INTRODUCTION TO LOGIC

Lecture 6
Natural Deduction
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There’s nothing you can’t prove if your outlook is only sufficiently limited  
_Dorothy L. Sayers_

Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $L_2$-sentences

<table>
<thead>
<tr>
<th>$[Pa]$</th>
<th>$\forall y (Py \rightarrow Qy)$</th>
<th>$\forall z (Qz \rightarrow Rz)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Pa \rightarrow Qa$</td>
<td>$Qa$</td>
<td>$Qa \rightarrow Ra$</td>
</tr>
<tr>
<td>$Ra$</td>
<td>$Pa \rightarrow Ra$</td>
<td>$\forall y (Py \rightarrow Rz)$</td>
</tr>
</tbody>
</table>

- The root of the tree is the conclusion
- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
- The rules depend purely on the syntax of the sentences
  ...not on their semantic properties.

Rules for $\wedge$

$\wedge$Intro
The result of appending $\phi \wedge \psi$ to a proof of $\phi$ and a proof of $\psi$ is a proof of $\phi \wedge \psi$.

\[
\begin{array}{c}
\vdots \\
\phi \\
\psi \\
\hline \\
\phi \wedge \psi \\
\end{array}
\]

$\wedge$Intro

$\wedge$Elim1 and $\wedge$Elim2

(1) The result of appending $\phi$ to a proof of $\phi \wedge \psi$ is a proof of $\phi$.
(2) The result of appending $\psi$ to a proof of $\phi \wedge \psi$ is a proof of $\psi$.

\[
\begin{array}{c}
\vdots \\
\phi \wedge \psi \\
\hline \\
\phi \\
\end{array}
\]

$\wedge$Elim1

\[
\begin{array}{c}
\vdots \\
\phi \wedge \psi \\
\hline \\
\psi \\
\end{array}
\]

$\wedge$Elim2

Example

$(P \wedge Q) \wedge R \vdash P$
6.1 Propositional logic

Example
\[ Qb \land Pa, Ra \vdash Pa \land Ra \]

Rules for →

→ELIM
The result of appending \( \psi \) to a proof of \( \phi \) and a proof of \( \phi \rightarrow \psi \) is a proof of \( \psi \).

\[
\vdots \\
\phi \\
\phi \rightarrow \psi \\
\vdots \\
\psi
\]

→Elim

This rule is often called ‘Modus Ponens’.

→INTRO
The result of appending \( \phi \rightarrow \psi \) to a proof of \( \psi \) and discharging all assumptions of \( \phi \) in the proof of \( \psi \) is a proof of \( \phi \rightarrow \psi \).

\[
\vdots \\
[\phi] \\
\vdots \\
\phi \rightarrow \psi \rightarrow \text{Intro}
\]

Conditional proof in informal reasoning.

(1) If it’s poison and Quintus took it, then he needs to be readmitted.
(2) It’s poison
So (C) if Quintus took it, he need to be readmitted.

Informal proof. Suppose Quintus took it.
Then (by 2) It’s poison and he took it.
Then (by 1 and MP) he needs to be readmitted.

So (by conditional proof) if Quintus took it, he needs to be readmitted.
6.1 Propositional logic

Example

\( P, (P \land Q) \rightarrow R \vdash Q \rightarrow R \)

We can now define \( \Gamma \vdash \phi \).

Let \( \Gamma \) be a set of sentences and \( \phi \) a sentence.

**Definition (Provable)**

The sentence \( \phi \) is **provable** from \( \Gamma \) if and only if:

- there is a proof of \( \phi \) with only sentences in \( \Gamma \) as non-discharged assumptions.

**Notation**

- \( \Gamma \vdash \phi \) is short for \( \phi \) is provable from \( \Gamma \)
- \( \vdash \phi \) is short for \( \emptyset \vdash \phi \)
- \( \psi_1, \ldots, \psi_n \vdash \phi \) is short for \( \{\psi_1, \ldots, \psi_n\} \vdash \phi \).

**Return to the rule of assumption.**

**ASSUMPTION RULE**

The occurrence of a sentence \( \phi \) with no sentence above it is an assumption. An assumption of \( \phi \) is a proof of \( \phi \).

This may seem odd.

Suppose I assume, the following:

\[ \exists x \exists y (Rxy \lor P) \]

By the rule, this counts as a proof of \( \exists x \exists y (Rxy \lor P) \)

But it is not an outright proof of \( \exists x \exists y (Rxy \lor P) \)

- This proof does not show \( \vdash \exists x \exists y (Rxy \lor P) \)
- Instead it shows \( \exists x \exists y (Rxy \lor P) \vdash \exists x \exists y (Rxy \lor P) \)

**Rules for \( \lor \)**

The introduction rules are straightforward.

\[
\begin{align*}
\phi & \vdash \phi \lor \psi \\
\psi & \vdash \phi \lor \psi
\end{align*}
\]

\[
\begin{align*}
\frac{\phi}{\phi \lor \psi} & \text{ \lor\text{-}Intro1} \\
\frac{\psi}{\phi \lor \psi} & \text{ \lor\text{-}Intro2}
\end{align*}
\]
The elimination rule is a little more complex.

\[
\frac{\phi \lor \psi}{\chi} \quad \lor \text{Elim}
\]

**Proof by cases in informal reasoning**

1. Either you don’t play and you quit or you do something quiet and don’t play.
   So, (C) You don’t play.

*Informal proof.* Suppose (1)

Case (i): You don’t play and you quit. So: you don’t play

Case (ii): You do something quiet and don’t play. So: you don’t play.

Either way then, (C) follows: you don’t play.

**The rules for \( \neg \)**

Here are the rules for \( \neg \).

\[
\frac{\phi}{\neg \phi} \quad \neg \text{Intro} \quad \frac{\neg \phi}{\psi} \quad \neg \text{Intro} \quad \frac{\phi}{\neg \psi} \quad \neg \text{Elim}
\]

The proof technique is known as *reductio ad absurdum*. 

**Example**

\( \neg P \land Q \lor (\exists x \ Q x \land \neg P) \vdash \neg P \)

**Example**

\( \neg (P \rightarrow Q) \vdash \neg Q \)
6.1 Propositional logic

Rules for \( \leftrightarrow \)

These are reminiscent of the rules for \( \rightarrow \)

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\phi \leftrightarrow \psi \\
\phi \leftrightarrow \psi
\end{array}
\]

\( \leftrightarrow \) Intro

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\phi \leftrightarrow \psi \\
\phi
\end{array}
\]

\( \leftrightarrow \) Elim1

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\psi \\
\phi
\end{array}
\]

\( \leftrightarrow \) Elim2

6.2 Predicate logic

Rules for \( \forall \)

\[
\begin{array}{c}
\forall \phi \\
\vdots \\
\hline
\phi[t/v]
\end{array}
\]

\( \forall \) Elim

In this rule:

- \( \phi \) is a formula in which only the variable \( v \) occurs freely
- \( t \) is a constant
- \( \phi[t/v] \) is the sentence obtained by replacing all free occurrences of \( v \) in \( \phi \) by \( t \).

Substitution

\( \phi[t/v] \) is the sentence obtained by replacing all free occurrences of \( v \) in \( \phi \) by \( t \).

- Recall: a free occurrence of \( v \) is one not bound by \( \forall v \) or \( \exists v \)

Compute the following

- \( Px[a/x] = \)
- \( \forall xPx[a/x] = \)
- \( \forall y(\exists xPx \lor Qx \rightarrow Py)[a/x] = \)

Example

\( \forall (Px \rightarrow Qx), Pa \vdash Qa \)
Here’s the introduction rule for $\forall$

\[
\frac{\phi[t/v]}{\forall v \phi} \quad \forall \text{Intro}
\]

side conditions:

1. the constant $t$ does not occur in $\phi$ and
2. $t$ does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

**Informal reasoning with arbitrary names**

(C) Every pedestrian is either a qualified driver or a pedestrian

*Informal proof.* Let an arbitrary thing be given. Call it ‘Jane Doe’.

Clearly, if Jane Doe is a pedestrian, then Jane Doe is either a qualified driver or a pedestrian.

So: every pedestrian is either a qualified driver or a pedestrian.

**Example**

\[\forall y (Py \rightarrow Qy), \forall z (Qz \rightarrow Rz) \vdash \forall y (Py \rightarrow Ry)\]

**Example**

\[\vdash \forall z (Pz \rightarrow Qz \lor Pz)\]

**Rules for $\exists$**

The introduction rule is straightforward.

\[
\frac{\phi[t/v]}{\exists v \phi} \quad \exists \text{Intro}
\]
The elimination rule is as follows.

\[ \exists \psi \quad \exists \text{Elim} \]

Side conditions:
(i) \( t \) does not occur in \( \exists \phi \)
(ii) \( t \) does not occur in \( \psi \),
(iii) \( t \) does not occur in any undischarged assumption other than \( \phi[t/v] \) in the proof of \( \psi \).

**Dummy names again**

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

*Informal Proof.* Let Smith be an Albanian penny.

By (2), Smith is a quindarka.

So, something is a quindarka.

So (C), follows from (1) and (2).

Let \( \Gamma \) be a set of \( \mathcal{L}_2 \)-sentences and \( \phi \) a \( \mathcal{L}_2 \)-sentence.

**Two notions of consequence**

\( \Gamma \vdash \phi \) iff there is a proof of \( \phi \) with only sentences in \( \Gamma \) as non-discharged assumptions.

\( \Gamma \models \phi \) iff there is no \( \mathcal{L}_2 \)-structure in which all sentences in \( \Gamma \) are true and \( \phi \) is false.

**Theorem**

(a) Soundness: \( \Gamma \vdash \phi \) only if \( \Gamma \models \phi \)

(b) Completeness: \( \Gamma \models \phi \) only if \( \Gamma \vdash \phi \)

*Proof.* Elements of Deductive Logic.