Part A Electromagnetism

Problem Sheet 1 (of 2)

1. Consider two point charges $\pm q$, with $-q$ placed at the origin and $+q$ placed at the position $\mathbf{d}$. What is the electrostatic potential and electric field for this configuration? Roughly sketch the field lines.

   Using Taylor's theorem show that in an open ball, not containing the origin and centred at a point $\mathbf{r}$, one has

$$\frac{1}{|\mathbf{r}-\mathbf{d}|} = \frac{1}{r} - \mathbf{d}\cdot\nabla\left(\frac{1}{r}\right) + O(d^2)$$

   where $d = |\mathbf{d}|$. If we now define $\mathbf{p} = q\,\mathbf{d}$ and take the limit $q \to \infty$, $d \to 0$ with $q\,d = |\mathbf{p}|$ fixed, show that the electrostatic potential in this limit is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} .$$

   This is the electrostatic field of a *dipole*. Notice the potential falls off as $1/r^2$, which is faster than the $1/r$ behaviour of a point charge.

2. Consider a spherically symmetric distribution of charge in which the charge density ρ is zero everywhere except for the region $a \leq r \leq b$, in which it is a constant K. Use the symmetry of the problem to argue that the electric field $\mathbf{E}$ must be radial, and thence deduce the electric field $\mathbf{E}$ everywhere using Gauss' law.

   Using your answer, deduce that

   (a) the electric field *outside* a ball of constant charge density and total charge Q is the same as that generated by a point charge Q at the centre of the ball.

   (b) in the limit $a \to b$, $K \to \infty$, with $(b-a)K = \sigma$ kept finite, the normal component of the electric field is discontinuous across the resulting spherical shell of charge, jumping by σ/ϵ_0.

   Notice that, in the process of answering this question, you have shown that Gauss' law implies Coulomb's law.

3. Let ρ be a bounded continuous function on $\mathbb{R}^3$ with support $\{\mathbf{r} \in \mathbb{R}^3 \mid \rho(\mathbf{r}) \neq 0\} \subset R \subset \mathbb{R}^3$ in a bounded region R, and define

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{r}'\in R}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' .$$

   Show that $\phi = O(1/r)$ and $\hat{\mathbf{u}}\cdot\mathbf{E} = -\hat{\mathbf{u}}\cdot\nabla\phi = O(1/r^2)$ as $r \to \infty$, where $\hat{\mathbf{u}}$ is any unit vector. Hence show that

$$\lim_{r\to\infty}\int_{B_r}\nabla\cdot(\phi\mathbf{E})\,\mathrm{d}V = 0 .$$

   where B_r is a ball of radius r [*Hint*: R being bounded means it is contained in a ball B_a for some radius a. Your proof should then involve some simple inequalities].

4. Consider a circular disc of radius a and constant surface charge density σ. Compute the electrostatic potential, and also the electric field, on the axis of symmetry at a distance z from the centre. Hence show that the discontinuity in the normal component of $\mathbf{E}$ at the centre of the disc is σ/ϵ_0.

5. Show that the electrostatic energy of a sphere of radius b and constant charge density is

$$W = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 b} ,$$

where Q is the total charge [*Hint*: use results from question 2]. The energy of a point charge, obtained by taking $b \to 0$, is thus infinite!

6. Consider a current density of the form

$$\mathbf{J}(\mathbf{r},t) = \mathbf{r}\,[a \sin\Omega(\mathbf{r},t) + b\cos\Omega(\mathbf{r},t)]$$

where $\Omega(\mathbf{r},t) = \omega t - \mathbf{k}\cdot\mathbf{r}$, a, b, ω are non-zero constants, and $\mathbf{k}$ is a constant vector. Show that the continuity equation may be satisfied by taking

$$\rho(\mathbf{r},t) = f(\mathbf{r})\sin\Omega(\mathbf{r},t) + g(\mathbf{r})\cos\Omega(\mathbf{r},t)$$

for appropriate functions f and g.

7. In 3-dimensional Cartesian coordinates (x, y, z), define $\lambda = (x^2 + y^2)^{1/2}$ and $\mathbf{A} = (0, 0, -k\log\lambda)$. Show that, where $\lambda \neq 0$,

$$\nabla\cdot\mathbf{A} = 0 , \qquad \nabla^2\mathbf{A} = 0 .$$

Now define $\mathbf{B} = \nabla\wedge\mathbf{A}$. Write out $\mathbf{B}$ explicitly in coordinates. Deduce (preferably *not* by using this coordinate expression) that, where $\lambda \neq 0$,

$$\nabla\cdot\mathbf{B} = 0 , \qquad \nabla\wedge\mathbf{B} = 0 .$$

Using the coordinate expression for $\mathbf{B}$ show that

$$\int_C \mathbf{B}\cdot \mathrm{d}\mathbf{r} = 2\pi k$$

for any closed curve C which winds once around the z-axis anticlockwise [*Hint*: first take C to be a circle in the plane $\{z = 0\}$ centred at the origin, then use Stokes' Theorem to obtain the general result].

8. Consider a circular loop of wire of radius a carrying a steady current I. If the loop is placed in the $x - y$ plane $\{z = 0\}$, centred at the origin, show using the Biot-Savart law that the magnetic field at the point $(0, 0, z)$ on the axis of symmetry is $\mathbf{B} = (0, 0, B(z))$ where

$$B(z) = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} .$$

Please send comments and corrections to `sparks@maths.ox.ac.uk`. A solution set is available.