Part A Quantum Theory: Problem Sheet 1 (of 2)

1. A laser pointer emits red light with a wavelength $\lambda = 6.50 \times 10^{-7}\,\mathrm{m}$. Taking $\hbar \simeq 1.05 \times 10^{-34}\,\mathrm{J\,s}$ and the speed of light as $c \simeq 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$, find the angular frequency and show that the photons have energy $\simeq 3.04 \times 10^{-19}\,\mathrm{J}$.

   The laser pointer has a power of $1.00 \times 10^{-3}\,\mathrm{W}$ ($1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}}$). Assuming it is 4% efficient, estimate the average number of photons emitted per second.

2. A particle of mass m moves in the interval $[-a, a]$ where the potential $V = V_0$ is constant. Using the stationary state Schrödinger equation show that the energy levels of the system are

$$E_n = V_0 + \frac{n^2\pi^2\hbar^2}{8ma^2} \ ,$$

   where n is a positive integer, and find the corresponding normalized wave functions.

3. A particle of mass m moves freely in the interval $[0, a]$ on the x-axis (so that the potential $V = 0$ within the interval). Initially the wave function is

$$\frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right)\left[1 + 2\cos\left(\frac{\pi x}{a}\right)\right] \ .$$

   Show that at a later time t the wave function is

$$\frac{1}{\sqrt{a}} \mathrm{e}^{-\mathrm{i}\pi^2\hbar t/2ma^2} \sin\left(\frac{\pi x}{a}\right)\left[1 + 2\mathrm{e}^{-3\mathrm{i}\pi^2\hbar t/2ma^2} \cos\left(\frac{\pi x}{a}\right)\right] \ .$$

   [*Formulae for the normalized wave functions $\Psi_n(x,t)$ and energy levels E_n may be quoted from lectures.*] Hence find the probability that at time t the particle lies within the interval $\left[0, \tfrac{1}{2}a\right]$.

4. Consider a particle of mass m confined to a box in three dimensions, with potential

$$V(x,y,z) \quad = \quad \begin{cases} 0 \ , & 0 < x < a\,,\ 0 < y < b\,,\ 0 < z < c\ , \\ +\infty \ , & \text{otherwise} \ , \end{cases}$$

   where (x, y, z) are Cartesian coordinates. By separating variables in the stationary state Schrödinger equation, show that the allowed energies of the particle are

$$E_{n_1,n_2,n_3} \quad = \quad \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right) \ ,$$

   where n_1, n_2, n_3 are positive integers, and find the corresponding normalized wave functions.

5. A particle of mass m moving on the x-axis has a (non-normalized) ground state wave function $1/\cosh^2 x$ with energy $-2\hbar^2/m$. Show that the potential is $V(x) = -\frac{3\hbar^2}{m}\,\mathrm{sech}^2\, x$. An excited state wave function for the particle is $\sinh x/\cosh^2 x$ What is the energy of this state?

6. A particle of mass m moves on the x-axis in a potential $V(x)$. Let $\psi(x)$ be a normalized wave function satisfying the stationary state Schrödinger equation with energy E. Show that if V is an even function (that is, $V(x) = V(-x)$), then $\widetilde{\psi}(x) \equiv \psi(-x)$ is also a normalized wave function. By considering the wave functions $\psi_\pm = \psi \pm \widetilde{\psi}$, or otherwise, deduce that there is either an odd or an even wave function (or both) satisfying the same Schrödinger equation.

7. Suppose that $\Psi(x,t)$ satisfies the one-dimensional time-dependent Schrödinger equation with potential $V(x)$. Let $\rho(x,t) = |\Psi(x,t)|^2$ be the probability density for the particle and

$$j(x,t) = \frac{\mathrm{i}\hbar}{2m}\left(\Psi\frac{\partial\overline{\Psi}}{\partial x} - \overline{\Psi}\frac{\partial\Psi}{\partial x}\right)$$

be the probability current. Show that

$$\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \ .$$

Show further that j vanishes identically if and only if there exists a nowhere zero function $\lambda(t)$ such that $\lambda(t)\Psi(x,t)$ takes only real values.

8. The first excited state wave function for a harmonic oscillator of frequency ω is

$$\psi_1(x) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2\xi\,\mathrm{e}^{-\xi^2/2} \ ,$$

where $\xi = \sqrt{\frac{m\omega}{\hbar}}x$. Show that $\psi_1(x)$ is normalized. Compute the expected values of x and $|x|$ in the state ψ_1.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.