Part A Electromagnetism

Problem Sheet 2 (of 2)

1. Consider an infinitely long cylindrical conductor of radius a, its axis of symmetry being the z-axis, carrying a uniform current in the z-direction of constant density J. Using Ampère's law, and assuming that the magnetic field is of the form $\mathbf{B} = B(s)\mathbf{e}_\phi$ in cylindrical coordinates (s, ϕ, z), show that

$$B(s) = \begin{cases} \dfrac{\mu_0 J s}{2} & s \leq a \\ \dfrac{\mu_0 J a^2}{2s} & s \geq a \, . \end{cases}$$

Hence show that the magnetic field outside the conductor is the same as that (derived in lectures) for an infinitely long wire carrying the same current I.

2. The *Helmholtz decomposition* states that any vector field $\mathbf{f}$ on $\mathbb{R}^3$ that is sufficiently differentiable and decays sufficiently rapidly at infinity may be written

$$\mathbf{f} = \nabla \wedge \mathbf{a} - \nabla \psi$$

where

$$\begin{aligned} \mathbf{a}(\mathbf{r}) &= \frac{1}{4\pi} \int \frac{\nabla' \wedge \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V' \\ \psi(\mathbf{r}) &= \frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V' \end{aligned}$$

and the integrals are over the whole of $\mathbb{R}^3$. Assuming this, show that Maxwell's equations, specialized to magnetostatics, imply the Biot-Savart law in the integral form involing the current density $\mathbf{J}$ (equation (2.18) in lectures). Similarly, show that Maxwell's equations, specialized to electrostatics, imply the integral expression for $\mathbf{E}$ in terms of the charge density ρ (equation (1.14) in lectures).

3. Assuming that a solution ψ of the following *wave-equation-with-source*

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = F$$

exists for any function F which you encounter, show that it is possible by means of a gauge transformation to impose the condition

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

on the potentials ϕ and $\mathbf{A}$ (this is called the *Lorenz gauge*). If this is done, show from the Maxwell equations with sources that the potentials satisfy wave-equations-with-sources as follows:

$$\begin{aligned} \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi &= \rho / \epsilon_0 \\ \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} &= \mu_0 \, \mathbf{J} \, . \end{aligned}$$

4. Show that if $\mathbf{E}$ and $\mathbf{B}$ satisfy the source-free ($\rho = 0$, $\mathbf{J} = \mathbf{0}$) Maxwell equations, then so do

$$\mathbf{E}' = \cos\alpha\,\mathbf{E} - \sin\alpha\, c\,\mathbf{B}, \qquad \mathbf{B}' = \frac{1}{c}\sin\alpha\,\mathbf{E} + \cos\alpha\,\mathbf{B}\ ,$$

for any constant α (this is called a *duality rotation*).

5. Let $\mathbf{a}$, $\mathbf{b}$ be constant complex 3-vectors, and define $\Omega = \omega(t - \mathbf{k}\cdot\mathbf{r})$, where ω and $\mathbf{k}$ are real and constant. Under what conditions do the *complex* vectors

$$\mathbf{E} = \mathbf{a}\,\mathrm{e}^{-\mathrm{i}\Omega} \qquad \mathbf{B} = \frac{1}{c}\mathbf{b}\,\mathrm{e}^{-\mathrm{i}\Omega}$$

satisfy the source-free Maxwell equations? Show that every monochromatic plane wave can be expressed as the real part of such a complex solution. What are the conditions on $\mathbf{a}$ for the plane wave to be (i) linearly polarized, and (ii) circularly polarized?

6. An electric field $\mathbf{E}$ is defined in Cartesian coordinates (x, y, z) and time t by the equation

$$\mathbf{E} = E(\cos(\omega(t - z/c)), \sin(\omega(t - z/c)), 0)\ ,$$

where E, ω and c are constants, c being the speed of light in vacuum. Find the corresponding magnetic field strength $\mathbf{B}$, and verify that the source-free Maxwell equations are satisfied.

A particle of mass m and charge Q moves with velocity $\mathbf{v}$ in this electromagnetic field. What forces act on the particle? If the particle is confined to a plane of constant z, on which it can move smoothly, show that the magnetic field does not influence the motion and that the particle can move in circles, whose radius you should find.

Please send comments and corrections to `sparks@maths.ox.ac.uk`. A solution set is available.