Part B Classical Mechanics: Specimen exam questions

This Part B course was new in the 2014/2015 academic year. However, Lagrangian and rigid body mechanics were covered in the old Part A Classical Mechanics course, and some of the past paper questions for this course serve as good specimen questions for the new Part B course. I recommend in particular

- AO2 Classical Mechanics 2008 Q.H2
- AO2 Classical Mechanics 2010 Q.H2
- AO2 Classical Mechanics 2011 Q.H2

Solutions to these may be found on the department webpage (Oxford students only).

The old Part A course covered Lagrangian mechanics in less detail, and in particular didn't include Noether's Theorem. There was no Hamiltonian mechanics. I've therefore written a specimen question on the Hamiltonian formalism, with solution:

1. (a) [8 marks] Consider a system with n degrees of freedom, generalized coordinates $q_1, \ldots, q_n$ and Lagrangian $L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$. Explain how the Lagrangian is related to the *Hamiltonian* $H = H(q_1, \ldots, q_n, p_1, \ldots, p_n)$ for such a system.

   The action is

$$S = \int_{t_1}^{t_2} \left(\sum_{a=1}^{n} p_a \dot{q}_a - H \right) \mathrm{d}t \ .$$

   Show that extremizing this action with respect to both q_a and p_a leads to Hamilton's equations of motion.

   (b) [4 marks] Let $f = f(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be a function on phase space. Show that Hamilton's equations imply

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{f, H\} \ ,$$

   where the right hand side is the *Poisson bracket*, which you should define.

   (c) [6 marks] Consider a coordinate transformation $q \to Q = Q(q,p)$, $p \to P = P(q,p)$ for a system with one degree of freedom. Denote the Poisson brackets in the two coordinate systems by $\{f, g\}_{q,p}$ and $\{f, g\}_{Q,P}$, respectively, where f and g are arbitrary functions on phase space. How are these Poisson brackets related for a *canonical transformation*? Show that the transformation is canonical if and only if $\{Q, P\}_{q,p} = 1$. If $H = H(q,p)$ is the Hamiltonian in the original coordinates, what is the Hamiltonian $K = K(Q,P)$ in the new coordinates?

   (d) [7 marks] Consider a system with one degree of freedom and canonical coordinates q, p. Show that

$$q \ \to \ Q \ = \ \log\left(\frac{\sinh q}{p}\right) \ , \qquad p \ \to \ P \ = \ p\,\frac{\cosh q}{\sinh q}$$

   is a canonical transformation. If the original Hamiltonian is $H = \frac{1}{2}p^2$, find the Hamiltonian in the new coordinates.

Solution

1. (a) First the generalized momenta p_a are defined by

$$p_a = \frac{\partial L}{\partial \dot{q}_a} .$$

The Hamiltonian is then

$$H = \sum_{a=1}^{n} p_a \dot{q}_a - L ,$$

where we interpret the left hand side as a function of q_a and p_a by inverting $p_a = p_a(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ as $\dot{q}_a = \dot{q}_a(q_1, \ldots, q_n, p_1, \ldots, p_n)$. This is a Legendre transformation with respect to the generalized velocities.

Next we write $\delta q_a(t) = \epsilon u_a(t)$, $\delta p_a(t) = \epsilon w_a(t)$, and expand the action to first order in ϵ. Using the chain rule this leads to the first order variation

$$\begin{aligned} \delta S &= \epsilon \sum_{a=1}^{n} \int_{t_1}^{t_2} \left(w_a \dot{q}_a + p_a \dot{u}_a - \frac{\partial H}{\partial q_a} u_a - \frac{\partial H}{\partial p_a} w_a \right) \mathrm{d}t , \\ &= \epsilon \sum_{a=1}^{n} \left\{ \int_{t_1}^{t_2} \left[w_a \left(\dot{q}_a - \frac{\partial H}{\partial p_a} \right) - u_a \left(\dot{p}_a + \frac{\partial H}{\partial q_a} \right) \right] \mathrm{d}t + \Big[p_a u_a \Big]_{t_1}^{t_2} \right\} , \end{aligned}$$

where we have integrated the second term by parts. In extremizing the action the endpoints of the path are held fixed, so $u_a(t_1) = 0 = u_a(t_2)$ and the boundary term is zero. Since $\delta S = 0$ for all variations we thus deduce Hamilton's equations

$$\dot{q}_a = \frac{\partial H}{\partial p_a} , \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a} .$$

(b) Using the chain rule

$$\begin{aligned} \dot{f} &= \sum_{a=1}^{n} \left(\frac{\partial f}{\partial q_a} \dot{q}_a + \frac{\partial f}{\partial p_a} \dot{p}_a \right) \\ &= \sum_{a=1}^{n} \left(\frac{\partial f}{\partial q_a} \frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial H}{\partial q_a} \right) \\ &= \{f, H\} , \end{aligned}$$

where the Poisson bracket of two functions f, g on phase space is defined as

$$\{f, g\} = \sum_{a=1}^{n} \left(\frac{\partial f}{\partial q_a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q_a} \right) .$$

(c) The transformation is canonical if $\{f, g\}_{p,q} = \{f, g\}_{P,Q}$ holds for all functions f, g on phase space (so that Poisson brackets are preserved). Using the chain rule we compute

$$
\begin{aligned}
\{f,g\}_{q,p} &= \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \\
&= \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial q}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial p}\right) \\
&\quad - \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial p}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial q}\right) \\
&= \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)\left(\frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\right) \\
&= \{Q,P\}_{q,p}\cdot\{f,g\}_{Q,P}\ .
\end{aligned}
$$

Here in going from the second equality to the third we note that 4 of the 8 terms cancel in two pairs, and in the third line we've simply factorized what remains. This shows the transformation is canonical if and only if $\{Q,P\}_{q,p} = 1$. Since this is a time-independent canonical transformation, the new Hamiltonian is obtained by simply substituting the coordinate transformation, *i.e.* the new Hamiltonian is $K(Q,P) = H(q(Q,P), p(Q,P))$.

(d) From part (c) we simply have to check that $\{Q,P\}_{q,p} = 1$. We compute

$$
\begin{aligned}
\frac{\partial Q}{\partial q} &= \frac{\cosh q}{\sinh q}\ , \qquad & \frac{\partial Q}{\partial p} &= -\frac{1}{p}\ , \\
\frac{\partial P}{\partial q} &= -\frac{p}{\sinh^2 q}\ , \qquad & \frac{\partial P}{\partial p} &= \frac{\cosh q}{\sinh q}\ ,
\end{aligned}
$$

and so

$$
\{Q,P\}_{q,p} = \frac{\cosh^2 q}{\sinh^2 q} - \frac{1}{p}\cdot\frac{p}{\sinh^2 q} = 1\ .
$$

The original Hamiltonian is $H = \frac{1}{2}p^2$, so we need to find $p(Q,P)$. From the coordinate transformation we immediately deduce

$$
p = P\tanh q\ , \quad \text{and} \quad p = \mathrm{e}^{-Q}\sinh q\ .
$$

We may combine these by noting

$$
\frac{P^2}{p^2} = \frac{\cosh^2 q}{\sinh^2 q} = \frac{1+\sinh^2 q}{\sinh^2 q} = \frac{1+p^2\mathrm{e}^{2Q}}{p^2\mathrm{e}^{2Q}}\ .
$$

Thus

$$
P^2 = p^2 + \mathrm{e}^{-2Q}\ ,
$$

and so the new Hamiltonian is

$$
K = \frac{1}{2}P^2 - \frac{1}{2}\mathrm{e}^{-2Q}\ .
$$

Please send comments and corrections to `sparks@maths.ox.ac.uk`.