semantics: week 5)
How can we show that it is valid? (natural deduction: week 6)