INTRODUCTION TO LOGIC

Lecture 4

The Syntax of Predicate Logic

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I counsel you, dear friend, in sum,
That first you take collegium logicum.
Your spirit's then well broken in for you,
In Spanish boots laced tightly to,
That you henceforth may more deliberately keep
The path of thought and straight along it creep,
And not perchance criss-cross may go,
A- will-o'-wisping to and fro.
Then you'll be taught full many a day
What at one stroke you've done alway,
Like eating and like drinking free,
It now must go like: One! Two! Three!
Goethe, Faust I

Outline

1. Introduction to $\mathcal{L}_2$
2. Straightforward predicate formalisation
3. The syntax of $\mathcal{L}_2$

Recall argument 2 from week 1.

Argument 2 **Valid**

(1) Zeno is a tortoise.
(2) All tortoises are toothless.
Therefore, (C) Zeno is toothless.

Is the argument **propositionally valid**?

Propositional Formalisation **Not Valid**

(1) T

(2) A

(C) L

Dictionary: T: Zeno is a tortoise.
A: All tortoises are toothless. L: Zeno is toothless.

But $T, A \not\models L$.

Subject-Predicate form.

To capture the validity of Argument 2 we need a richer formal language:
One that can capture some of the internal structure of sentences like 'Zeno is a tortoise' and 'All tortoises are toothless'.

$$\underbrace{\text{Zeno}}_{\text{Designator}}\ \underbrace{\text{is a tortoise}}_{\text{Predicate}}$$

The language we use is the **language of predicate logic**: $\mathcal{L}_2$

Designators/Constants

In English, designators are singular noun phrases that (purport to) refer to a single thing.

English designators

- Proper names: 'Zeno', 'London', 'Elizabeth II'
- Definite descriptions: 'The capital of France','The Queen', 'England's capital', 'Volker's tortoise'.

In $\mathcal{L}_2$, the equivalent of designators are **constants**

$\mathcal{L}_2$ constants

- a, b, c, etc.

Predicates

In English, a predicate is the result of deleting one or more designators from a sentence.

English predicates

- ___ is a tortoise — Arity: 1
- ___ is bigger than ___ — Arity: 2
- ___ opened ___ with ___ — Arity: 3

Arity

A predicate's **arity** is the number of occurrences of designators it takes to make a whole sentence again.

In $\mathcal{L}_2$, we simply stipulate what the predicates are.

$\mathcal{L}_2$-predicates

- P^1, Q^1, R^1, P^1_1, Q^1_{36}, etc — (1-place, unary) Arity: 1
- P^2, P^2_1, R^2_{43}, etc. — (2-place, binary) Arity: 2
- P^3, P^3_1, R^3_{43}, etc — (3-place, ternary) Arity: 3

Arity

The superscript (or 'arity index') tells us what the arity is.

Combining n occurrences of constants with an n-ary predicate makes an $\mathcal{L}_2$-sentence.

Some $\mathcal{L}_2$-sentences

Examples: P^1a, R^2ab, R^2aa, $P^4_{23}abca$

Predicate formalisation

We can now formalise the first premise

Formalise: Zeno is a tortoise

Formalisation: P^1a

Dictionary: a: Zeno. P^1: ...is a tortoise.

More examples	Formalisation
(1) Alice paid Beatrice.	P^2ab
(2) Beatrice paid Alice.	P^2ba
(3) Alice was paid by Beatrice. Paraphrase: Beatrice paid Alice.	P^2ba
(4) Alice paid Beatrice or was paid by Beatrice Paraphrase: Alice paid Beatrice or Beatrice paid Alice	$P^2ab \vee P^2ba$

Dictionary: a: Alice. b: Beatrice. P^2: ...paid ....

Argument 2 revisited, again

What about the second premiss?

> All tortoises are toothless

This cannot be formalised as a designator-predicate sentence. To formalise this sentence we additionally need to deal with quantifiers and pronouns.

$$\underbrace{\text{Everything is such that}}_{\textbf{Quantifier}}\ \underbrace{\text{it}}_{\textbf{Pronoun}}\ \underbrace{\text{has mass}}_{\textbf{Predicate}}$$

4.1 Predicates and Quantification

Pronouns/Variables

In English, pronouns are expressions used instead of nouns

English pronouns

‘she’, ‘him’, ‘it’

Distinguish two uses of pronouns: **lazy** and **quantificational**.

Lazy use

Tom likes Miranda Fitzwilliam-Carter and **she** likes **him**.

The pronouns each refer to a particular person

Quantificational use

If a girl likes Tom, Tom likes **her**.

The pronoun cannot be replaced by the noun to which it refers back (without changing the meaning).
The pronoun ‘her’ does not refer to a single person: it is used to make a general claim.

4.1 Predicates and Quantification

In English, we rely on contextual clues to keep track of what pronouns and other expressions are referring back to.

‘Part’ expresses a transitive relation

If an object is part of another object and this object is part of yet another object, then it is part of that object.

We could instead express this as:

> If an object x_1 is part of another object x_2 and x_2 is part of yet another object x_3, then x_1 is part of x_3.

NB: not idiomatic English.
In $\mathcal{L}_2$, the equivalent of pronouns are variables.

$\mathcal{L}_2$ variables

- $x, y, z, x_1, y_1, z_1, x_2, \ldots$

4.1 Predicates and Quantification

Quantifiers

In English, quantifiers are noun phrases typically used to make general claims

English quantifiers

(1) ‘Everything is such that’, ‘Something is such that’
(2) ‘Every philosopher’, ‘All tortoises’, ‘No people’
(3) ‘Most people’, ‘Over half of all tortoises’

The $\mathcal{L}_2$-quantifiers correspond to the quantifiers in (1).

$\mathcal{L}_2$-quantifiers

(1) $\forall x$, $\exists x$; also: $\exists y$, $\exists z_{34}$, etc.

$\mathcal{L}_2$ can formalise the quantifiers in (2) via paraphrase.
$\mathcal{L}_2$ cannot formalise the quantifiers in (3).

Formalising general sentences

Formalise

$\underbrace{\text{Everything is such that}}_{\forall x}\ \underbrace{\text{it}}_{x}\ \underbrace{\text{has mass}}_{M}$

$\forall x$ corresponds to 'Everything is such that'
x corresponds to 'it'
Let M correspond to 'has mass'
Then Mx corresponds to 'it has mass'

Formalisation: $\forall x Mx$
Dictionary: M: ...has mass.

Note: it's fine to omit the arity index when formalising.

More examples	**Formalisation**
(1) Everything has mass.	$\forall x Mx$
(2) Something has mass.	$\exists x Mx$
(3) Some person has mass.	$\exists x(Px \land Mx)$
(4) Every person has mass.	$\forall x(Px \to Mx)$

Dictionary: M:...has mass. P ...is a person.

Paraphrases.

(1) Everything is such that: it has mass.
Every x is such that: x has mass.

(2) Some x is such that: x has mass.

(3) Something that is a person has mass.
Something is such that (it is a person and it has mass).
Some x is such that (x is a person and x has mass).

(4) Everything that is a person has mass.
Every x is such that (if x is a person, then x has mass).

Common variants

$\mathcal{L}_2$	standard connective	some other formulations
$\forall x$	**everything**	**every**, **all**, **any**, **a**
$\exists x$	**something**	**some**, **at least one**, **any**, **a**

Examples	**Formalisations**
(1) A dog barked.	$\exists x(Dx \land B_1 x)$
(2) A dog barks.	$\forall x(Dx \to B_2 x)$

Dictionary. D: ...is a dog. B_1: ...barked. B_2: ...barks

Paraphrase (of the most natural readings)

(1) Some dog barked
Some x is such that (x is a dog and x barked).

(2) Every dog barks
Every x is such that (if x is dog, then x barks).

Quantifiers at the end of sentences

Examples	**Formalisation**
(1) Everyone loves Zuleika	$\forall x(Px \to Lxa)$
(2) Zuleika loves everyone	$\forall x(Px \to Lax)$

Dictionary: a: Zuleika. P: ... is a person. L: ... loves ....

Paraphrases

(1) Every x is such that
(if x is a person, then x **loves Zuleika**)

(2) Everyone is loved by Zuleika
Every x is such that
(if x is a person, then x **is loved by Zuleika**)
Every x is such that
(if x is a person, then **Zuleika loves** x)

We use the same trick to deal with multiple quantifiers.

Example	**Formalisation**
Someone loves everyone	$\exists x(Px \wedge \forall y(Py \to Lxy))$

Dictionary: P: ... is a person. L: ... loves ....

Paraphrases

Some x is such that (x is a person and x **loves everyone**)
Some x is such that (x is a person and **everyone is loved by** x)
Some x is such that
(x is a person and **everything is such that:**
(if it is a person, then it is loved by x**))**
Some x is such that
(x is a person and **every** y **is such that:**
(if y **is a person, then** y **is loved by** x**))**
Some x is such that
(x is a person and every y is such that:
(if y is a person, then x **loves** y))

The syntax of $\mathcal{L}_2$

Here's the official syntax of $\mathcal{L}_2$40

Definition (Predicate letters)

P^k_n, Q^k_n, or R^k_n are **predicate letters**, where k and n are either missing (no symbol) or a numeral '1', '2', '3', ...

P	Q	R	P_1	Q_1	R_1	P_2	...	Sentence letters
P^1	Q^1	R^1	P^1_1	Q^1_1	R^1_1	P^1_2	...	Unary predicates
P^2	Q^2	R^2	P^2_1	Q^2_1	R^2_1	P^2_2	...	Binary predicates
$\vdots$								

Definition

The following are **variables**: x, y, z, x_1, y_1, z_1, x_2, ...
The following are **constants**: a, b, c, a_1, b_1, c_1, a_2, ...

Definition (atomic formulae of $\mathcal{L}_2$)

If Z is a predicate letter of arity n and each of $t_1, \ldots, t_n$ is a variable or a constant, then $Zt_1 \ldots t_n$ is an **atomic formula** of $\mathcal{L}_2$.

Definition (formulae of $\mathcal{L}_2$)

1. All atomic formulae of $\mathcal{L}_2$ are formulae of $\mathcal{L}_2$.
2. If ϕ and ψ are formulae of $\mathcal{L}_2$, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ and $(\phi \leftrightarrow \psi)$ are formulae of $\mathcal{L}_2$.
3. If v is a variable and ϕ is a formula, then $\forall v\, \phi$ and $\exists v\, \phi$ are formulae of $\mathcal{L}_2$.

Worked example: which of these are $\mathcal{L}_2$-formulae?

(i) Pa

(ii) Q^3axy

(iii) P

(iv) $\forall v \phi$

(No bracketing conventions, etc. have been applied.)

Formulae vs Sentences

An $\mathcal{L}_2$ formula ϕ is a **sentence** if every occurrence of each variable v in ϕ is 'bound' by a quantifier $\exists v$ or $\forall v$.

Examples

- P^1x is not a sentence. No quantifier binds the variable.
- $\forall x P^1 x$ is a sentence: x is bound by $\forall x$.
- $\forall y P^1 x$ is not a sentence: x is not bound by $\forall y$
- $(\forall x P^1 x \vee P^1 x)$ is not a sentence: only the first occurrence is bound by $\forall x$

Bound variables

- An occurrence of a variable v in ϕ is **bound** if it occurs within the scope of $\forall v$ or $\exists v$; otherwise it is **free**.
- The **scope** of an occurrence of a quantifier in ϕ is the smallest subformula of ϕ that contains this occurrence.

Underline the free variables in the following:

$\forall x(\forall y(P^1x \to P^1y) \to R^2xy)$

Notational conventions

These aren't part of the official syntax of $\mathcal{L}_2$.
But it is permitted to apply the following conventions to make it easier to read $\mathcal{L}_2$-formulae.
(We've been applying them already.)

- The bracketing conventions from $\mathcal{L}_1$ may be applied to drop brackets.
- Arity indices may be dropped.
  e.g Pxy is used to abbreviate P^2xy
- Letters other than the official P, Q, R and a, b, c may be used in formalisations.
- But you should stick to letters early in the alphabet for constants and letters late in the alphabet (start with: x, y, z) for variables.

What of argument 2?

Argument 2 **Valid**

(1) Zeno is a tortoise.
(2) All tortoises are toothless.
Therefore, (C) Zeno is toothless.

Formalisation

(1) Ta
(2) $\forall x(Tx \to Lx)$
(C) La

Dictionary: a: Zeno. T:...is a tortoise. L: ...is toothless

What is it for this $\mathcal{L}_2$-argument to be valid? (semantics: week 5)
How can we show that it is valid? (natural deduction: week 6)

http://logicmanual.philosophy.ox.ac.uk