INTRODUCTION TO LOGIC Lecture 6 Natural Deduction Dr. James Studd

There's nothing you can't prove if your outlook is only sufficiently limited Dorothy L. Sayers

Outline

- Proof
- **2** Rules for connectives
- 3 Rules for quantifiers
- Adequacy

	$\forall \! y \left(Py \to Qy \right)$	
[Pa]	$Pa \rightarrow Qa$	$\forall z \left(Qz \to Rz \right)$
	Qa	$Qa \to Ra$
	Ra	
	$Pa \rightarrow Ra$	
	$\forall \! y \left(Py \to Ry \right)$	

• Proofs in Natural Deduction are trees of \mathcal{L}_2 -sentences

	$\forall \! y \left(Py \rightarrow Qy \right)$	
[Pa]	$Pa \rightarrow Qa$	$\forall z \left(Qz \to Rz \right)$
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	Ra	
	$Pa \rightarrow Ra$	
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• The root of the tree is the conclusion

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	Qa	$Qa \rightarrow Ra$
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	$Pa \rightarrow Ra$	
	$\forall \! y \left(Py \to Ry \right)$	

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- The unbracketed sentences at the top are the premisses

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	$\frac{Ra}{Pa \rightarrow Ra}$	
	$\frac{Pa \rightarrow Ra}{\forall y (Py \rightarrow Ry)}$	

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- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
- The rules depend purely on the syntax of the sentences ... not on their semantic properties.

Rules for \wedge

$\wedge Intro$

The result of appending $\phi \wedge \psi$ to a proof of ϕ and a proof of ψ is a proof of $\phi \wedge \psi$.

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 $\frac{\phi \quad \psi}{\phi \land \psi}$ - ∧Intro

Rules for \land

∧Intro

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$$\frac{\vdots \qquad \vdots}{\phi \quad \psi} \land \text{Intro}$$

$\wedge \mathrm{ELIM1}$ and $\wedge \mathrm{ELIM2}$

The result of appending φ to a proof of φ ∧ ψ is a proof of φ.
 The result of appending ψ to a proof of φ ∧ ψ is a proof of ψ.

$$\begin{array}{c} \vdots \\ \hline \phi \wedge \psi \\ \hline \phi \\ \hline \phi \\ \end{array} \wedge Elim1 \\ \begin{array}{c} \vdots \\ \hline \phi \wedge \psi \\ \psi \\ \hline \psi \\ \wedge Elim2 \\ \end{array}$$

$$(P \land Q) \land R \vdash P$$

$$(P \land Q) \land R \vdash P$$

$(P \wedge Q) \wedge R$

First, assume the premiss. This is covered by the

ASSUMPTION RULE

The occurrence of a sentence ϕ with no sentence above it is an assumption. An assumption of ϕ is a proof of ϕ .

You may assume any sentence. (But choosing the right assumptions is important.)

$$(P \land Q) \land R \vdash P$$

Next apply a rule \land Elim1.

$$\frac{(P \land Q) \land R}{P \land Q}$$

 $\frac{ \stackrel{.}{\overset{.}{}}}{ \frac{\phi \wedge \psi}{\phi}} \wedge Elim1$

$$(P \land Q) \land R \vdash P$$

Next apply the same rule a second time.

$$\frac{(P \land Q) \land R}{\frac{P \land Q}{P}}$$

 $\frac{\vdots}{\frac{\phi \land \psi}{\phi} \land Elim1}$

$$(P \land Q) \land R \vdash P$$

That's it! We have a complete proof.

$$\frac{(P \land Q) \land R}{\frac{P \land Q}{P}}$$

 $(P \land Q) \land R \vdash \mathbf{P}$

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$$\frac{(P \land Q) \land R}{\frac{P \land Q}{P}}$$

• The conclusion is the sentence at the root.

 $(P \land Q) \land R \vdash P$

That's it! We have a complete proof.

 $\frac{(P \land Q) \land R}{\frac{P \land Q}{P}}$

- The conclusion is the sentence at the root.
- The premiss is the sentence at the top

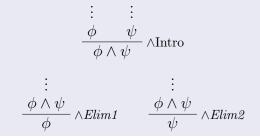
$$(P \land Q) \land R \vdash P$$

$$\frac{(P \land Q) \land R}{\frac{P \land Q}{P}}$$

- The conclusion is the sentence at the root.
- The premiss is the sentence at the top
- Each line is a correct application of a Natural Deduction rule.

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

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$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

First assume the first premiss.

 $Qb \wedge Pa$

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

Next apply a rule for \wedge .

 $\frac{Qb \wedge Pa}{Pa}$

 $\begin{array}{c} \vdots \\ \hline \phi \wedge \psi \\ \hline \psi \\ \hline \psi \\ \hline \end{array} \wedge Elim2 \\ \end{array}$

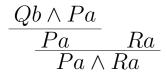
$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

Now assume the second premiss



$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

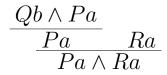
Now apply the introduction rule for \wedge .



 $\frac{\vdots}{\phi} \frac{\vdots}{\psi} \wedge \text{Intro}$

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

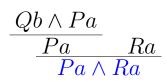
The proof is complete.



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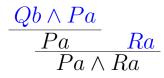
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$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

 $\frac{Qb \wedge Pa}{Pa} \frac{Ra}{Pa \wedge Ra}$

The proof is complete.

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Rules for \rightarrow

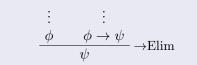
$\rightarrow E LIM$

The result of appending ψ to a proof of ϕ and a proof of $\phi \rightarrow \psi$ is a proof of ψ .

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Rules for \rightarrow

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$$\frac{\vdots \qquad \vdots}{\phi \qquad \phi \rightarrow \psi}_{\psi} \rightarrow \text{Elim}$$

This rule is often called 'Modus Ponens'.

$$\exists y \, Py \to Qa, \exists y \, Py \vdash Qa$$

$\exists y \: Py \to Qa, \exists y \: Py \vdash Qa$

$$\begin{array}{ccc} \vdots & \vdots \\ \phi & \phi \to \psi \\ \hline \psi & \end{array} \to \operatorname{Elim} \end{array}$$

$$\exists y \, Py \to Qa, \exists y \, Py \vdash Qa$$

Assume both premisses.

$\exists y \, Py \qquad \exists y \, Py \to Qa$

$$\exists y \, Py \to Qa, \exists y \, Py \vdash Qa$$

Apply the elimination rule.

$$\frac{\exists y \, Py \qquad \exists y \, Py \to Qa}{Qa}$$

 $\begin{array}{ccc} \vdots & \vdots \\ \phi & \phi \to \psi \\ \hline \end{array} \to \text{Elim}$ ٠ ψ

$$\exists y \, Py \to Qa, \exists y \, Py \vdash Qa$$

Finished!

$$\frac{\exists y \, Py \qquad \exists y \, Py \to Qa}{Qa}$$

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Finished!

• The conclusion is the sentence at the root.

$$\frac{\exists y \, Py \qquad \exists y \, Py \to Qa}{Qa}$$

 $\exists y \, Py \to Qa, \exists y \, Py \vdash Qa$

$$\frac{\exists y \, Py \qquad \exists y \, Py \to Qa}{Qa}$$

Finished!

- The conclusion is the sentence at the root.
- The only assumptions are premisses.

$$\exists y \, Py \to Qa, \exists y \, Py \vdash Qa$$

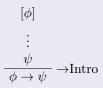
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Finished!

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The result of appending $\phi \to \psi$ to a proof of ψ and discharging all assumptions of ϕ in the proof of ψ is a proof of $\phi \to \psi$.

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$$\begin{matrix} [\phi] \\ \vdots \\ \psi \\ \hline \phi \to \psi \end{matrix} \to \text{Intro}$$

Conditional proof in informal reasoning.

(1) If it's poison and Quintus took it, then he needs to be readmitted.(2) It's poison

So (C) if Quintus took it, he need to be readmitted.

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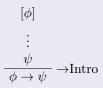
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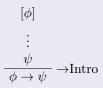
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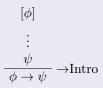
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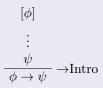
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So (by conditional proof) if Quintus took it, he needs to be readmitted.

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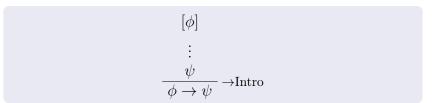
Example

$$P, (P \land Q) \to R \vdash Q \to R$$

25

Example

$$P, (P \land Q) \to R \vdash Q \to R$$



$$P, (P \land Q) \to R \vdash Q \to R$$

Assume the first premiss

P

 $P, (P \land Q) \to R \vdash Q \to R$

$P \qquad Q$

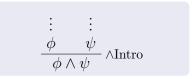
Next assume Q.

- This is the standard way to prove a conditional conclusion.
- We assume the antecedent and prove the consequent.

$$P, (P \land Q) \to R \vdash Q \to R$$

Apply \land Intro.





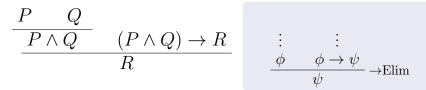
$$P, (P \land Q) \to R \vdash Q \to R$$

Assume the second premiss.

$$\frac{P \quad Q}{P \land Q} \quad (P \land Q) \to R$$

$$P, (P \land Q) \to R \vdash Q \to R$$

Apply \rightarrow Elim.



$$P, (P \land Q) \to R \vdash Q \to R$$

$$\frac{P \quad Q}{P \land Q} \quad (P \land Q) \to R$$

$$\frac{R}{R}$$

• Assuming the antecedent Q we've reached the consequent R.

• So we may apply
$$\rightarrow$$
Intro

$$\begin{array}{c} [\phi] \\ \vdots \\ \hline \psi \\ \hline \phi \to \psi \end{array} \to \text{Intro} \end{array}$$

$$P, (P \land Q) \to R \vdash Q \to R$$

$$\frac{P \quad Q}{P \land Q} \quad (P \land Q) \to R \\
 \frac{R}{Q \to R}$$

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$$P, (P \land Q) \to R \vdash Q \to R$$

$$\frac{\begin{array}{ccc}
P & [Q] \\
\hline P \land Q & (P \land Q) \to R \\
\hline \hline R \\
\hline Q \to R
\end{array}$$

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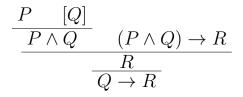
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$$\begin{array}{c} [\phi] \\ \vdots \\ \psi \\ \hline \phi \rightarrow \psi \end{array} \rightarrow \text{Intro}$$

• We discharge the assumption of Q.

$$P, (P \land Q) \to R \vdash Q \to R$$

The proof is complete



$$P, (P \land Q) \to R \vdash Q \to R$$

$$\frac{\begin{array}{ccc}
P & [Q] \\
\hline P \land Q & (P \land Q) \rightarrow R \\
\hline R \\
\hline Q \rightarrow R
\end{array}$$

The proof is complete

• The conclusion is at the root.

$$P, (P \land Q) \to R \vdash Q \to R$$

$$\frac{\begin{array}{cc} P & [Q] \\ \hline P \land Q & (P \land Q) \to R \\ \hline \hline R \\ \hline Q \to R \end{array}$$

The proof is complete

- The conclusion is at the root.
- The only *undischarged* assumptions are premisses.

$$P, (P \land Q) \to R \vdash Q \to R$$

$$\frac{\begin{array}{c|c}
P & [Q] \\
\hline P \land Q & (P \land Q) \to R \\
\hline \hline R \\
\hline Q \to R
\end{array}$$

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- The conclusion is at the root.
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- Discharged assumptions don't need to be amongst the premisses.

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The sentence ϕ is *provable* from Γ if and only if:

 there is a proof of φ with only sentences in Γ as non-discharged assumptions.

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Notation

- $\Gamma \vdash \phi$ is short for ϕ is provable from Γ
- $\vdash \phi$ is short for $\emptyset \vdash \phi$
- $\psi_1, \ldots, \psi_n \vdash \phi$ is short for $\{\psi_1, \ldots, \psi_n\} \vdash \phi$.

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But it is not an *outright* proof of $\exists x \exists y (Rxy \lor P)$

Return to the rule of assumption.

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- This proof does not show $\vdash \exists x \exists y (Rxy \lor P)$
- Instead it shows $\exists x \exists y (Rxy \lor P) \vdash \exists x \exists y (Rxy \lor P)$

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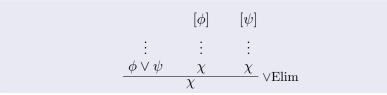
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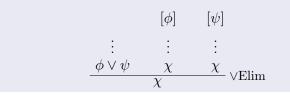
- This proof does not show $\vdash \exists x \exists y (Rxy \lor P)$
- Instead it shows $\exists x \exists y (Rxy \lor P) \vdash \exists x \exists y (Rxy \lor P)$

Rules for \lor

The introduction rules are straightforward.

$$\begin{array}{c} \vdots \\ \hline \phi \\ \hline \phi \lor \psi \end{array} \lor \text{Intro1} \qquad \begin{array}{c} \vdots \\ \hline \psi \\ \hline \phi \lor \psi \\ \hline \psi \text{Intro2} \end{array}$$

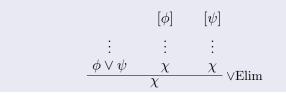




Proof by cases in informal reasoning

(1) Either you don't play and you quit or you do something quiet and don't play.

So, (C) you don't play.

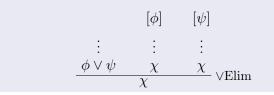


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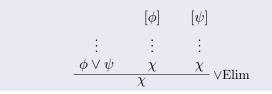
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Case (i): You don't play and you quit.



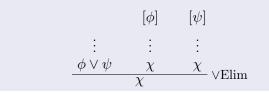
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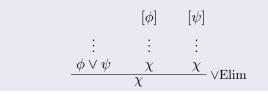
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Case (i): You don't play and you quit. So: you don't play

Case (ii): You do something quiet and don't play.



Proof by cases in informal reasoning

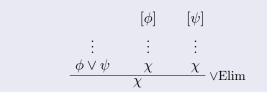
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Case (i): You don't play and you quit. So: you don't play

Case (ii): You do something quiet and don't play. So: you don't play.



Proof by cases in informal reasoning

(1) Either you don't play and you quit or you do something quiet and don't play.

So, (C) you don't play.

Informal proof. Suppose (1)

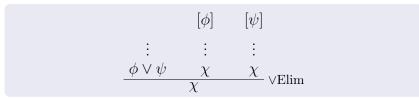
Case (i): You don't play and you quit. So: you don't play

Case (ii): You do something quiet and don't play. So: you don't play.

Either way then, (C) follows: you don't play.

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$



$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P)$$

Assume the premiss

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$\neg P \land Q$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P)$$

Case 1: Assume $\neg P \land Q$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \qquad \frac{\neg P \land Q}{\neg P}$$

Apply \land Elim1

$$\frac{\vdots}{\frac{\phi \land \psi}{\phi} \land Elim1}$$

That completes case 1.

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \qquad \frac{\neg P \land Q}{\neg P} \qquad \frac{\exists x \, Qx \land \neg P}{\neg P}$$

Case 2: Assume $\exists xQx \land \neg P$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$\frac{\neg P \land Q}{(\neg P \land Q) \lor (\exists x \, Qx \land \neg P)} \quad \frac{\neg P \land Q}{\neg P} \quad \frac{\exists x \, Qx \land \neg P}{\neg P}$$

Apply \land Elim1 once more

$$\frac{\vdots}{\frac{\phi \land \psi}{\phi} \land Elim1}$$

That completes case 2.

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \qquad \frac{\neg P \land Q}{\neg P} \qquad \frac{\exists x \, Qx \land \neg P}{\neg P}$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$\begin{array}{c|c} \neg P \land Q \\ \hline (\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \\ \hline \neg P \\ \hline \hline \neg P \\ \end{array} \begin{array}{c} \exists x \, Qx \land \neg P \\ \hline \neg P \\ \hline \end{array}$$

$$\begin{array}{ccc} [\phi] & [\psi] \\ \vdots & \vdots & \vdots \\ \hline \frac{\phi \lor \psi & \chi & \chi}{\chi} \lor \text{Elim} \end{array}$$

$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$\frac{(\neg P \land Q) \lor (\exists x \, Qx \land \neg P)}{\neg P} \quad \frac{\exists x \, Qx \land \neg P}{\neg P}}{\neg P}$$

$$\begin{array}{ccc} [\phi] & [\psi] \\ \vdots & \vdots & \vdots \\ \hline \frac{\phi \lor \psi & \chi & \chi}{\chi} \lor \text{Elim} \end{array}$$

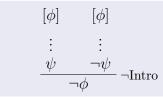
$$(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P$$

$$\frac{[\neg P \land Q]}{\neg P} \quad \frac{[\neg P \land Q]}{\neg P} \quad \frac{[\exists x \, Qx \land \neg P]}{\neg P}$$

$$\begin{array}{ccc} [\phi] & [\psi] \\ \vdots & \vdots & \vdots \\ \hline \frac{\phi \lor \psi & \chi & \chi}{\chi} \lor \text{Elim} \end{array}$$

The rules for \neg

Here are the rules for \neg .



The proof technique is known as *reductio ad absurdum*.

The rules for \neg

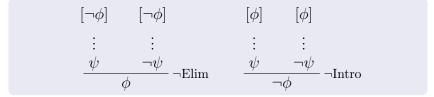
Here are the rules for \neg .



The proof technique is known as *reductio ad absurdum*.

$$\neg(P \to Q) \vdash \neg Q$$

$$\neg(P \to Q) \vdash \neg Q$$



$$\neg (P \to Q) \vdash \neg Q$$

Q

We want to prove $\neg Q$ be *reductio*. So start by assuming Q, and we'll go for a contradiction.

$$\neg (P \to Q) \vdash \neg Q$$

 $Q \qquad \neg (P \to Q)$

We can always safely assume the premiss.

$$\neg(P \to Q) \vdash \neg Q$$

$$\frac{Q}{P \to Q} \quad \neg (P \to Q)$$

Next apply \rightarrow Intro to get a contradiction

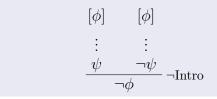
$$\begin{array}{c} [\phi] \\ \vdots \\ \psi \\ \overline{\phi \to \psi} \to \text{Intro} \end{array}$$

(Note this rule can be applied even when we haven't assumed the antecedent)

$$\neg (P \to Q) \vdash \neg Q$$

$$\frac{Q}{P \to Q} \quad \neg (P \to Q) \\ \neg Q$$

Now we apply \neg Intro

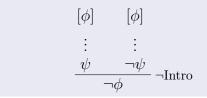


And we're done.

$$\neg (P \to Q) \vdash \neg Q$$

$$\begin{array}{c} [Q] \\ \hline P \rightarrow Q & \neg (P \rightarrow Q) \\ \hline \neg Q \end{array}$$

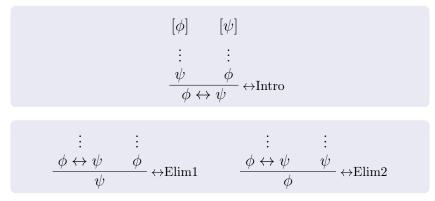
Now we apply \neg Intro



And we're done.

Rules for \leftrightarrow

These are reminiscent of the rules for \rightarrow



Rules for \forall

÷ $\frac{\forall v \, \phi}{\phi[t/v]} \, \forall \text{Elim}$

Rules for \forall

$$\frac{\vdots}{\phi[t/v]} \forall \text{Elim}$$

In this rule:

- ϕ is a formula in which only the variable v occurs freely
- t is a constant
- $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

Substitution

 $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

• Recall: a free occurrence of v is one not bound by $\forall v \text{ or } \exists v$

Compute the following

• Px[a/x] =

•
$$\forall x P x[a/x] =$$

•
$$\forall y (\exists x P x \lor Q x \to P y) [a/x] =$$

40

Substitution

 $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

• Recall: a free occurrence of v is one not bound by $\forall v \text{ or } \exists v$

Compute the following

• Px[a/x] = Pa

•
$$\forall x P x[a/x] =$$

•
$$\forall y (\exists x Px \lor Qx \to Py)[a/x] =$$

Substitution

 $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

• Recall: a free occurrence of v is one not bound by $\forall v \text{ or } \exists v$

Compute the following

• Px[a/x] = Pa

•
$$\forall x P x[a/x] = \forall x P x$$

•
$$\forall y (\exists x P x \lor Q x \to P y) [a/x] = 40$$

Substitution

 $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

• Recall: a free occurrence of v is one not bound by $\forall v \text{ or } \exists v$

Compute the following

- Px[a/x] = Pa
- $\forall x P x[a/x] = \forall x P x$
- $\forall y (\exists x Px \lor Qx \to Py)[a/x] = \forall y (\exists x Px \lor Qa \to Py)$ 40

$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

 $\frac{\vdots}{\phi[t/v]} \forall \text{Elim}$

$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

Assume the first premiss.

 $\forall x \left(Px \to Qx \right)$

$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

Apply $\forall \mathrm{Elim}$

$$\frac{\forall x \left(Px \to Qx \right)}{Pa \to Qa}$$

$$\begin{array}{c} \vdots \\ \hline \forall v \ \phi \\ \hline \phi[t/v] \end{array} \forall \text{Elim} \end{array}$$

To apply the rule: delete $\forall x$ and by replace all occurrences of x in the formula by the constant a.

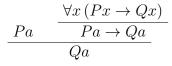
$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

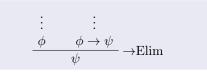
Assume the other premiss

$$Pa \qquad \frac{\forall x \left(Px \to Qx \right)}{Pa \to Qa}$$

$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

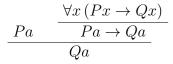
Apply modus ponens.





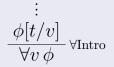
$$\forall x \, (Px \to Qx), Pa \vdash Qa$$

And we're done



side conditions:

- (i) the constant t does not occur in ϕ and
- (ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.



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Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

 $\frac{\phi[t/v]}{\forall v \phi}$ ∀Intro

∀Intro

 $\frac{\phi[t/v]}{\forall v \phi}$

side conditions:

- (i) the constant t does not occur in ϕ and
- (ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian *Informal proof.*

∀Intro

 $\frac{\phi[t/v]}{\forall v \phi}$

side conditions:

- (i) the constant t does not occur in ϕ and
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Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

Informal proof. Let an arbitrary thing be given. Call it 'Jane Doe'.

∀Intro

 $\frac{\phi[t/v]}{\forall v \phi}$

side conditions:

- (i) the constant t does not occur in ϕ and
- (ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

Informal proof. Let an arbitrary thing be given. Call it 'Jane Doe'.

Clearly, *if* Jane Doe is a pedestrian, then Jane Doe is either a qualified driver or a pedestrian.

 $\frac{\phi[t/v]}{\forall v \phi}$

-∀Intro

side conditions:

- (i) the constant t does not occur in ϕ and
- (ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

Informal proof. Let an arbitrary thing be given. Call it 'Jane Doe'.

Clearly, *if* Jane Doe is a pedestrian, then Jane Doe is either a qualified driver or a pedestrian.

So: every pedestrian is either a qualified driver or a pedestrian.

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

$$\begin{array}{c} \vdots \\ \hline \phi[t/v] \\ \hline \forall v \phi \end{array} \forall \text{Intro} \end{array}$$

$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$

I assume Pa. (We'll try to prove $Pa \rightarrow Qa \lor Pa$ without making assumptions about a)

Pa

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

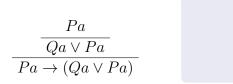
Apply \lor Intro2.



٠ : $\frac{\psi}{\phi \lor \psi} \lor \text{Intro2}$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

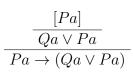
 $\text{Apply} \rightarrow \text{Intro}$



$$\begin{array}{c} [\phi] \\ \vdots \\ \hline \psi \\ \hline \phi \rightarrow \psi \end{array} \rightarrow \text{Intro} \end{array}$$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

 $\text{Apply} \rightarrow \text{Intro}$



$$\begin{array}{c} [\phi] \\ \vdots \\ \hline \psi \\ \hline \phi \rightarrow \psi \end{array} \rightarrow \text{Intro} \end{array}$$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

Finally we want to apply the rule for introducing \forall .

$$\begin{array}{c} \hline [Pa] \\ \hline Qa \lor Pa \\ \hline Pa \to (Qa \lor Pa) \\ \hline \forall z \left(Pz \to (Qz \lor Pz) \right) \end{array} \end{array}$$

÷ $\frac{\phi[t/v]}{\forall v \phi} \forall \text{Intro}$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

Finally we want to apply the rule for introducing \forall .

$$\begin{array}{c} [Pa] \\ \hline \hline Qa \lor Pa \\ \hline Pa \to (Qa \lor Pa) \\ \hline \forall z \left(Pz \to (Qz \lor Pz) \right) \end{array}$$

$$\begin{array}{c} \vdots \\ \hline \phi[t/v] \\ \hline \forall v \phi \end{array} \forall \text{Intro} \end{array}$$

But we also need to check the side conditions are met.

for t = a; $\phi = (Pz \rightarrow (Qz \lor Pz))$

(i) t does not occur in ϕ

i.e. a does not occur in $(Pz \rightarrow (Qz \lor Pz))$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

Finally we want to apply the rule for introducing \forall .

$$\begin{array}{c} [Pa] \\ \hline \hline Qa \lor Pa \\ \hline \hline Pa \to (Qa \lor Pa) \\ \hline \forall z \left(Pz \to (Qz \lor Pz) \right) \end{array} \end{array}$$

$$\begin{array}{c} \vdots \\ \hline \phi[t/v] \\ \hline \forall v \phi \end{array} \forall Intro$$

But we also need to check the side conditions are met. for t = a; $\phi = (Pz \rightarrow (Qz \lor Pz))$

or $t = a; \phi = (Pz \to (Qz \lor Pz))$

(i) t does not occur in ϕ

i.e. a does not occur in $(Pz \rightarrow (Qz \lor Pz))$

$$\vdash \forall z \left(Pz \to Qz \lor Pz \right)$$

Finally we want to apply the rule for introducing \forall .

$$\begin{array}{c} [Pa] \\ \hline Qa \lor Pa \\ \hline Pa \to (Qa \lor Pa) \\ \hline \forall z \left(Pz \to (Qz \lor Pz) \right) \end{array}$$

$$\begin{array}{c} \vdots \\ \hline \phi[t/v] \\ \hline \forall v \phi \end{array} \forall \text{Intro} \end{array}$$

But we also need to check the side conditions are met.

for t = a; $\phi = (Pz \rightarrow (Qz \lor Pz))$

- (ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.
- **i.e.** *a* does not occur in undischarged assumptions.

Rules for \exists

The introduction rule is straightforward.

 $\frac{\phi[t/v]}{\exists v \phi} \exists \text{Intro}$

$$Rcc \vdash \exists y Rcy$$

 $Rcc \vdash \exists y \, Rcy$

Rcc

Assume the premiss.

 $Rcc \vdash \exists y \, Rcy$

Rcc

Apply \exists Intro

$$\frac{\phi[t/v]}{\exists v \phi} \exists \text{Intro}$$

All we need to do is to choose the right ϕ and v

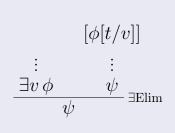
 $Rcc \vdash \exists y \, Rcy$

$$\frac{Rcc}{\exists y \, Rcy}$$

Apply \exists Intro

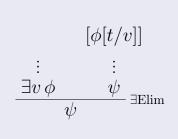
$$\frac{\phi[t/v]}{\exists v \phi} \exists \text{Intro}$$

All we need to do is to choose the right ϕ and vLet $\phi = Rcy, v = y$ $\phi[c/y] = Rcc$ $\exists v\phi = \exists yRcy$



Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

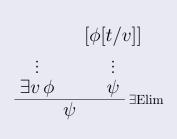


Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.



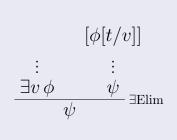
Side conditions:

- (i) t does not occur in $\exists v \phi$
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Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof.



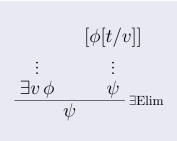
Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.



Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

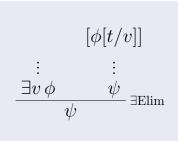
Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka.

The elimination rule is as follows.



Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

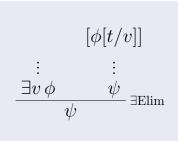
Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka. So, something is a quindarka.

The elimination rule is as follows.



Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka.

So, something is a quindarka.

So (C), follows from (1) and (2).

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$

Pc

The standard way to reason from $\exists x P x$ is to assume Pt (for t a new constant)

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

$$\forall x \left(Px \to Qx \right) \\ Pc$$

Assume the second premiss.

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

$$Pc \qquad \frac{\forall x \left(Px \to Qx \right)}{Pc \to Qc}$$

Apply $\forall \text{Elim.}$

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

$$\begin{array}{c} Pc & \overline{ \begin{array}{c} \forall x \left(Px \rightarrow Qx \right) \\ Pc \rightarrow Qc \end{array} } \\ \hline Qc \end{array} \end{array}$$

Apply \rightarrow Elim.

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

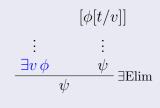
Apply \exists Intro.

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

Now we've reached a conclusion assuming Pc (and making no other assumptions about c) we can apply \exists Elim.

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

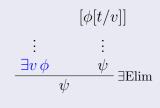
$$\begin{array}{c} Pc & \frac{\forall x \left(Px \to Qx \right)}{Pc \to Qc} \\ \hline \hline \frac{Qc}{\exists x Qx} \\ \hline \end{array}$$



- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

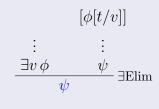
$$\begin{array}{c} Pc & \frac{\forall x \left(Px \to Qx \right)}{Pc \to Qc} \\ \hline \hline \frac{Qc}{\exists x Qx} \\ \hline \exists x Qx \end{array}$$



- (i) c does not occur in $\exists x P x$
- (ii) t does not occur in ψ
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

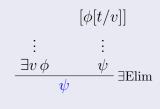
$$\frac{Pc}{\exists x Px} \frac{\begin{array}{c} \forall x (Px \to Qx) \\ Pc \to Qc \\ \hline \hline Qc \\ \exists x Qx \end{array}}{\exists x Qx}$$



- (i) c does not occur in $\exists x P x$
- (ii) t does not occur in ψ
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

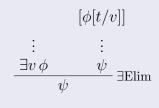
$$\frac{Pc}{\exists x Px} \frac{\begin{array}{c} \forall x (Px \to Qx) \\ Pc \to Qc \\ \hline \hline Qc \\ \exists x Qx \end{array}}{\exists x Qx}$$



- (i) c does not occur in $\exists x P x$
- (ii) c does not occur in $\exists x Q x$
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

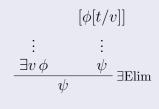
$$\begin{array}{c} Pc & \frac{\forall x \left(Px \to Qx \right)}{Pc \to Qc} \\ \hline \\ \exists x Px & \frac{Qc}{\exists x Qx} \\ \hline \\ \exists x Qx \end{array}$$



- (i) c does not occur in $\exists x P x$
- (ii) c does not occur in $\exists x Q x$
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

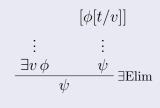
$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$

$$\begin{array}{ccc}
Pc & \overline{\forall x \left(Px \to Qx \right)} \\
Pc \to Qc \\
\hline
Qc \\
\exists x Qx \\
\hline
\exists x Qx \\
\end{array}$$



- (i) c does not occur in $\exists x P x$
- (ii) c does not occur in $\exists x Q x$
- (iii) c does not occur in any undischarged assumption other than Pc in the proof of $\exists xQx$.

$$\exists x \, Px, \forall x \, (Px \to Qx) \vdash \exists x \, Qx$$



- (i) c does not occur in $\exists x P x$
- (ii) c does not occur in $\exists x Q x$
- (iii) c does not occur in any undischarged assumption other than Pc in the proof of $\exists xQx$.

Let Γ be a set of \mathcal{L}_2 -sentences and ϕ a \mathcal{L}_2 -sentence.

Two notions of consequence

 $\Gamma \vdash \phi$ iff there is a proof of ϕ with only sentences in Γ as non-discharged assumptions.

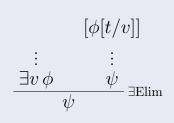
 $\Gamma \vDash \phi$ iff there is no \mathcal{L}_2 -structure in which all sentences in Γ are true and ϕ is false.

Theorem

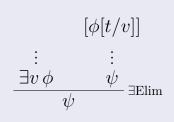
- (a) Soundness: $\Gamma \vdash \phi$ only if $\Gamma \vDash \phi$
- (b) Completeness: $\Gamma \vDash \phi$ only if $\Gamma \vdash \phi$

Х

Proof. Elements of Deductive Logic.



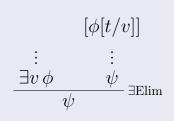
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .



Side conditions:

- **X** t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

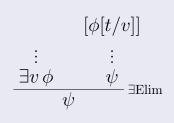
Raa



Side conditions:

- **X** t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

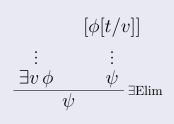
 $\frac{Raa}{\exists x R x x}$



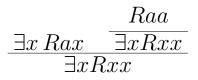
Side conditions:

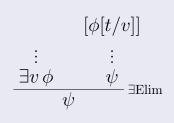
- **X** t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

 $\exists x \, Rax \quad \frac{Raa}{\exists x Rxx}$



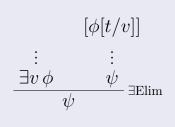
- **X** t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .





- **X** t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\exists x \, Rax}{\exists x Rxx} \frac{[Raa]}{\exists x Rxx}$$

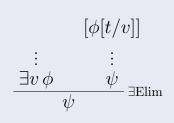


Side conditions:

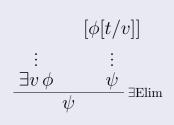
- **X** t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\exists x \, Rax}{\exists x Rxx} \frac{[Raa]}{\exists x Rxx}$$

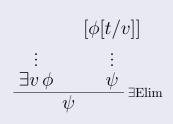
But clearly, $\exists x Rax \not\models \exists x Rxx$. Without (i), ND is not sound.



- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .



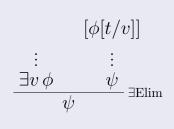
- (i) t does not occur in $\exists v \phi$
 - **X** t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .



Side conditions:

- (i) t does not occur in $\exists v \phi$
- **X** t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

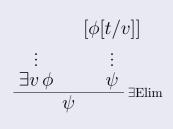
Pa



Side conditions:

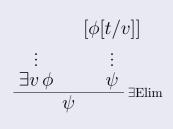
- (i) t does not occur in $\exists v \phi$
- **X** t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

 $\exists x P x P a$



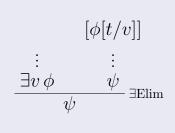
- (i) t does not occur in $\exists v \phi$
- **X** t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\exists x \, Px \quad Pa}{Pa}$$



- (i) t does not occur in $\exists v \phi$
- **X** t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\exists x \, Px \quad [Pa]}{Pa}$$

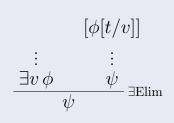


Side conditions:

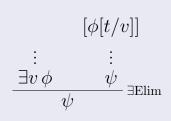
- (i) t does not occur in $\exists v \phi$
- **X** t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\exists x \, Px \quad [Pa]}{Pa}$$

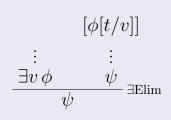
But clearly, $\exists x P x \not\models P a$. Without (ii), ND is not sound.



- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
- (iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .



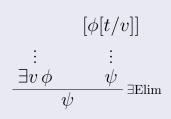
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .



Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

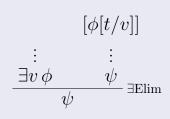
Pa



Side conditions:

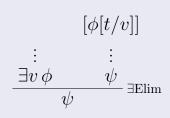
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

 $Pa \qquad Qa$



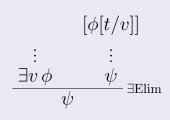
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\begin{array}{c|c} Pa & Qa \\ \hline Pa \wedge Qa \end{array}$$



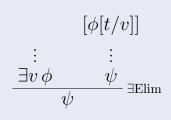
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\begin{array}{cc} Pa & Qa \\ \hline Pa \wedge Qa \\ \hline \exists x (Px \wedge Qx) \end{array}$$



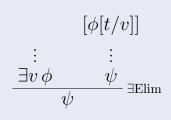
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{Pa \quad Qa}{Pa \wedge Qa}$$
$$\exists x Px \quad \exists x (Px \wedge Qx)$$



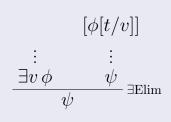
- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{\begin{array}{ccc} Pa & Qa \\ \hline Pa \wedge Qa \\ \hline \hline \exists x Px & \hline \exists x (Px \wedge Qx) \\ \hline \exists x (Px \wedge Qx) \end{array}}{\begin{array}{c} \exists x (Px \wedge Qx) \end{array}}$$



- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{ \begin{bmatrix} Pa \end{bmatrix} Qa}{Pa \land Qa} \\
\frac{\exists x Px}{\exists x (Px \land Qx)} \\
\frac{\exists x (Px \land Qx)}{\exists x (Px \land Qx)}$$



Side conditions:

- (i) t does not occur in $\exists v \phi$
- (ii) t does not occur in ψ ,
 - **X** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ .

$$\frac{ \begin{bmatrix} Pa \end{bmatrix} Qa}{Pa \land Qa} \\
\frac{\exists x Px}{\exists x (Px \land Qx)} \\
\frac{\exists x (Px \land Qx)}{\exists x (Px \land Qx)}$$

But $\exists x P x, Qa \not\vDash \exists x (Px \land Qx)$. Without (iii), ND is not sound.

http://logicmanual.philosophy.ox.ac.uk