INTRODUCTION TO LOGIC

Lecture 6
Natural Deduction
Dr. James Studd

There’s nothing you can’t prove if your outlook is only sufficiently limited

Dorothy L. Sayers
Outline

1. Proof
2. Rules for connectives
3. Rules for quantifiers
4. Adequacy
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

\[
\begin{array}{c}
\forall y \, (P_y \rightarrow Q_y) \\
[Pa] \quad \\ \\
Pa \rightarrow Qa \\
Qa \\
\forall z \, (Qz \rightarrow Rz) \\
Qa \rightarrow Ra \\
Ra \\
Pa \rightarrow Ra \\
\forall y \, (P_y \rightarrow R_y)
\end{array}
\]
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

$$
\begin{align*}
\forall y \, (Py \rightarrow Qy) & \\
[Pa] & \\
Pa \rightarrow Qa & \\
\forall z \, (Qz \rightarrow Rz) & \\
Qa & \\
Qa \rightarrow Ra & \\
R_a & \\
Pa \rightarrow Ra & \\
\forall y \, (Py \rightarrow Ry)
\end{align*}
$$

- The root of the tree is the conclusion
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

$\forall y (Py \rightarrow Qy)$

$[Pa] \quad Pa \rightarrow Qa \quad \forall z (Qz \rightarrow Rz)$

$Qa \quad Qa \rightarrow Ra$

$Ra \quad Pa \rightarrow Ra$

$\forall y (Py \rightarrow Ry)$

- The root of the tree is the conclusion
- The unbracketed sentences at the top are the premisses
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of \( \mathcal{L}_2 \)-sentences

\[
\begin{align*}
\forall y (Py \to Qy) \\
[Pa] & \quad Pa \to Qa \\
\hline \\
& \quad Qa \\
& \quad \forall z (Qz \to Rz) \\
\hline \\
& \quad Qa \to Ra \\
\hline \\
& \quad Ra \\
\hline \\
& \quad Pa \to Ra \\
\hline \\
& \quad \forall y (Py \to Ry)
\end{align*}
\]

- The root of the tree is the conclusion
- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

\[
\begin{array}{ccc}
[P_a] & \forall y (Py \rightarrow Qy) & \forall z (Qz \rightarrow Rz) \\
Pa & Pa \rightarrow Qa & Qa \rightarrow Ra \\
Qa & Ra & Pa \rightarrow Ra \\
Ra & & \forall y (Py \rightarrow Ry)
\end{array}
\]

- The root of the tree is the conclusion
- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
- The rules depend purely on the syntax of the sentences
Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

\[
\begin{align*}
[Pa] & \quad \forall y (Py \to Qy) \\
Pa & \quad Pa \to Qa \\
Qa & \quad Qa \to Ra \\
Ra & \quad Qa \to Ra \\
Pa \to Ra & \quad Ra \\
\forall y (Py \to Ry)
\end{align*}
\]

- The root of the tree is the conclusion
- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
- The rules depend purely on the syntax of the sentences
  ... not on their semantic properties.
6.1 Propositional logic

Rules for $\land$

$\land\text{Intro}$

The result of appending $\phi \land \psi$ to a proof of $\phi$ and a proof of $\psi$ is a proof of $\phi \land \psi$. 
Rules for $\land$

$\land\text{Intro}$

The result of appending $\phi \land \psi$ to a proof of $\phi$ and a proof of $\psi$ is a proof of $\phi \land \psi$.

\[
\begin{array}{c}
\vdots \quad \vdots \\
\phi \\ \psi \\
\hline \\
\phi \land \psi
\end{array}
\]

$\land\text{Intro}$
Rules for $\land$

$\textbf{\land Intro}$

The result of appending $\phi \land \psi$ to a proof of $\phi$ and a proof of $\psi$ is a proof of $\phi \land \psi$.

\[
\begin{array}{c}
\vdots \\
\phi \\
\psi \\
\hline
\phi \land \psi
\end{array}
\]

$\textbf{\land Elim1 and \land Elim2}$

(1) The result of appending $\phi$ to a proof of $\phi \land \psi$ is a proof of $\phi$.

(2) The result of appending $\psi$ to a proof of $\phi \land \psi$ is a proof of $\psi$.

\[
\begin{array}{c}
\vdots \\
\phi \land \psi \\
\hline
\phi
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\phi \land \psi \\
\hline
\psi
\end{array}
\]
Example

\[(P \land Q) \land R \vdash P\]
First, assume the premiss. This is covered by the

ASSUMPTION RULE

The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.

You may assume any sentence. (But choosing the right assumptions is important.)
Example

$(P \land Q) \land R \vdash P$

Next apply a rule $\land$Elim1.

\[
\frac{(P \land Q) \land R}{P \land Q} \quad \frac{\phi \land \psi}{\phi} \quad \land\text{Elim1}
\]
Example

\[(P \land Q) \land R \vdash P\]

Next apply the same rule a second time.

\[
\begin{align*}
(P \land Q) \land R \\
\quad \quad \quad P \land Q \\
\quad \quad \quad \quad \quad P
\end{align*}
\]

\[
\vdash \quad \phi \land \psi \\
\quad \quad \quad \phi \\
\quad \quad \quad \quad \quad \land Elim1
\]
Example

\[(P \land Q) \land R \vdash P\]

That’s it! We have a complete proof.

\[
\begin{array}{c}
(P \land Q) \land R \\
\hline
P \land Q \\
\hline
P
\end{array}
\]
Example

\[(P \land Q) \land R \vdash P\]

That’s it! We have a complete proof.

- The conclusion is the sentence at the root.
Example

\[(P \land Q) \land R \vdash P\]

That’s it! We have a complete proof.

- The conclusion is the sentence at the root.
- The premiss is the sentence at the top.
Example

\[(P \land Q) \land R \vdash P\]

That’s it! We have a complete proof.

- The conclusion is the sentence at the root.
- The premiss is the sentence at the top
- Each line is a correct application of a Natural Deduction rule.
Example

\[ Qb \land Pa, Ra \vdash Pa \land Ra \]
Example

\( Qb \land Pa, Ra \vdash Pa \land Ra \)

\[
\begin{align*}
\vdots & \quad \vdots \\
\phi & \quad \psi \\
\hline 
\phi \land \psi & \quad \text{\^Intro} \\
\vdots & \quad \vdots \\
\phi \land \psi & \quad \text{\^Elim1} \\
\phi & \\
\vdots & \quad \vdots \\
\psi & \quad \text{\^Elim2}
\end{align*}
\]
Example

\(Qb \land Pa, Ra \vdash Pa \land Ra\)

First assume the first premiss.

\(Qb \land Pa\)
Example

\[ Qb \land Pa, Ra \vdash Pa \land Ra \]

Next apply a rule for \( \land \).

\[ \frac{ Qb \land Pa }{ Pa } \]

\[ \vdots \]

\[ \frac{ \phi \land \psi }{ \psi } \quad ^{\text{\^Elim2}} \]
Example

\( Qb \land Pa, Ra \vdash Pa \land Ra \)

Now assume the second premiss

\[
\begin{align*}
Qb \land Pa \\
\hline
Pa \\
Ra
\end{align*}
\]
Example

$Qb \land Pa, Ra \vdash Pa \land Ra$

Now apply the introduction rule for $\land$.

$$
\begin{array}{c}
Qb \land Pa \\
\hline
Pa \\
Ra
\end{array}
\begin{array}{c}
Pa \\
\hline
Pa \land Ra
\end{array}
$$
The proof is complete.

\[ Qb \land Pa, Ra \vdash Pa \land Ra \]

\[ Qb \land Pa \]

\[ \frac{Pa}{Pa} \]

\[ Ra \]

\[ \frac{Pa \land Ra}{Pa \land Ra} \]
Example

\[ Qb \land Pa, Ra \vdash Pa \land Ra \]

The proof is complete.

- The conclusion is the sentence at the root.

\[
\begin{array}{c}
Qb \land Pa \\
\hline
Pa \quad Ra \\
\hline
Pa \land Ra
\end{array}
\]
Example

\( Qb \land Pa, Ra \vdash Pa \land Ra \)

The proof is complete.

- The conclusion is the sentence at the root.
- The premisses are the sentences at the top.
Example

\( Qb \land Pa, Ra \vdash Pa \land Ra \)

The proof is complete.

- The conclusion is the sentence at the root.
- The premisses are the sentences at the top.
- Each line is a correct application of a Natural Deduction rule.
### Rules for $\rightarrow$

<table>
<thead>
<tr>
<th><strong>$\rightarrow$ Elim</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>The result of appending $\psi$ to a proof of $\phi$ and a proof of $\phi \rightarrow \psi$ is a proof of $\psi$.</td>
</tr>
</tbody>
</table>

This rule is often called 'Modus Ponens'.
Rules for \( \rightarrow \)

\[ \rightarrow \text{Elim} \]

The result of appending \( \psi \) to a proof of \( \phi \) and a proof of \( \phi \rightarrow \psi \) is a proof of \( \psi \).

\[
\begin{array}{c}
\vdots \\
\phi \\
\phi \rightarrow \psi \\
\hline
\psi \\
\rightarrow \text{Elim}
\end{array}
\]
Rules for $\rightarrow$

**→Elim**

The result of appending $\psi$ to a proof of $\phi$ and a proof of $\phi \rightarrow \psi$ is a proof of $\psi$.

\[
\begin{array}{c}
\vdots \\
\phi \\
\phi \rightarrow \psi \\
\hline
\psi
\end{array}
\rightarrow\text{Elim}
\]

This rule is often called ‘Modus Ponens’.
Example

\[ \exists y \, P y \rightarrow Q a, \exists y \, P y \vdash Q a \]
Example
\[ \exists y \, P_y \rightarrow Q a, \exists y \, P_y \vdash Q a \]

\[ \vdots \vdots \]

\[ \phi \quad \phi \rightarrow \psi \]

\[ \psi \rightarrow \text{Elim} \]
**Example**

\[ \exists y \ P y \rightarrow Qa, \ \exists y \ P y \vdash Qa \]

Assume both premisses.

\[ \exists y \ P y \quad \exists y \ P y \rightarrow Qa \]
Example

$$\exists y Py \rightarrow Qa, \exists y Py \vdash Qa$$

Apply the elimination rule.

$$\exists y Py \quad \exists y Py \rightarrow Qa$$

$$\begin{array}{c}
\vdash \\
\vdash \\
\phi \\
\phi \rightarrow \psi \\
\psi
\end{array} \quad \rightarrow \text{Elim}$$
Example

\[ \exists y Py \to Qa, \exists y Py \vdash Qa \]

Finished!

\[
\exists y Py \quad \exists y Py \to Qa \\
\hline
Qa
\]

The conclusion is the sentence at the root. The only assumptions are premisses. Each line is a correct application of a Natural Deduction rule.
Example

\( \exists y Py \rightarrow Qa, \exists y Py \vdash Qa \)

\[\begin{array}{c}
\exists y Py \\
\exists y Py \rightarrow Qa
\end{array}\]

\[\begin{array}{c}
Qa
\end{array}\]

Finished!

- The conclusion is the sentence at the root.
### Example

\[ \exists y \, P_y \rightarrow Q_a, \exists y \, P_y \vdash Q_a \]

---

**Finished!**

- The conclusion is the sentence at the root.
- The only assumptions are premisses.
Example

\( \exists y \, Py \rightarrow Qa, \exists y \, Py \vdash Qa \)

\[
\begin{array}{c}
\exists y \, Py \\
\exists y \, Py \rightarrow Qa
\end{array}
\]

Finished!

- The conclusion is the sentence at the root.
- The only assumptions are premisses.
- Each line is a correct application of a Natural Deduction rule.
The result of appending $\phi \rightarrow \psi$ to a proof of $\psi$ and discharging all assumptions of $\phi$ in the proof of $\psi$ is a proof of $\phi \rightarrow \psi$. 

---

**Intro**

Conditional proof in informal reasoning.

(1) If it's poison and Quintus took it, then he needs to be readmitted.

(2) It's poison

So (C) if Quintus took it, he needs to be readmitted.

Informal proof.

Assume Quintus took it.

Then (by 2) it's poison and he took it.

Then (by 1 and MP) he needs to be readmitted.

So (by conditional proof) if Quintus took it, he needs to be readmitted.
The result of appending \( \phi \rightarrow \psi \) to a proof of \( \psi \) and discharging all assumptions of \( \phi \) in the proof of \( \psi \) is a proof of \( \phi \rightarrow \psi \).
The result of appending $\phi \rightarrow \psi$ to a proof of $\psi$ and discharging all assumptions of $\phi$ in the proof of $\psi$ is a proof of $\phi \rightarrow \psi$.

Conditional proof in informal reasoning.
(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.
The result of appending \( \phi \to \psi \) to a proof of \( \psi \) and discharging all assumptions of \( \phi \) in the proof of \( \psi \) is a proof of \( \phi \to \psi \).

\[
\begin{align*}
[\phi] \\
\vdots \\
\psi \\
\hline
\phi \to \psi \\
\to \text{Intro}
\end{align*}
\]

Conditional proof in informal reasoning.

(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.

*Informal proof.*
→Intro

The result of appending $\phi \rightarrow \psi$ to a proof of $\psi$ and discharging all assumptions of $\phi$ in the proof of $\psi$ is a proof of $\phi \rightarrow \psi$.

$\begin{align*}
[\phi] \\
\vdots \\
\psi \\
\hline
\phi \rightarrow \psi \rightarrow \text{Intro}
\end{align*}$

Conditional proof in informal reasoning.

(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.

*Informal proof.* Assume Quintus took it.
6.1 Propositional logic

→Intro

The result of appending \( \phi \rightarrow \psi \) to a proof of \( \psi \) and discharging all assumptions of \( \phi \) in the proof of \( \psi \) is a proof of \( \phi \rightarrow \psi \).

\[
\begin{align*}
[\phi] \\
\vdots \\
\psi \\
\hline
\phi \rightarrow \psi 
\end{align*}
\rightarrow \text{Intro}
\]

Conditional proof in informal reasoning.

(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.

Informal proof. Assume Quintus took it.
Then (by 2) it’s poison \textit{and} he took it.
The result of appending $\phi \rightarrow \psi$ to a proof of $\psi$ and discharging all assumptions of $\phi$ in the proof of $\psi$ is a proof of $\phi \rightarrow \psi$.

\[
\begin{array}{c}
[\phi] \\
\vdots \\
\psi \\
\hline 
\phi \rightarrow \psi 
\end{array}
\rightarrow \text{Intro}
\]

Conditional proof in informal reasoning.

(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.

_Informal proof_. Assume Quintus took it.
Then (by 2) it’s poison _and_ he took it.
Then (by 1 and MP) he needs to be readmitted.
The result of appending $\phi \rightarrow \psi$ to a proof of $\psi$ and discharging all assumptions of $\phi$ in the proof of $\psi$ is a proof of $\phi \rightarrow \psi$.

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\phi \rightarrow \psi
\end{array}
\rightarrow \text{Intro}
\]

Conditional proof in informal reasoning.

(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.

*Informal proof.* Assume Quintus took it.
Then (by 2) it’s poison *and* he took it.
Then (by 1 and MP) he needs to be readmitted.
So (by conditional proof) *if* Quintus took it, he needs to be readmitted.
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]
Example

\( P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \)

\[ [\phi] \]

\[ \vdots \]

\[ \psi \]

\[ \frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \rightarrow \text{Intro} \]
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

Assume the first premiss

\( P \)
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

Next assume \( Q \).

- This is the standard way to prove a conditional conclusion.
- We assume the antecedent and prove the consequent.
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

Apply \(\land\)Intro.

\[ \begin{array}{c}
P \\
Q \\
\hline
P \land Q
\end{array} \]

\[ \begin{array}{cc}
\phi & \psi \\
\hline
\phi \land \psi
\end{array} \]

\(\land\)Intro
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

Assume the second premiss.

\[
\begin{array}{c}
P \\
Q
\end{array} \quad \begin{array}{c}
P \land Q \\
(P \land Q) \rightarrow R
\end{array}
\]
### Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

\[
\frac{P \quad Q}{P \land Q} \quad \frac{(P \land Q) \rightarrow R}{R}
\]

Apply \( \rightarrow \)Elim.

\[
\vdash \quad \vdash
\frac{\phi \quad \phi \rightarrow \psi}{\psi} \quad \rightarrow \text{Elim}
\]
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

\[
\begin{array}{c}
P \quad Q \\
\hline
P \land Q
\end{array}
\]

\[
(P \land Q) \rightarrow R
\]

\[
R
\]

- Assuming the antecedent \( Q \) we’ve reached the consequent \( R \).
- So we may apply \( \rightarrow \text{Intro} \)

\[
[\phi] \\
\vdots \\
\phi \rightarrow \psi \\
\hline
\phi \rightarrow \psi \rightarrow \text{Intro}
\]
Example

\[
P, (P \land Q) \rightarrow R \vdash Q \rightarrow R
\]

\[
\begin{array}{c}
P \\
Q \\
\hline \\
P \land Q
\end{array}
\]

\[
(P \land Q) \rightarrow R
\]

\[
\begin{array}{c}
R \\
\hline \\
Q \rightarrow R
\end{array}
\]

- Assuming the antecedent \( Q \) we’ve reached the consequent \( R \).
- So we may apply \( \rightarrow \text{Intro} \)

\[
[\phi] \\
\vdots \\
\psi \\
\phi \rightarrow \psi \rightarrow \text{Intro}
\]
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

\[
\begin{array}{c}
P \\
\hline 
[Q] \\
\hline 
P \land Q \\
\hline 
(P \land Q) \rightarrow R \\
\hline 
R \\
\hline 
Q \rightarrow R
\end{array}
\]

- Assuming the antecedent \( Q \) we’ve reached the consequent \( R \).
- So we may apply \( \rightarrow \)Intro

We discharge the assumption of \( Q \).
Example

$P, (P \land Q) \rightarrow R \vdash Q \rightarrow R$

The proof is complete

$$
\begin{array}{c}
P \\
\hline
P \land [Q] \\
P \land Q \\
\hline
(P \land Q) \rightarrow R \\
\hline
R \\
\hline
Q \rightarrow R
\end{array}
$$
6.1 Propositional logic

Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

\[
\begin{align*}
P & \quad \boxed{[Q]} \\
\hline
P \land Q & \\
\hline
(P \land Q) \rightarrow R \\
\hline
R \\
\hline
Q \rightarrow R
\end{align*}
\]

The proof is complete.

- The conclusion is at the root.
Example

\[ P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \]

\[
\begin{array}{c}
\hline
P & [Q] \\
\hline
\hline
P \land Q & (P \land Q) \rightarrow R \\
R & R \\
\hline
Q \rightarrow R \\
\hline
\end{array}
\]

The proof is complete.

\begin{itemize}
\item The conclusion is at the root.
\item The only undischarged assumptions are premisses.
\end{itemize}
The proof is complete

- The conclusion is at the root.
- The only undischarged assumptions are premisses.
- Discharged assumptions don’t need to be amongst the premisses.
We can now define $\Gamma \vdash \phi$. 
We can now define $\Gamma \vDash \phi$.
Let $\Gamma$ be a set of sentences and $\phi$ a sentence.

**Definition (Provable)**

The sentence $\phi$ is *provable* from $\Gamma$ if and only if:
- there is a proof of $\phi$ with only sentences in $\Gamma$ as non-discharged assumptions.
We can now define $\Gamma \vdash \phi$.
Let $\Gamma$ be a set of sentences and $\phi$ a sentence.

**Definition (Provable)**

The sentence $\phi$ is *provable* from $\Gamma$ if and only if:

- there is a proof of $\phi$ with only sentences in $\Gamma$ as non-discharged assumptions.

**Notation**

- $\Gamma \vdash \phi$ is short for $\phi$ is provable from $\Gamma$
- $\vdash \phi$ is short for $\emptyset \vdash \phi$
- $\psi_1, \ldots, \psi_n \vdash \phi$ is short for $\{\psi_1, \ldots, \psi_n\} \vdash \phi$. 
Return to the rule of assumption.
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$. 
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.

This may seem odd.
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.

This may seem odd.
Suppose I assume, the following:

$$\exists x \exists y (Rxy \lor P)$$
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence φ with no sentence above it is an assumption. An assumption of φ is a proof of φ.

This may seem odd.
Suppose I assume, the following:

\[ \exists x \exists y (Rxy \lor P) \]

By the rule, this counts as a proof of \( \exists x \exists y (Rxy \lor P) \)
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.

This may seem odd.
Suppose I assume, the following:

$$\exists x \exists y (Rxy \lor P)$$

By the rule, this counts as a proof of $\exists x \exists y (Rxy \lor P)$

But it is not an *outright* proof of $\exists x \exists y (Rxy \lor P)$
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.

This may seem odd. Suppose I assume, the following:

$$\exists x \exists y (Rxy \lor P)$$

By the rule, this counts as a proof of $\exists x \exists y (Rxy \lor P)$

But it is not an *outright* proof of $\exists x \exists y (Rxy \lor P)$

- This proof does *not* show $\vdash \exists x \exists y (Rxy \lor P)$
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence \( \phi \) with no sentence above it is an assumption. An assumption of \( \phi \) is a proof of \( \phi \).

This may seem odd. Suppose I assume, the following:

\[
\exists x \exists y (Rxy \lor P)
\]

By the rule, this counts as a proof of \( \exists x \exists y (Rxy \lor P) \)

But it is not an *outright* proof of \( \exists x \exists y (Rxy \lor P) \)

- This proof does *not* show \( \vdash \exists x \exists y (Rxy \lor P) \)
- Instead it shows \( \exists x \exists y (Rxy \lor P) \vdash \exists x \exists y (Rxy \lor P) \)
Return to the rule of assumption.

**ASSUMPTION RULE**

The occurrence of a sentence \( \phi \) with no sentence above it is an assumption. An assumption of \( \phi \) is a proof of \( \phi \).

This may seem odd.
Suppose I assume, the following:

\[
\exists x \exists y (Rxy \lor P)
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By the rule, this counts as a proof of \( \exists x \exists y (Rxy \lor P) \)

But it is not an *outright* proof of \( \exists x \exists y (Rxy \lor P) \)

- This proof does *not* show \( \vdash \exists x \exists y (Rxy \lor P) \)
- Instead it shows \( \exists x \exists y (Rxy \lor P) \vdash \exists x \exists y (Rxy \lor P) \)
Rules for $\lor$

The introduction rules are straightforward.

\[
\therefore \\
\frac{\phi}{\phi \lor \psi} \quad \lor\text{Intro1} \\
\frac{\psi}{\phi \lor \psi} \quad \lor\text{Intro2}
\]
The elimination rule is a little more complex.

\[
\begin{array}{ccc}
\phi & \psi \\
\vdots & \vdots & \vdots \\
\phi \lor \psi & \chi & \chi \\
\hline
\chi & \chi & \chi \\
\end{array}
\]
\(\land\text{Elim}\)
The elimination rule is a little more complex.

\[
\begin{array}{c}
[\phi] & [\psi] \\
\vdots & \vdots & \vdots \\
\phi \lor \psi & \chi & \chi \\
\hline
\chi & \chi & \lor \text{Elim}
\end{array}
\]

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.
So, (C) you don’t play.
The elimination rule is a little more complex.

\[
\begin{array}{c}
\phi \\
\psi \\
\chi
\end{array}
\]

\[
\frac{\phi \lor \psi}{\chi} \quad \frac{\chi}{\chi} \quad \text{ Elim}
\]

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.
So, (C) you don’t play.

*Informal proof.* Suppose (1)
The elimination rule is a little more complex.

\[
\begin{array}{c}
\vdots \\
\phi \lor \psi \\
\chi \\
\chi \\
\hline
\chi
\end{array}
\]

\[\chi \quad \text{Elim} \]

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.
So, (C) you don’t play.

*Informal proof.* Suppose (1)

Case (i): You don’t play and you quit.
The elimination rule is a little more complex.

\[
\begin{array}{cccc}
\phi & \psi & \chi & \chi \\
\hline
\phi \vee \psi & \chi & \chi & \text{\textbackslash Elim} \\
\end{array}
\]

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.
So, (C) you don’t play.

*Informal proof.* Suppose (1)

Case (i): You don’t play and you quit. So: you don’t play.
The elimination rule is a little more complex.

\[
\begin{array}{c c c}
[\phi] & [\psi] \\
\vdots & \vdots & \vdots \\
\phi \lor \psi & \chi & \chi \\
\hline
\chi & \chi & \lor\text{Elim}
\end{array}
\]

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.
So, (C) you don’t play.

*Informal proof.* Suppose (1)

Case (i): You don’t play and you quit. So: you don’t play

Case (ii): You do something quiet and don’t play.
The elimination rule is a little more complex.

\[
\begin{array}{c}
[\phi] & [\psi] \\
\quad & \\
\quad & \\
\phi \lor \psi & \chi & \chi \\
\hline
\chi & \\
\hline
\text{\lorElim}
\end{array}
\]

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.
So, (C) you don’t play.

*Informal proof.* Suppose (1)

Case (i): You don’t play and you quit. So: you don’t play
Case (ii): You do something quiet and don’t play. So: you don’t play.
The elimination rule is a little more complex.

\[
\begin{array}{c}
\phi \\
\psi \\
\hline
\phi \lor \psi \\
\chi \\
\hline
\chi \\
\hline
\end{array}
\]

\(\vee\text{Elim}\)

**Proof by cases in informal reasoning**

(1) Either you don’t play and you quit or you do something quiet and don’t play.

So, (C) you don’t play.

*Informal proof.* Suppose (1)

Case (i): You don’t play and you quit. So: you don’t play

Case (ii): You do something quiet and don’t play. So: you don’t play.

Either way then, (C) follows: you don’t play.
Example

\[(\neg P \land Q) \lor (\exists x\ Qx \land \neg P) \vdash \neg P\]
Example

$$(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P$$

\[
\begin{array}{c|c}
[\phi] & [\psi] \\
\hline
\vdash & \vdash \\
\hline
\phi \lor \psi & \chi \\
\hline
\chi & \chi \quad \text{\textbackslash{}Elim}
\end{array}
\]
Example

\[(\neg P \land Q) \lor (\exists x \ Q\ x \land \neg P) \vdash \neg P\]

\[(\neg P \land Q) \lor (\exists x \ Q\ x \land \neg P)\]

Assume the premiss
Example

$$(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P$$

\[
(\neg P \land Q) \lor (\exists x \ Qx \land \neg P)
\]

Case 1: Assume $\neg P \land Q$
Example

\((\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P\)

\[
(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \\
\frac{\neg P \land Q}{\neg P}
\]

Apply \(\land\)Elim1

\[
\vdots \\
\phi \land \psi \\
\phi \quad \land\text{Elim1}
\]

That completes case 1.
Example

\((\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P\)

\[
\begin{align*}
(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) & \quad \frac{\neg P \land Q}{\neg P} \quad \frac{\exists x \ Qx \land \neg P}{\neg P}
\end{align*}
\]

Case 2: Assume \(\exists x Qx \land \neg P\)
Example

\[ (\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P \]

\[
(\neg P \land Q) \lor (\exists x \ Qx \land \neg P)
\]

\[
\dfrac{\neg P \land Q \quad \exists x \ Qx \land \neg P}{\neg P}
\]

Apply \ AndElim1\ once more

\[
\vdots
\]

\[
\dfrac{\phi \land \psi}{\phi} \quad \text{\AndElim1}
\]

That completes case 2.
Example

\[(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \vdash \neg P\]

\[
\begin{array}{c}
(\neg P \land Q) \lor (\exists x \, Qx \land \neg P) \\
\hline
\neg P \land Q \quad \exists x \, Qx \land \neg P \\
\hline
\neg P \\
\neg P
\end{array}
\]

We’ve reached \(\neg P\) from each disjunct.
We can now apply \(\lor\)Elim

\[
\begin{array}{c}
\phi \lor \psi \\
\hline
\chi \quad \chi
\end{array}
\]

\(\lor\)Elim
Example

$$(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P$$

$$
\begin{array}{c}
(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \\
\frac{\neg P \land Q}{\neg P} \quad \frac{\exists x \ Qx \land \neg P}{\neg P}
\end{array}
$$

We’ve reached $\neg P$ from each disjunct.
We can now apply $\lor$Elim

$$
\begin{array}{c}
\phi \\
\vdots \\
\phi \lor \psi \chi \\
\frac{\chi \lor \chi}{\chi} \lor\text{Elim}
\end{array}
$$
Example

\((\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P\)

\[
\frac{(\neg P \land Q) \lor (\exists x \ Qx \land \neg P)}{\neg P}
\]

We’ve reached \(\neg P\) from each disjunct. We can now apply \(\lor\)Elim

\[
\frac{[\phi]}{\phi \lor \psi} \quad \frac{[\psi]}{\chi \lor \chi}
\]

\(\lor\)Elim
Example

$$(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \vdash \neg P$$

\[
\begin{array}{c}
(\neg P \land Q) \lor (\exists x \ Qx \land \neg P) \\
\hline \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \equiv
The rules for \( \neg \)

Here are the rules for \( \neg \).

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi & \neg \psi \\
\hline \\
\neg \phi & \neg \text{Intro}
\end{array}
\]

The proof technique is known as *reductio ad absurdum*. 
The rules for $\neg$

Here are the rules for $\neg$.

<table>
<thead>
<tr>
<th>$[\phi]$</th>
<th>$[\phi]$</th>
<th>$[\neg\phi]$</th>
<th>$[\neg\phi]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$\neg\psi$</td>
<td>$\psi$</td>
<td>$\neg\psi$</td>
</tr>
<tr>
<td>$\neg\phi$</td>
<td>$\neg\phi$</td>
<td>$\phi$</td>
<td>$\neg\psi$</td>
</tr>
</tbody>
</table>

$\neg$Intro $\neg$Elim

The proof technique is known as *reductio ad absurdum*. 
Example

\neg(P \rightarrow Q) \vdash \neg Q
Example

\[ \neg (P \rightarrow Q) \vdash \neg Q \]
We want to prove \( \neg Q \) be \textit{reductio}.
So start by assuming \( Q \), and we’ll go for a contradiction.
Example

\((P \to Q) \vdash \neg Q\)

We can always safely assume the premiss.
Example

\[\neg (P \rightarrow Q) \vdash \neg Q\]

\[
\begin{array}{c}
Q \\
P \\
\hline
P \rightarrow Q
\end{array}
\]

\[\neg (P \rightarrow Q)\]

Next apply \(\rightarrow\)Intro to get a contradiction

\[
\begin{array}{c}
[\phi] \\
\vdots \\
\psi \\
\hline
\phi \rightarrow \psi
\end{array} \rightarrow \text{Intro}
\]

(Note this rule can be applied even when we haven’t assumed the antecedent)
Example

\[ \neg(P \rightarrow Q) \vdash \neg Q \]

\[
\begin{array}{c}
Q \\
\hline
P \rightarrow Q
\end{array}
\]

\[
\begin{array}{c}
\neg(P \rightarrow Q) \\
\hline
\neg Q
\end{array}
\]

Now we apply \( \neg \text{Intro} \)

\[
\begin{array}{c}
[\phi] \\
\hline
\psi
\end{array}
\]

\[
\begin{array}{c}
[\phi] \\
\hline
\neg \psi
\end{array}
\]

\[
\begin{array}{c}
\neg \phi
\end{array}
\]

\( \neg \text{Intro} \)

And we’re done.
6.1 Propositional logic

Example

\[\neg(P \rightarrow Q) \vdash \neg Q\]

\[
\frac{[Q]}{P \rightarrow Q} \quad \neg(P \rightarrow Q) \\
\hline
\neg Q
\]

Now we apply \text{\neg Intro}

\[
\begin{array}{c}
\vdash [\phi] \\
\vdash [\phi] \\
\vdash \vdash \vdash \\
\vdash \psi \quad \neg \psi \\
\hline
\neg \phi \quad \neg \text{Intro}
\end{array}
\]

And we’re done.
Rules for $\leftrightarrow$

These are reminiscent of the rules for $\rightarrow$

\[
\begin{array}{cc}
\phi & \psi \\
\hline
\phi \leftrightarrow \psi \\
\end{array}
\]

$\phi \leftarrow \psi$  $\leftrightarrow$ Intro

$\phi \leftrightarrow \psi$  $\leftrightarrow$ Elim1

$\phi \leftrightarrow \psi$  $\psi$  $\leftrightarrow$ Elim2
Rules for $\forall$

\[\vdots\]

\[\forall v \phi\]

\[\phi[t/v]\]

$\forall$Elim
Rules for $\forall$

\[
\vdash
\begin{array}{c}
\forall v \phi \\
\hline
\phi[t/v]
\end{array}
\quad \forall\text{Elim}
\]

In this rule:

- $\phi$ is a formula in which only the variable $v$ occurs freely.
- $t$ is a constant.
- $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of $v$ in $\phi$ by $t$. 
Substitution

\( \phi[t/v] \) is the sentence obtained by replacing all free occurrences of \( v \) in \( \phi \) by \( t \).

- Recall: a free occurrence of \( v \) is one not bound by \( \forall v \) or \( \exists v \)

**Compute the following**

- \( Px[a/x] = \) 
- \( \forall xPx[a/x] = \) 
- \( \forall y(\exists xPx \lor Qx \rightarrow Py)[a/x] = \)
Substitution

\[ \phi[t/v] \] is the sentence obtained by replacing all free occurrences of \( v \) in \( \phi \) by \( t \).

- Recall: a free occurrence of \( v \) is one not bound by \( \forall v \) or \( \exists v \)

Compute the following

- \( Px[a/x] = Pa \)
- \( \forall xPx[a/x] = \)
- \( \forall y(\exists xPx \lor Qx \rightarrow Py)[a/x] = \)
Substitution

\( \phi[t/v] \) is the sentence obtained by replacing all free occurrences of \( v \) in \( \phi \) by \( t \).

- Recall: a free occurrence of \( v \) is one not bound by \( \forall v \) or \( \exists v \)

Compute the following

\[
\begin{align*}
Px[a/x] &= Pa \\
\forall xPx[a/x] &= \forall xPx \\
\forall y(\exists xPx \lor Qx \to Py)[a/x] &= \end{align*}
\]
Substitution

$\phi[t/v]$ is the sentence obtained by replacing all free occurrences of $v$ in $\phi$ by $t$.

- Recall: a free occurrence of $v$ is one not bound by $\forall v$ or $\exists v$

Compute the following

- $Px[a/x] = Pa$
- $\forall x Px[a/x] = \forall x Px$
- $\forall y (\exists x Px \lor Qx \to Py)[a/x] = \forall y (\exists x Px \lor Qa \to Py)$
Example

\( \forall x (Px \rightarrow Qx), Pa \vdash QA \)
\[ \forall x (Px \rightarrow Qx), Pa \vdash Qa \]

\[
\vdots
\]

\[
\begin{array}{c}
\forall v \phi \\
\phi[t/v]
\end{array}
\]

\[ \forall \text{Elim} \]
Example

\[ \forall x \ (P x \rightarrow Q x), \ Pa \vdash Q a \]

Assume the first premiss.

\[ \forall x \ (P x \rightarrow Q x) \]
Example

\[ \forall x (P x \rightarrow Q x), P a \vdash Q a \]

\[ \begin{array}{c}
\forall x (P x \rightarrow Q x) \\
\hline
P a \rightarrow Q a
\end{array} \]

Apply \( \forall \text{Elim} \)

\[ \begin{array}{c}
\forall v \phi \\
\hline
\phi[t/v] \quad \forall \text{Elim}
\end{array} \]

To apply the rule: delete \( \forall x \) and by replace all occurrences of \( x \) in the formula by the constant \( a \).
Example

\( \forall x (Px \rightarrow Qx), Pa \vdash Qa \)

Assume the other premiss

\[
\begin{align*}
Pa & \quad \forall x (Px \rightarrow Qx) \\
\hline
Pa & \rightarrow Qa
\end{align*}
\]
Example
\( \forall x (P x \rightarrow Q x), Pa \vdash Q a \)

Apply modus ponens.

\[
\begin{array}{c}
Pa \\
\hline
Pa (P x \rightarrow Q x) \\
\hline
Pa \rightarrow Q a \\
\hline
Q a
\end{array}
\]

\[
\begin{array}{c}
\phi \\
\hline
\phi \rightarrow \psi \\
\hline
\psi \rightarrow \text{Elim}
\end{array}
\]
Example

\[ \forall x (P x \rightarrow Q x), \, P a \vdash Q a \]

And we’re done

\[
\begin{array}{c}
\forall x (P x \rightarrow Q x)
\hline
P a \\
\hline
P a \rightarrow Q a
\hline
Q a
\end{array}
\]
Here’s the introduction rule for $\forall$

\[ \therefore \phi[t/v] \quad \forall \text{Intro} \]

side conditions:

(i) the constant $t$ does not occur in $\phi$ and

(ii) $t$ does not occur in any undischarged assumption in the proof of $\phi[t/v]$. 

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

Informal proof.
Let an arbitrary thing be given. Call it 'Jane Doe'. Clearly, if Jane Doe is a pedestrian, then Jane Doe is either a qualified driver or a pedestrian. So: every pedestrian is either a qualified driver or a pedestrian.
Here’s the introduction rule for $\forall$

$$
\begin{array}{c}
\phi[t/v] \\
\forall v \phi
\end{array}
\xrightarrow{\forall \text{Intro}}
$$

side conditions:

(i) the constant $t$ does not occur in $\phi$ and

(ii) $t$ does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian
Here’s the introduction rule for ∀

\[
\begin{align*}
\vdots \\
\phi[t/v] \\
\frac{\forall v \phi}{\forall \forall \phi} \quad \forall \text{Intro}
\end{align*}
\]

side conditions:

(i) the constant \( t \) does not occur in \( \phi \) and

(ii) \( t \) does not occur in any undischarged assumption in the proof of \( \phi[t/v] \).

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

Informal proof.
Here’s the introduction rule for $\forall$

\[
\begin{align*}
\vdash & \\
\phi[t/v] & \phi[t/v] \quad \forall \text{Intro}
\end{align*}
\]

side conditions:

(i) the constant $t$ does not occur in $\phi$ and

(ii) $t$ does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

**Informal reasoning with arbitrary names**

(C) Every pedestrian is either a qualified driver or a pedestrian

*Informal proof.* Let an arbitrary thing be given. Call it ‘Jane Doe’.
Here’s the introduction rule for $\forall$

\[
\begin{array}{c}
\vdots \\
\phi[t/v] \\
\hline
\forall v \phi
\end{array}
\]

\text{\textit{\&Intro}}

side conditions:

\begin{enumerate}[label=(\roman*)]
    
    \item the constant $t$ does not occur in $\phi$ and
    \item $t$ does not occur in any undischarged assumption in the proof of $\phi[t/v]$.
\end{enumerate}

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

\textit{Informal proof}. Let an arbitrary thing be given. Call it ‘Jane Doe’.

Clearly, if Jane Doe is a pedestrian, then Jane Doe is either a qualified driver or a pedestrian.
Here’s the introduction rule for $\forall$

\[
\begin{array}{c}
\vdash \\
\phi[t/v] \\
\hline
\forall v \phi \\
\end{array}
\text{\textup{\small \textbf{\textit{\textbf{\textbf{\small \forall Intro}}}}}}
\]

side conditions:

(i) the constant $t$ does not occur in $\phi$ and

(ii) $t$ does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every pedestrian is either a qualified driver or a pedestrian

\textit{Informal proof}. Let an arbitrary thing be given. Call it ‘Jane Doe’.

Clearly, \textit{if} Jane Doe is a pedestrian, then Jane Doe is either a qualified driver or a pedestrian.

So: every pedestrian is either a qualified driver or a pedestrian.
Example

\[ \vdash \forall z (Pz \rightarrow Qz \lor Pz) \]
Example

\[ \vdash \forall z (Pz \rightarrow Qz \lor Pz) \]

\[ \vdash \forall v \phi \]

\[ \phi[t/v] \]

\[ \forall v \phi \]

\[ \forall \text{Intro} \]
<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊢ ∀z (Pz → Qz ∨ Pz)</td>
</tr>
</tbody>
</table>

I assume Pa.
(We’ll try to prove Pa → Qa ∨ Pa without making assumptions about a)
Example

\[ \vdash \forall z (Pz \rightarrow Qz \lor Pz) \]

\[
\begin{array}{c}
Pa \\
\hline 
Qa \lor Pa
\end{array}
\]

Apply \lor Intro2.

\[
\begin{array}{c}
\vdash \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi \lor \psi
\hline 
\phi \lor \psi
\end{array}
\lor Intro2
\]
Example

\[ \vdash \forall z (Pz \rightarrow Qz \lor Pz) \]

Apply \(\rightarrow\)Intro

\[
\begin{array}{c}
P_a \\
\hline
Q_a \lor P_a \\
\hline
P_a \rightarrow (Q_a \lor P_a)
\end{array}
\]

\[
\begin{array}{c}
[\phi] \\
\vdots
\hline
\psi \\
\hline
\phi \rightarrow \psi \rightarrow \text{Intro}
\end{array}
\]
Example

\[ \vdash \forall z\ (Pz \rightarrow Qz \lor Pz) \]

Apply \( \rightarrow \)Intro

\[
\frac{\begin{array}{c}
[Pa] \\
\hline
Qa \lor Pa \\
\hline
Pa \rightarrow (Qa \lor Pa)
\end{array}}{}
\]

\[
[\phi] \\
\vdots \\
\psi \\
\hline
\phi \rightarrow \psi \quad \rightarrow \text{Intro}
\]

\[ \vdash \forall z\ (Pz \rightarrow Qz \lor Pz) \]
Example

\[ \vdash \forall z \,(Pz \rightarrow Qz \lor Pz) \]

Finally we want to apply the rule for introducing \( \forall \).

\[
\begin{array}{c}
[Pa] \\
\hline
Qa \lor Pa \\
\hline
Pa \rightarrow (Qa \lor Pa) \\
\hline
\forall z \,(Pz \rightarrow (Qz \lor Pz))
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\phi[t/v] \\
\hline
\forall v \phi
\end{array}
\]

\( \forall \text{Intro} \)
Example

\[ \vdash \forall z (Pz \rightarrow Qz \lor Pz) \]

Finally we want to apply the rule for introducing \( \forall \).

\[
\begin{array}{c}
\frac{[Pa]}{Qa \lor Pa} \\
\frac{Pa \rightarrow (Qa \lor Pa)}{\forall z (Pz \rightarrow (Qz \lor Pz))}
\end{array}
\]

But we also need to check the side conditions are met.

for \( t = a; \phi = (Pz \rightarrow (Qz \lor Pz)) \)

\( (i) \) \( t \) does not occur in \( \phi \)

i.e. \( a \) does not occur in

\( (Pz \rightarrow (Qz \lor Pz)) \)
Example

\[ \vdash \forall z (Pz \rightarrow Qz \lor Pz) \]

Finally we want to apply the rule for introducing \( \forall \).

\[
\begin{align*}
\text{[Pa]} & \quad \text{\dfrac{Qa \lor Pa}{Pa \rightarrow (Qa \lor Pa)}} \quad \text{∀Intro} \\
\forall z (Pz \rightarrow (Qz \lor Pz))
\end{align*}
\]

But we also need to check the side conditions are met.

for \( t = a; \phi = (Pz \rightarrow (Qz \lor Pz)) \)

\( i \) \( t \) does not occur in \( \phi \)

\( \text{i.e.} \) \( a \) does not occur in

\( (Pz \rightarrow (Qz \lor Pz)) \)
Example

\[ \vdash \forall z (Pz \rightarrow Qz \vee Pz) \]

Finally we want to apply the rule for introducing \( \forall \).

\[ \begin{array}{c}
 [Pa] \\
 \frac{Qa \vee Pa}{Pa \rightarrow (Qa \vee Pa)} \\
 \frac{\forall z (Pz \rightarrow (Qz \vee Pz))}{\forall v \phi} \]
\end{array} \]

But we also need to check the side conditions are met.
for \( t = a; \phi = (Pz \rightarrow (Qz \vee Pz)) \)

(ii) \( t \) does not occur in any undischarged assumption in the proof of \( \phi[t/v] \).

i.e. \( a \) does not occur in undischarged assumptions.
Rules for $\exists$

The introduction rule is straightforward.

$$\frac{\phi[t/v]}{\exists v \phi} \exists \text{Intro}$$
Example

\( Rcc \vdash \exists y \ Rcy \)
Example

\[ Rcc \vdash \exists y \, Rc y \]

\[ Rcc \]

Assume the premiss.
Example

$Rcc \vdash \exists y Rcy$

Apply $\exists$Intro

$$\frac{\phi[t/v]}{\exists v \phi} \exists$Intro$

All we need to do is to choose the right $\phi$ and $v$
Example

\( Rcc \vdash \exists y \, Rcy \)

\[ \frac{Rcc}{\exists y \, Rcy} \]

Apply \( \exists \text{Intro} \)

\[
\frac{\phi[t/v]}{\exists v \, \phi} \quad \exists \text{Intro}
\]

All we need to do is to choose the right \( \phi \) and \( v \).

Let \( \phi = Rcy \), \( v = y \)

\( \phi[c/y] = Rcc \)

\( \exists v \phi = \exists y \, Rcy \)
The elimination rule is as follows.

<table>
<thead>
<tr>
<th>$[\phi[t/v]]$</th>
<th>Side conditions:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash \exists v \phi$</td>
<td>(i) $t$ does not occur in $\exists v \phi$</td>
</tr>
<tr>
<td>$\vdash \psi$</td>
<td>(ii) $t$ does not occur in $\psi$,</td>
</tr>
<tr>
<td>$\exists v \phi \quad \psi$</td>
<td>(iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.</td>
</tr>
</tbody>
</table>

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

(iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$. 

Informal Proof. Let Smith be an Albanian penny.
By (2), Smith is a quindarka.
So, something is a quindarka.
So (C), follows from (1) and (2).
The elimination rule is as follows.

\[
\begin{array}{c}
\phi[t/v] \\
\vdots \\
\exists v \phi \\
\hline
\psi \\
\end{array}
\]

\[\exists \text{Elim}\]

Side conditions:

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

**Dummy names again**

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.
The elimination rule is as follows.

\[
\exists v \phi \quad \vdash \quad \exists \ell \text{Elim}
\]

Side conditions:

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

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(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

*Informal Proof.*
The elimination rule is as follows.

\[
\frac{\exists v \phi \quad \psi}{\psi} \quad \exists \text{Elim}
\]

Side conditions:

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.
The elimination rule is as follows.

\[
\frac{\exists v \phi}{\psi} \quad \exists \text{Elim}
\]

[\phi[t/v]]

\vdots \quad \vdots

\exists v \phi \quad \psi

\text{Side conditions:}

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

**Dummy names again**

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*Informal Proof.* Let Smith be an Albanian penny.

By (2), Smith is a quindarka.
The elimination rule is as follows.

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \\
\exists v \phi \\
\hline \\
\psi \\
\hline \\
\exists \text{Elim} \\
\psi
\end{array}
\]

Side conditions:

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

**Dummy names again**

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

*Informal Proof.* Let Smith be an Albanian penny.

By (2), Smith is a quindarka. So, something is a quindarka.
The elimination rule is as follows.

\[
\begin{array}{c}
\exists v \phi \\
\vdots \\
\exists v \phi & \psi \\
\hline \\
\psi \\
\hline
\end{array}
\]

\[\phi[t/v]\]

Side conditions:

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

**Dummy names again**

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

*Informal Proof.* Let Smith be an Albanian penny.

By (2), Smith is a quindarka.

So, something is a quindarka.

So (C), follows from (1) and (2).
Example

\[ \exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x \]
Example

\[ \exists x \, P x, \forall x \, (P x \to Q x) \vdash \exists x \, Q x \]

The standard way to reason from \( \exists x \, P x \) is to assume \( P t \) (for \( t \) a new constant)
Example

$\exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x$

$\forall x \, (P x \rightarrow Q x)$

$Pc$

Assume the second premiss.
Example

\[ \exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x \]

\[
\begin{align*}
\forall x \, (P x \rightarrow Q x) \\
P_c \\
\hline \\
P_c \rightarrow Q_c
\end{align*}
\]

Apply \( \forall \)Elim.
Example

\[ \exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x \]

\[
\begin{array}{c}
Pc \quad \forall x \,(P x \rightarrow Q x) \\
\hline
Pc \rightarrow Qc \\
\hline
Qc 
\end{array}
\]

Apply \rightarrow \text{Elim.}
Example

\[\exists x \, Px, \forall x \, (Px \rightarrow Qx) \vdash \exists x \, Qx\]

\[
\begin{array}{c}
Pc \\
\hline
\forall x \, (Px \rightarrow Qx) \\
Pc \rightarrow Qc \\
\hline
Qc \\
\hline
\exists x \, Qx
\end{array}
\]

Apply \(\exists\text{Intro}\).
Example

\[ \exists x \, P \, x, \forall x \, (P \, x \rightarrow Q \, x) \vdash \exists x \, Q \, x \]

\[
\begin{array}{c}
\forall x \, (P \, x \rightarrow Q \, x) \\
\hline
P \, c \\
\hline
P \, c \rightarrow Q \, c \\
\hline
Q \, c \\
\hline
\exists x \, Q \, x
\end{array}
\]

Now we’ve reached a conclusion assuming \( P \, c \) (and making no other assumptions about \( c \)) we can apply \( \exists \text{Elim} \).
Example

\[ \exists x \, P x, \, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x \]

\[ \exists x \, P x \]

\[ \forall x \, (P x \rightarrow Q x) \]

\[ P c \]

\[ \exists x \, Q x \]

\[ \forall x \, (P x \rightarrow Q x) \]

\[ Q c \]

\[ \exists x \, Q x \]

\[ [\phi[t/v]] \]

\[ \vdash \exists v \, \phi \]

\[ \exists v \, \phi \]

\[ \vdash \psi \]

\[ \exists Elim \]

\[ \vdash \psi \]

\[ \psi \]

(i) \( t \) does not occur in \( \exists v \, \phi \)

(ii) \( t \) does not occur in \( \psi \)

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).
Example

\[ \exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x \]

\[
\begin{array}{c}
\forall x \, (P x \rightarrow Q x) \\
  \hline
  \forall x \, (P x \rightarrow Q x) \\
  P c \\
  \hline
  P c \rightarrow Q c \\
  Q c \\
  \hline
  \exists x \, P x \\
  \exists x \, Q x \\
\end{array}
\]

\[ \exists x \, Q x \]

\[
\begin{array}{l}
[\phi[t/v]] \\
\vdots \\
\exists v \, \phi \\
\hline
\psi \\
\exists \text{Elim} \\
\hline
\psi
\end{array}
\]

(i) \( c \) does not occur in \( \exists x \, P x \)

(ii) \( t \) does not occur in \( \psi \)

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).
Example

$$
\exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x
$$

$$
\begin{align*}
\forall x \, (P x \rightarrow Q x) \\
P c \\
\hline
P c \rightarrow Q c \\
\hline
Q c \\
\hline
\exists x \, P x \\
\hline
\exists x \, Q x
\end{align*}
$$

$$
[\phi[t/v]] \\
\vdots \\
\exists v \, \phi \\
\psi \\
\hline
\psi \\
\exists \text{Elim}
$$

(i) $c$ does not occur in $\exists x \, P x$

(ii) $t$ does not occur in $\psi$

(iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$. 
### Example

$$\exists x \, P x, \forall x \, (P x \to Q x) \vdash \exists x \, Q x$$

\[
\begin{array}{c}
\forall x \, (P x \to Q x) \\
\hline
P c \\
\hline
P c \to Q c \\
\hline
Q c \\
\hline
\exists x \, P x \\
\hline
\exists x \, Q x
\end{array}
\]

\[
\exists x \, Q x
\]

\[
\begin{align*}
[\phi[t/v]] \\
\vdots \\
\exists v \, \phi \\
\psi \\
\hline
\exists \text{Elim}
\end{align*}
\]

\((i)\) \quad c \text{ does not occur in } \exists x \, P x

\((ii)\) \quad c \text{ does not occur in } \exists x \, Q x

\((iii)\) \quad t \text{ does not occur in any undischarged assumption other than } \phi[t/v] \text{ in the proof of } \psi.
Example

\[ \exists x \, Px, \forall x \, (Px \rightarrow Qx) \vdash \exists x \, Qx \]

\[
\begin{array}{c}
\forall x \,(Px \rightarrow Qx) \\
Pc \\
\hline
Pc \rightarrow Qc \\
\hline
Qc \\
\hline
\exists x \, Px \\
\hline
\exists x \, Qx \\
\hline
\exists x \, Qx
\end{array}
\]

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \\
\exists v \, \phi \\
\vdots \\
\psi \\
\hline
\exists \text{Elim} \\
\psi \\
\end{array}
\]

(i) \( c \) does not occur in \( \exists x \, Px \)

(ii) \( c \) does not occur in \( \exists x \, Qx \)

(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).
Example

\[ \exists x \, P x, \forall x \, (P x \to Q x) \vdash \exists x \, Q x \]

\[
\begin{array}{c}
\forall x \, (P x \to Q x) \\
Pc \\
\hline
Pc \to Qc \\
\hline
Qc \\
\hline
\exists x \, Q x
\end{array}
\]

\[
\begin{array}{c}
\exists x \, P x \\
\hline
\exists x \, Q x
\end{array}
\]

\[ \exists x \, Q x \]

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \\
\exists v \, \phi, \psi \\
\hline
\psi, \exists \text{Elim}
\end{array}
\]

(i) \( c \) does not occur in \( \exists x \, P x \)

(ii) \( c \) does not occur in \( \exists x \, Q x \)

(iii) \( c \) does not occur in any undischarged assumption other than \( Pc \) in the proof of \( \exists x \, Q x \).
Example

\( \exists x \, P x, \forall x \, (P x \rightarrow Q x) \vdash \exists x \, Q x \)

\[
\begin{array}{c}
\forall x \, (P x \rightarrow Q x) \\
\hline
[Pc] \\
\hline
P c \rightarrow Q c \\
\hline
Q c \\
\hline
\exists x \, P x \\
\hline
\exists x \, Q x
\end{array}
\]

\( \exists x \, Q x \)

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \quad \vdots \\
\exists v \, \phi \\
\hline
\psi \\
\hline
\psi \\
\hline
\exists Elim
\end{array}
\]

(i) \( c \) does not occur in \( \exists x \, P x \)

(ii) \( c \) does not occur in \( \exists x \, Q x \)

(iii) \( c \) does not occur in any undischarged assumption other than \( P c \) in the proof of \( \exists x \, Q x \).
Let $\Gamma$ be a set of $\mathcal{L}_2$-sentences and $\phi$ a $\mathcal{L}_2$-sentence.  

**Two notions of consequence**

$\Gamma \vdash \phi$ iff there is a proof of $\phi$ with only sentences in $\Gamma$ as non-discharged assumptions.

$\Gamma \models \phi$ iff there is no $\mathcal{L}_2$-structure in which all sentences in $\Gamma$ are true and $\phi$ is false.

**Theorem**

(a) Soundness: $\Gamma \vdash \phi$ only if $\Gamma \models \phi$

(b) Completeness: $\Gamma \models \phi$ only if $\Gamma \vdash \phi$

*Proof.* Elements of Deductive Logic.
Why we need the side conditions on $\exists$Elim

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

(iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$. 

\[
\begin{array}{c}
\exists v \phi \\
\vdots \\
\exists x \exists x
\end{array}
\]
Why we need the side conditions on $\exists\text{Elim}$

Side conditions:

- $\boxtimes$ $t$ does not occur in $\exists v \phi$
- $t$ does not occur in $\psi$,
- $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

$\forall a a$

Adequacy
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $t$ does not occur in $\psi$,
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

$$\frac{[\phi[t/v]]}{\exists \phi} \vdash \psi$$

$$\exists \text{Elim}$$

$$\quad Raa$$

$$\exists x Rxx$$
Why we need the side conditions on \( \exists \text{Elim} \)

Side conditions:

- \( \exists v \phi \) \quad (i) \quad \text{t does not occur in } \exists v \phi
- \psi \quad (ii) \quad \text{t does not occur in } \psi
- \psi \quad (iii) \quad \text{t does not occur in any undischarged assumption other than } \phi[t/v] \text{ in the proof of } \psi.

\[
\begin{align*}
\exists v \phi && \psi \\
\vdash && \vdash \\
\exists \text{Elim} &&\psi
\end{align*}
\]

But clearly, \( \exists x Rax \not \models \exists x Rxx \). Without (i), ND is not sound.

\[
\begin{align*}
\exists x Rax && \vdash \exists x Rxx
\end{align*}
\]
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $t$ does not occur in $\psi$,
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
\begin{align*}
\phi[t/v] & \\
\vdots & \\
\exists v \phi & \\
\psi & \exists \text{Elim} \\
\psi & \\
Raa & \\
\exists x Rax & \exists x Rxx \\
\exists x Rxx & \exists x Rxx
\end{align*}
\]
Why we need the side conditions on $\exists$Elim

Side conditions:

- $\forall t \text{ does not occur in } \exists v \phi$
- (ii) $t$ does not occur in $\psi$,
- (iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
\begin{align*}
\exists v \phi & \quad \psi \\
\psi & \quad \exists \text{Elim}
\end{align*}
\]

But clearly, $\exists x Rax \not\models \exists x Rxx$. Without (i), ND is not sound.
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $t$ does not occur in $\psi$,
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

But clearly, $\exists x Rax \not\models \exists x Rxx$. Without (i), ND is not sound.
Why we need the side conditions on $\exists$Elim

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \\
\exists v \phi \\
\psi
\end{array}
\begin{array}{c}
\therefore \\
\psi
\end{array}
\exists$Elim \\
\]

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

(iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$. 

But clearly, $\exists x P x \not\models P a$. Without (ii), ND is not sound.
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $x$ does not occur in $\psi$,
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

But clearly, $\exists x P x \not \models P a$. Without (ii), ND is not sound.
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $t$ does not occur in $\psi$,
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

But clearly, $\exists x P x \not\models Pa$. Without (ii), ND is not sound.
Why we need the side conditions on $\exists$Elim

Side conditions:

- (i) $t$ does not occur in $\exists v \phi$
- (ii) $t$ does not occur in $\psi$,
- (iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

$\exists x \, P \, x \quad P \, a$
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $t$ does not occur in $\exists x Px$.
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

Given

\[ [\phi[t/v]] \]

\[ \vdots \]

\[ \exists v \phi \quad \psi \]

$\exists$Elim

\[ \psi \]

But clearly, $\exists x Px \not\models Pa$. Without (ii), ND is not sound.
Why we need the side conditions on \( \exists \text{Elim} \)

Side conditions:

1. (i) \( t \) does not occur in \( \exists v \phi \)
2. \( t \) does not occur in \( \psi \),
   - Crossed out: \( t \) does not occur in \( \psi \),
3. (iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \\
\exists v \phi \\
\vdots \\
\psi
\end{array}
\]
\( \exists \text{Elim} \)

\[
\exists x \ P x \quad [P a]
\]
\( P a \)
Why we need the side conditions on $\exists \text{Elim}$

Side conditions:

1. $t$ does not occur in $\exists v \phi$  
2. $t$ does not occur in $\psi$,  
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
\begin{array}{c}
\exists v \phi \\
\vdots \\
\psi
\end{array} \quad \exists \text{Elim}
\]

But clearly, $\exists x Px \not\vdash Pa$. Without (ii), ND is not sound.
Why we need the side conditions on $\exists$Elim

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

(iii) $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$. 

$\exists v \phi \vdash \psi$
Why we need the side conditions on $\exists$Elim

Side conditions:

1. $t$ does not occur in $\exists v \phi$
2. $t$ does not occur in $\psi$,
3. $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$. 

\[
\begin{array}{c}
[\phi[t/v]] \\
\vdots \quad \vdots \\
\exists v \phi \\
\hline
\psi \quad \exists Elim \\
\psi
\end{array}
\]
Why we need the side conditions on $\exists$Elim

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

$\times$ $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.


$Pa$
Why we need the side conditions on \( \exists \text{Elim} \)

**Side conditions:**

1. (i) \( t \) does not occur in \( \exists v \phi \)
2. (ii) \( t \) does not occur in \( \psi \),
3. \( \times \) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

\[\begin{array}{c}
\exists v \phi \\
\vdots \\
\exists \vdots \\
\exists \phi[t/v] \\
\psi \\
\hline
\psi \\
\exists \text{Elim}
\end{array}\]

But \( \exists x Px, Qa \not\models \exists x (Px \land Qx) \).

Without (iii), ND is not sound.

\[Pa \quad Qa\]
Why we need the side conditions on $\exists$Elim

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$, $X$ $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
[\phi[t/v]] : \vdots \vdots \exists v \phi \quad \psi \quad \exists \text{Elim} \quad \psi
\]

\[
P_a \quad Q_a
\]

\[
\frac{}{P_a \land Q_a}
\]
Why we need the side conditions on $\exists$Elim

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

$\times$ $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
\begin{array}{c}
\phi[t/v] \\
\vdots \\
\exists v \phi \\
\psi
\end{array}
\]

\[
\exists v \phi \quad \psi \\
\hline
\psi
\]

\[
\frac{\text{Pa} \quad Qa}{\text{Pa} \land Qa} \\
\frac{\text{Pa} \land Qa}{\exists x (Px \land Qx)}
\]
Why we need the side conditions on $\exists \text{Elim}$

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

$\times$ $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
\begin{array}{c}
\exists v \phi \\
\vdots \\
[\phi[t/v]] \\
\vdots \\
\exists \text{Elim}
\end{array}
\]

\[
\begin{array}{c}
P a \\
\hline
Q a \\
\hline
P a \land Q a
\end{array}
\]

\[
\begin{array}{c}
\exists x \ P x \\
\hline
\exists x (P x \land Q x)
\end{array}
\]
Why we need the side conditions on $\exists$Elim

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$,

$\times$ $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

\[
\begin{array}{c}
\exists v \phi \\
\vdots \\
\phi[t/v] \\
\vdots \\
\exists v \phi \\
\hline
\psi
\end{array}
\]

\[
Pa \quad Qa
\]

\[
\begin{array}{c}
Pa \\
\hline
Pa \land Qa
\end{array}
\]

\[
\begin{array}{c}
\exists x Px \\
\hline
\exists x (Px \land Qx)
\end{array}
\]

\[
\begin{array}{c}
\exists x (Px \land Qx) \\
\hline
\exists x (Px \land Qx)
\end{array}
\]
Why we need the side conditions on $\exists \text{Elim}$

Side conditions:

(i) $t$ does not occur in $\exists v \phi$

(ii) $t$ does not occur in $\psi$

$\times$ $t$ does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

$$
\begin{align*}
[\phi[t/v]] & \quad [Pa] \quad Qa \\
\vdots & \quad \vdots \\
\exists v \phi & \quad \psi \\
\psi & \quad \exists \text{Elim}
\end{align*}
$$

$$
\begin{align*}
\exists x \, Px & \quad \exists x \, (Px \land Qx) \\
Pa \land Qa & \quad \exists x \, (Px \land Qx)
\end{align*}
$$
Why we need the side conditions on \( \exists \text{Elim} \)

Side conditions:

(i) \( t \) does not occur in \( \exists v \phi \)

(ii) \( t \) does not occur in \( \psi \),

\( \times \) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

\[
\begin{align*}
[\phi[t/v]] & \quad \psi \\
\vdots & \quad \vdots \\
\exists v \phi & \quad \psi \quad \exists \text{Elim}
\end{align*}
\]

\[
\begin{align*}
[Pa] & \quad Qa \\
\hline
Pa \wedge Qa \\
\exists x \quad Px & \quad \exists x(Px \wedge Qx) \\
\hline
\exists x(Px \wedge Qx)
\end{align*}
\]

But \( \exists xPx, Qa \not\models \exists x(Px \wedge Qx) \). Without (iii), ND is not sound.