INTRODUCTION TO LOGIC

Lecture 5

The Semantics of Predicate Logic

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We could forget about philosophy.
Settle down and maybe get into semantics.
Woody Allen
'Mr. Big'

Outline

1. Validity.
2. Semantics for simple English sentences.
3. $\mathcal{L}_2$-structures.
4. Semantics for $\mathcal{L}_2$-formulae.

What of argument 2?

Argument 2 **Valid**

(1) Zeno is a tortoise.
(2) All tortoises are toothless.
Therefore, (C) Zeno is toothless.

Formalisation

(1) Ta
(2) $\forall x(Tx \to Lx)$
(C) La

Dictionary: a: Zeno. T:...is a tortoise. L: ...is toothless

What is it for this $\mathcal{L}_2$-argument to be valid?

Validity

Recall the definition of validity for $\mathcal{L}_1$.
Let Γ be a set of sentences of $\mathcal{L}_1$ and ϕ a sentence of $\mathcal{L}_1$

Definition

The argument with all sentences in Γ as premisses and ϕ as conclusion is valid if and only if there is no $\mathcal{L}_1$-structure under which:

(i) all sentences in Γ are true; and
(ii) ϕ is false.

We use an exactly analogous definition for $\mathcal{L}_2$, replacing '$\mathcal{L}_1$' everywhere above with '$\mathcal{L}_2$'
It remains to define: $\mathcal{L}_2$-structure, truth in an $\mathcal{L}_2$-structure

Structures

Structures interpret non-logical expressions.

$\mathcal{L}_1$-structures

- Non-logical expressions in $\mathcal{L}_1$: $P, Q, R, \ldots$.
- An $\mathcal{L}_1$-structure $\mathcal{A}$ assigns each sentence letter a semantic value (specifically, a truth-value: T or F)

$\mathcal{L}_2$ is a richer language. This calls for richer structures.

$\mathcal{L}_2$-structures

- Non-logical expressions: $P^1, Q^1, R^1, \ldots$
  $P^2, Q^2, R^2, \ldots$
  $\vdots$
  $a, b, c, \ldots$
- An $\mathcal{L}_2$-structure $\mathcal{A}$ assigns each predicate and constant a semantic value (specifically, what?)

Semantics in English

Start with a semantics for simple English sentences.

'Bertrand Russell is a philosopher'

The sentence is true (i.e.: its semantic value is: T).
. . . because of the relationship between the semantic values of its constituents.

expression	*semantic value*
'Bertrand Russell'	Russell
'is a philosopher'	the property of *being a philosopher*

. . . because Russell has the property of *being a philosopher*.
. . . because |'Bertrand Russell'| has |'is a philosopher'|.

Notation

When e is an expression, we write $|e|$ for its semantic value.

Similarly:

'Alonzo Church reveres Bertrand Russell' is true iff
Church stands in the relation of *revering* to Russell

In other words:

|'Alonzo Church reveres Bertrand Russell'| = T iff
|'Alonzo Church'| stands in |'reveres'| to |'Bertrand Russell'|

Semantic values for English expressions

expression	*semantic value*
designator	object
unary predicate	property (alias: unary relation)
binary predicate	binary relation

Examples

- |'Bertrand Russell'| = Russell
- |'is a philosopher'| = the property of *being a philosopher*
- |'reveres'| = the relation of *revering*

We'll take this one step further, by saying more about properties and relations.

Properties

In logic, we identify properties with sets.

Property (alias: unary relation)

A unary relation $\boldsymbol{P}$ is a set of zero or more objects.

Specifically, $\boldsymbol{P}$ is the set of objects that have the property.

Informally: $d \in \boldsymbol{P}$ indicates that d has property $\boldsymbol{P}$.

Example

The property of *being a philosopher*
$= $ the set of philosophers
$= \{d : d \text{ is a philosopher}\}$
$= \{\text{Descartes, Kant, Russell}, \ldots\}$

Relations

Recall that we identify binary relations with sets of pairs.

Binary relation

A binary relation $\boldsymbol{R}$ is a set of zero or more pairs of objects.

$\boldsymbol{R}$ is the set of pairs $\langle d, e\rangle$ such that d stands in $\boldsymbol{R}$ to e.

Informally: $\langle d, e\rangle \in \boldsymbol{R}$ indicates that d bears $\boldsymbol{R}$ to e.

Example

The relation of *revering* $= \{\langle d, e\rangle : d \text{ reveres } e\}$

Similarly:

A ternary (3-ary) relation is a set of triples (3-tuples).
A quaternary (4-ary) relation is a set of quadruples (4-tuples).
etc.

Putting this all together:

'Bertrand Russell is a philosopher' is true
iff |'Bertrand Russell'| has |'is a philosopher'|
iff Russell $\in$ the set of philosophers

Similarly:

'Alonzo Church reveres Russell' is true
iff |'Alonzo Church'| stands in |'reveres'| to |'Russell'|
iff $\langle \text{Church, Russell}\rangle \in \{\langle d, e\rangle : d \text{ reveres } e\}$

Semantics for atomic $\mathcal{L}_2$-sentences

The semantics for atomic $\mathcal{L}_2$-sentences is similar.

An $\mathcal{L}_2$-structure specifies semantic values for $\mathcal{L}_2$-expressions:

$\mathcal{L}_2$-*expression*	*semantic value*
constant: a	object: $\lvert a\rvert_{\mathcal{A}}$
sentence letter: P	truth-value: $\lvert P\rvert$ (i.e. T or F)
unary predicate: P^1	unary relation: $\lvert P^1\rvert$ (i.e. a set)
binary predicate: P^2	binary relation: $\lvert P^2\rvert$ (a set of pairs)

- $|Pb| = T$ iff $|b|$ has $|P|$
  iff $|b| \in |P|$
- $|Rab| = T$ iff $|a|$ stands in $|R|$ to $|b|$
  iff $\langle |a|, |b|\rangle \in |R|$

Notation: $|e|_{\mathcal{A}}$ is the semantic value of e in $\mathcal{L}_2$-structure $\mathcal{A}$.

Semantics for atomic $\mathcal{L}_2$-formulae

We have the semantics for $\mathcal{L}_2$-*sentences* like Pa.
What about $\mathcal{L}_2$-*formulae* like Px?

In English:

- The designator 'Russell' has a constant semantic value.
- Pronouns, such as 'it', do not.
  'it' refers to different objects depending on the context.

Something similar happens in an $\mathcal{L}_2$-structure $\mathcal{A}$:

- $a, b, c, \ldots$ are assigned a constant semantic value in $\mathcal{A}$.
- Variables: $x, y, z, \ldots$ are not.

What object each variable denotes is specified with a variable assignment.

Variable assignments

Variable assignment

A variable assignment assigns an object to each variable.

One can think of a variable assignment as an infinite list

Example: the assignment α.

x	y	z	x_1	y_1	z_1	x_2	$\ldots$
Mercury	Venus	Venus	Neptune	Mars	Venus	Mars	

Notation

We write $|x|^\alpha$ for the object α assigns to x.
We use lower case Greek letters: α, β, γ for assignments.

e.g. $|x|^\alpha$ = Mercury; $|y|^\alpha$ =Venus; $|x_2|^\alpha$ = Mars.

Once x has been assigned an object, the semantics for Px are much like the semantics for Pa

We write $|e|^\alpha_\mathcal{A}$ for the semantic value of expression e in the structure $\mathcal{A}$ under the variable assignment α.

- $|Px|^\alpha_\mathcal{A} = \mathrm{T}$ iff $|x|^\alpha$ has $|P|_\mathcal{A}$ (NB: $|x|^\alpha_\mathcal{A} = |x|^\alpha$)
  iff $|x|^\alpha \in |P|_\mathcal{A}$
- $|Rxy|^\alpha_\mathcal{A} = \mathrm{T}$ iff $|x|^\alpha$ stands in $|R|_\mathcal{A}$ to $|y|^\alpha$
  iff $\langle |x|^\alpha, |y|^\alpha \rangle \in |R|_\mathcal{A}$

Note: semantic values of constants and predicates are unaffected by the assignment (e.g. $|P|^\alpha_\mathcal{A} = |P|_\mathcal{A}$, $|a|^\alpha_\mathcal{A} = |a|_\mathcal{A}$).

- $|Rab|^\alpha_\mathcal{A} = \mathrm{T}$ iff $\langle |a|_\mathcal{A}, |b|_\mathcal{A} \rangle \in |R|_\mathcal{A}$
- $|Rxb|^\alpha_\mathcal{A} = \mathrm{T}$ iff $\langle |x|^\alpha, |b|_\mathcal{A} \rangle \in |R|_\mathcal{A}$

Similarly for other atomic formulae.

Worked example

Let $\mathcal{L}_2$-structure $\mathcal{A}$ be such that:

- $|a|_\mathcal{A}$ = Alonzo Church
- $|b|_\mathcal{A}$ = Bertrand Russell
- $|P|_\mathcal{A}$ = {Frege, Russell}
- $|R|_\mathcal{A}$ = {⟨Church, Russell⟩}

Let assignments α and β be such that:

	x	y	z
α:	Frege	Russell	Wittgenstein
β:	Church	Church	Church

Compute the following:

$|x|^\alpha_\mathcal{A} =$ $\quad |x|^\beta_\mathcal{A} =$ $\quad |a|^\alpha_\mathcal{A} =$

$|Py|^\alpha_\mathcal{A} =$ $\quad |Py|^\beta_\mathcal{A} =$ $\quad |Pb|^\alpha_\mathcal{A} =$

$|Rxy|^\alpha_\mathcal{A} =$ $\quad |Rxy|^\beta_\mathcal{A} =$ $\quad |Rxb|^\alpha_\mathcal{A} =$

Semantics for quantifiers

In English, the truth-value of a quantified sentence depends on how widely the quantifiers range.

Almost everyone attended the first lecture.

The context supplies a 'domain' telling us who 'everyone' ranges over

Domain: the set of first-year Oxford philosophy students

Almost every first-year Oxford philosophy student attended the first lecture.

Domain: the set of everyone in the world

Almost everyone in the world attended the first lecture.

An $\mathcal{L}_2$-structure $\mathcal{A}$ specifies a non-empty set $D_{\mathcal{A}}$ as the domain.
An assignment over $\mathcal{A}$ assigns a member of $D_{\mathcal{A}}$ to each variable.

Semantics for $\forall/\exists$ (first approximation):

$|\forall x Px|_{\mathcal{A}} = \mathrm{T}$
iff every member of $D_{\mathcal{A}}$ has $|P|_{\mathcal{A}}$
iff every assignment α of x to a member of $D_{\mathcal{A}}$ is such that $|x|^{\alpha} \in |P|_{\mathcal{A}}$
iff every assignment α over $\mathcal{A}$ is such that $|Px|^{\alpha}_{\mathcal{A}} = T$

Similarly:

$|\exists x Px|_{\mathcal{A}} = \mathrm{T}$
iff some member of $D_{\mathcal{A}}$ has $|P|_{\mathcal{A}}$
iff some assignment α of x to a member of $D_{\mathcal{A}}$ is such that $|x|^{\alpha} \in |P|_{\mathcal{A}}$
iff some assignment α over $\mathcal{A}$ is such that $|Px|^{\alpha}_{\mathcal{A}} = T$

This is correct but the general case is more complex.

The semantics of quantifiers is complicated by the need to deal with multiple quantifiers in sentences such as $\forall x \exists y Rxy$

Suppose we try to evaluate this as before under $\mathcal{A}$ with domain $D_{\mathcal{A}}$

$|\forall x \exists y Rxy|_{\mathcal{A}} = T$
iff every assignment α over $\mathcal{A}$ is such that $|\exists y Rxy|^{\alpha}_{\mathcal{A}} = T$

To progress any further we need to be able evaluate $\exists y Rxy$ under an assignment α of an object to x.

How to determine $|\exists y Rxy|^{\alpha}_{\mathcal{A}}$?

$|\exists y Rxy|^{\alpha}_{\mathcal{A}} = \mathrm{T}$

iff some d in $D_{\mathcal{A}}$ is such that $|x|^{\alpha}$ stands in $|R|_{\mathcal{A}}$ to d

iff some assignment β over $\mathcal{A}$ is such that $|x|^{\alpha}$ stands in $|R|_{\mathcal{A}}$ to $|y|^{\beta}$

So we don't have to keep track of multiple assignments:

Say that β differs from α in y at most if $|v|^{\alpha} = |v|^{\beta}$ for all variables v with the possible exception of y.

$|\exists y Rxy|^{\alpha}_{\mathcal{A}} = \mathrm{T}$

iff some assignment β over $\mathcal{A}$ which differs from α in y at most is such that $|x|^{\beta}$ stands in $|R|_{\mathcal{A}}$ to $|y|^{\beta}$

iff some assignment β over $\mathcal{A}$ which differs from α in y at most is such that $|Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T}$

$\mathcal{L}_2$-structures

Here's the full specification of an $\mathcal{L}_2$-structure.

An $\mathcal{L}_2$-structure $\mathcal{A}$ supplies two things

(1) a domain: a non-empty set $D_{\mathcal{A}}$

(2) a semantic value for each predicate and constant.

$\mathcal{L}_2$-expression	*semantic value in $\mathcal{A}$*
constant: a	object: $\lvert a\rvert_{\mathcal{A}}$
sentence letter: P	truth-value: $\lvert P\rvert_{\mathcal{A}}$ (= T or F)
unary predicate: P^1	unary relation: $\lvert P^1\rvert_{\mathcal{A}}$ (i.e. a set)
binary predicate: P^2	binary relation: $\lvert P^2\rvert_{\mathcal{A}}$ (a set of pairs)
ternary predicate: P^3	ternary relation: $\lvert P^3\rvert_{\mathcal{A}}$ (a set of triples)
etc. etc.	

Summary of semantics of $\mathcal{L}_2$

Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure and α an assignment over $\mathcal{A}$.

Atomic formulae

Let Φ^n be a n-ary predicate letter ($n > 0$) and let t_1, t_2, ... be variables or constants.

- $|\Phi^n|^\alpha_{\mathcal{A}}$ is the n-ary relation assigned to Φ^n by $\mathcal{A}$.
- $|t|^\alpha_{\mathcal{A}}$ is the object t denotes in $\mathcal{A}$ if t is a constant.
- $|t|^\alpha_{\mathcal{A}}$ is the object assigned to t by α if t is a variable.

(i) $|\Phi^1 t_1|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|t_1|^\alpha_{\mathcal{A}} \in |\Phi^1|_{\mathcal{A}}$
$|\Phi^2 t_1 t_2|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $\langle |t_1|^\alpha_{\mathcal{A}}, |t_2|^\alpha_{\mathcal{A}}\rangle \in |\Phi^2|_{\mathcal{A}}$
$|\Phi^3 t_1 t_2 t_3|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $\langle |t_1|^\alpha_{\mathcal{A}}, |t_2|^\alpha_{\mathcal{A}}, |t_3|^\alpha_{\mathcal{A}}\rangle \in |\Phi^3|_{\mathcal{A}}$
etc.

The semantics for connectives are just like those for $\mathcal{L}_1$.

Semantics for connectives

(ii) $|\neg\phi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\alpha_{\mathcal{A}} = \mathrm{F}$.

(iii) $|\phi \wedge \psi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\alpha_{\mathcal{A}} = \mathrm{T}$ and $|\psi|^\alpha_{\mathcal{A}} = \mathrm{T}$.

(iv) $|\phi \vee \psi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\alpha_{\mathcal{A}} = \mathrm{T}$ or $|\psi|^\alpha_{\mathcal{A}} = \mathrm{T}$.

(v) $|\phi \to \psi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\alpha_{\mathcal{A}} = \mathrm{F}$ or $|\psi|^\alpha_{\mathcal{A}} = \mathrm{T}$.

(vi) $|\phi \leftrightarrow \psi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\alpha_{\mathcal{A}} = |\psi|^\alpha_{\mathcal{A}}$.

These are the semantic clauses for $\forall v$ and $\exists v$.

Quantifiers

(vii) $|\forall v\, \phi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\beta_{\mathcal{A}} = \mathrm{T}$ for all variable assignments β over $\mathcal{A}$ differing from α in v at most.

(viii) $|\exists v\, \phi|^\alpha_{\mathcal{A}} = \mathrm{T}$ if and only if $|\phi|^\beta_{\mathcal{A}} = \mathrm{T}$ for at least one variable assignment β over $\mathcal{A}$ differing from α in v at most.

Truth

Just one detail remains. 50

We haven't yet said what it is for a sentence to be true in an $\mathcal{L}_2$-structure $\mathcal{A}$.

We've said what it is for a formula to be true in an $\mathcal{L}_2$-structure $\mathcal{A}$ under an assignment over $\mathcal{A}$

(We've defined $|\phi|^{\alpha}_{\mathcal{A}}$; we want now to define $|\phi|_{\mathcal{A}}$.)

Fact about sentences

The truth-value of a sentence does *not* depend on the assignment. For α and β over $\mathcal{A}$: $|\phi|^{\alpha}_{\mathcal{A}} = |\phi|^{\beta}_{\mathcal{A}}$ (when ϕ is a sentence).

A sentence ϕ is true in an $\mathcal{L}_2$-structure $\mathcal{A}$ (in symbols: $|\phi|_{\mathcal{A}} = \text{T}$) iff $|\phi|^{\alpha}_{\mathcal{A}} = \text{T}$ for all variable assignments α over $\mathcal{A}$.

equivalently: $|\phi|^{\alpha}_{\mathcal{A}} = \text{T}$ for some variable assignment α over $\mathcal{A}$.

http://logicmanual.philosophy.ox.ac.uk