In this lecture I’ll wrap up my treatment of predicate logic by bringing together three strands:
- predicate logic and English (chapter 4)
- the semantics of predicate logic (chapter 5)
- Natural Deduction (chapter 6)

Consistency

Theorem (adequacy)
Assume that $\phi$ and all elements of $\Gamma$ are $L_2$-sentences. Then $\Gamma \vdash \phi$ if and only if $\Gamma \models \phi$.

Definition (syntactic consistency)
A set $\Gamma$ of $L_2$-sentences is **syntactically consistent** iff there is a sentence $\phi$ such that $\Gamma \not\vdash \phi$.

A set $\Gamma$ is syntactically **inconsistent** iff it’s not syntactically consistent.
Decidability

In contrast to $\mathcal{L}_1$, we still don't have a systematic method for checking whether an argument in $\mathcal{L}_2$ (with finitely many premisses) is valid or whether an $\mathcal{L}_2$-sentence is a logical truth or whether it is inconsistent.

**Theorem (Church 1936)**

There is no ‘recursive’ method for deciding whether an $\mathcal{L}_2$-sentence is logically true (or whether an $\mathcal{L}_2$-argument with finitely many premisses is valid).

That is, one cannot write a computer programme that tells one, applied to an $\mathcal{L}_2$-sentence, after finite time whether the sentence is logically true or not. That holds even if no restrictions are imposed on the memory, disk space, computation time etc.

Consequently, there is no method for deciding whether a given $\mathcal{L}_2$-sentence is provable.

How does the formal language $\mathcal{L}_2$ relate to English? In chapter 4 I have already sketched how one goes about formalisations of English sentences in $\mathcal{L}_2$.

**Definition**

An argument in English is valid in predicate logic if and only if its formalisation in the language $\mathcal{L}_2$ of predicate logic is valid.
Example
All concrete objects are located in space. The number 5 isn’t located in space. So the number 5 isn’t a concrete object.

FORMALISATION
∀x (Px → Qx), ¬Qa ⊨ ¬Pa

P: … is a concrete object
Q: … is located in space
a: the number 5

So the argument is valid in predicate logic.

Example: show the following argument is not valid
A belief is known only if it is true and justified. The belief that Jones is in Barcelona or Jones owns a Ford is true and justified. Therefore, it’s known.

Step (i): formalise
Premiss 1: ∀x (Px → (P₁x → (Qx ∧ Rx))).
Premiss 2: Pa ∧ Qa ∧ Ra.
Conclusion: P₁a.

Dictionary:
P: … is a belief
P₁: … is known
Q: … is true
R: … is justified
a: the belief that Jones is in Barcelona or Jones owns a Ford

Claim
∀x (Px → (P₁x → Qx ∧ Rx)), Pa ∧ Qa ∧ Ra ⊭ P₁a

(Because of the adequacy theorem ⊭ and ⊭ coincide.)

Here is a counterexample:
Let \( \mathcal{A} \) be an \( \mathcal{L}_2 \)-structure with \{1\} as its domain and

| \( |P|_\mathcal{A} = \{1\} \) |
| \( |P₁|_\mathcal{A} = \emptyset \) |
| \( |Q|_\mathcal{A} = \{1\} \) |
| \( |R|_\mathcal{A} = \{1\} \) |
| \( |a|_\mathcal{A} = 1 \) |

The premises are true, the conclusion is false in this structure.
Logical truth of English sentences in predicate logic etc. are defined in analogy to the notions of logical truth etc. in propositional logic:

**Definition**

1. An English sentence is **logically true in predicate logic** iff its formalisation in predicate logic is logically true.
2. An English sentence is a **contradiction in predicate logic** iff its formalisation in predicate logic is a contradiction.
3. A set of English sentences is **consistent in predicate logic** iff the set of their formalisations in predicate logic is semantically consistent.

To show that an English sentence is logically true in predicate logic, one can (try to) formalise the sentence as a sentence $\phi$ of $\mathcal{L}_2$ and prove that $\vdash \phi$.

To show that an English sentence is a contradiction in predicate logic one can formalise the sentence as a sentence $\phi$ of $\mathcal{L}_2$ and prove that $\vdash \neg \phi$.

To show that a set of English sentences is consistent in predicate logic, one can formalise all sentences in the set and show that the formalisations are all true in some $\mathcal{L}_2$-structure.

To show that a set of English sentences is inconsistent in predicate logic, one can formalise some of the sentences as $\phi_1, \ldots, \phi_n$ and show that $\{\phi_1, \ldots, \phi_n\} \vdash (P \land \neg P)$.

The language $\mathcal{L}_2$ is very powerful. Large parts of the sciences and mathematics can easily be formalised in it.

I return to the problem of formalising English sentences in $\mathcal{L}_2$. As we can now analyse more structural features of English sentences, we get new problems.

**FIRST PROBLEM**

**Arity of predicates**

*Hassan buttered the toast with a knife in the pantry.*

*Hassan buttered the toast with a knife.*

*Hassan buttered the toast.*

Should one formalise the predicate as a predicate letter of arity 4, 3, or 2?

Arguably, one could paraphrase the last sentence as

*Hassan buttered the toast with something in some place.*

and then use a predicate letter of arity 4 for the formalisation.
SECOND PROBLEM
Lexical ambiguity

The predicate expressions ‘is a bank’, ‘is a suit’ are ambiguous.

In $L_2$ there is no lexical ambiguity: in a $L_2$-structure the semantic value of a unary predicate letter is always a single set. Accordingly, one has to use different predicate letters for ‘is a bank’ (as a financial institution) and for ‘is a bank’ (as the edge of a river).

Example
All the books were taken by someone.

Arguably, there are two readings:
1. $\forall y (P y \rightarrow \exists x (Q x \land R x y))$ and
2. $\exists x (Q x \land \forall y (P y \rightarrow R x y))$

$P$: … is a book
$Q$: … is a person
$R$: … took …

THIRD PROBLEM
Structural ambiguity

The indefinite article can be used to make existential or universal claims.

Example
A New College student is clever.

This is ambiguous between:
1. $\exists x (P x \land Q x)$ and
2. $\forall x (P x \rightarrow Q x)$

$P$: … is a New College student
$Q$: … is clever

The kind of ambiguity in

All the books were taken by someone.

is known as scope ambiguity because the formalisations assign different scopes to the quantifier $\forall x$.

Definition (scope of a quantifier)
The scope of an occurrence of a quantifier in a sentence $\phi$ is (the occurrence of) the smallest $L_2$-formula that contains that quantifier and is part of $\phi$. 
FOURTH PROBLEM

Intensionality

Example

Sören believes in an almighty being. Therefore there is an almighty being.

INCORRECT FORMALISATION, Exercise 6.3 (i)

\[ \exists x (Rax \land Px) \vdash \exists x Px \]

\(a\): Sören
\(P\): … is an almighty being
\(R\): … believes in …

Probably this is not a good proof for the existence of an almighty being.

What’s going wrong here?

The English predicate expression ‘believes in’ doesn’t express a relation between the believer and another object. Its semantics is different from the semantics of a predicate letter of \(\mathcal{L}_2\).

So the \(\mathcal{L}_2\)-argument is valid.

Example

Bahareh wants to live in Oxford. Oxford is a city with high levels of air pollution. Therefore Bahareh wants to live in a city with high levels of air pollution.

INCORRECT FORMALISATION

\[ \exists x (Rax \land Qx) \]

\(R\): … wants to live in …
\(Q\): … is a city with high levels of air pollution
\(a\): Bahareh
\(b\): Oxford
If the formalisation were sound the argument would be valid in predicate logic as

**Claim**

\[ R_a, Q_b \vdash \exists x (R_a x \land Q_x) \]

\[
\begin{align*}
 & R_a, Q_b \\
 \hline \\
 & \ \\
 & R_a x \land Q_x \\
 \hline \\
 & \exists x (R_a x \land Q_x)
\end{align*}
\]

\[ \exists \text{Intro} \]

Again ‘wants to live in’ doesn’t express a relation between a person and a place. Hence it cannot be formalised using a binary predicate letter (at best one can formalise ‘wants to live in Oxford’ and ‘wants to live in a city with high levels of air pollution’ as two separate unary predicate letters.

This shows that in some cases the truth of an English sentence does not only depend on what a designator designates.

This is not the case in \( L_2 \).

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**Extensionality of \( L_2 \)**

If constants, sentence letters, and predicate letters are replaced in an \( L_2 \)-sentence by other constants, sentence letters, and predicate letters (respectively) that have the same extension in a given \( L_2 \)-structure, then the truth-value of the sentence in that \( L_2 \)-structure does not change.

That is, the truth value of an \( L_2 \)-sentence in an \( L_2 \)-structure depends only on the extension (semantic value) of the symbols in the sentence in that \( L_2 \)-structure.

In contrast to English, \( L_2 \) is an **extensional language**.

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**‘that’-sentences and ontology**

**Example**

Fred believes that 8 is (identical to) 8.
Fred believes that the number of planets is 8.

**Example**

It’s necessary that 8 is 8.
It’s necessary that the number of planets is 8.

These two examples show that

\[ \ldots \text{ believes that} \ldots \text{ is} \ldots \]
\[ \text{it’s necessary that} \ldots \text{ is} \ldots \]

... do not express relations and thus **must not** be formalised as predicate letters.
Some philosophers have proposed to analyse these sentences using propositions:

**Example**
Fred believes the proposition that 8 is 8.

**formalisation**
\[ R_{1ab} \]

\[ R_1: \text{... believes...} \]
\[ a: \text{Fred} \]
\[ b: \text{the proposition that 8 is 8} \]

In more sophisticated formalisations the constant \( b \) might be further analysed (even within predicate logic). At any rate we are now getting into metaphysical problems: What are propositions (if they exist at all)? How are propositions structured?

**Quotation**

The phrase

‘... has six letters’

is not extensional.

**Example**

‘London’ has six letters.
‘the capital of England’ has six letters.

Different philosophical views force different formalisations:
- If belief is a relation between a believer and a proposition, the formalisation \( R_{1ab} \) is adequate.
- If belief is merely a certain state of mind and the believer is not entering a relation with some object (proposition etc), then \( R_{1ab} \) is surely not adequate.

Looking back to Chapter 1 should shed some light on how to deal with quotation marks in formalisations.

**Example**

‘snow’ is a noun.

This isn’t a sentence about snow; this is a sentence about the word ‘snow’. Thus, the above sentence must not be formalised as \( Pa \)

with the following dictionary:
- \( P \): ‘...’ is a noun
- \( a \): snow

But it can be formalised as \( Qb \).

- \( Q \): ‘... is a noun
- \( b \): ‘snow’
The *spoken* sentence

*Tom is monosyllabic.*

is ambiguous. The ambiguity is made explicit by the following two formalisations.

<table>
<thead>
<tr>
<th>FORMALISATION I</th>
<th>Qa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q: ... is monosyllabic</td>
<td>a: ‘Tom’</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FORMALISATION II</th>
<th>Qb</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q: ... is monosyllabic</td>
<td>b: Tom</td>
</tr>
</tbody>
</table>

One might argue that ‘is monosyllabic’ is ambiguous at least in spoken English.