Part A Quantum Theory: Problem Sheet 2 (of 2)

1. A particle of mass m and charge q moves on the x-axis under the influence of a harmonic oscillator potential of angular frequency ω, and a constant electric field $\mathcal{E}$. The potential is

$$V(x) = \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}x \ .$$

Show that the energy levels are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{q^2\mathcal{E}^2}{2m\omega^2} \ ,$$

where n is a non-negative integer. [*Hint: change variable.*]

2. Consider the equation

$$\hat{H}\psi = E\psi \ ,$$

where $\hat{H}$ is the differential operator

$$\hat{H} = \frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right) \ ,$$

$W(x)$ is a real function, and E is a constant. Show that this is the stationary state Schrödinger equation with potential $V(x) = \frac{\hbar^2}{2m}\left(W^2 - \frac{\mathrm{d}W}{\mathrm{d}x}\right)$.

Show that

$$\int_{-\infty}^{\infty} \bar{\psi}\hat{H}\psi \,\mathrm{d}x \quad = \quad \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left|\frac{\mathrm{d}\psi}{\mathrm{d}x} + W\psi\right|^2 \mathrm{d}x - \frac{\hbar^2}{2m}\left[\bar{\psi}\left(\frac{\mathrm{d}\psi}{\mathrm{d}x} + W\psi\right)\right]_{-\infty}^{\infty} \ ,$$

and hence that, for wave functions going to zero sufficiently fast at infinity, the energy E is non-negative and that a solution with zero energy satisfies a first order equation.

By taking $W(x) = \lambda x$, use your results to find the ground state energy, and corresponding ground state wave function, for the one-dimensional harmonic oscillator with potential $V(x) = \frac{1}{2}m\omega^2 x^2$.

3. A particle of mass m moves in three dimensions under the influence of the potential

$$V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) \ .$$

Show that the energy levels have the form $\left(n + \frac{3}{2}\right)\hbar\omega$ where n is a non-negative integer, and find their degeneracies. Show that the ground state wave function is spherically symmetric.

4. A particle moves in two dimensions under the influence of the potential

$$V(x, y) = \frac{1}{2}m\omega^2\left(10x^2 + 12xy + 10y^2\right) \ .$$

By considering V in the rotated coordinates $u = (x + y)/\sqrt{2}$, $v = (x - y)/\sqrt{2}$, find the energy levels and the associated degeneracy of each level.

5. In a two-dimensional model of the hydrogen atom, the stationary state Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right]-\frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi\ ,$$

where (r,ϕ) are polar coordinates. By separating the equation via $\psi(r,\phi)=R(r)\Phi(\phi)$, show that $\Phi(\phi)$ is a constant linear combination of $\mathrm{e}^{il\phi}$ and $\mathrm{e}^{-il\phi}$, where l is a non-negative integer. [*Hint: Use the fact that* $\Phi(\phi+2\pi)=\Phi(\phi)$.]

By further substituting $R(r)=f(r)\mathrm{e}^{-\kappa r}$, where $\kappa=\sqrt{-2mE}/\hbar$, show that the radial equation becomes

$$f''+\left(\frac{1}{r}-2\kappa\right)f'-\left(\frac{l^2}{r^2}+\frac{\kappa-\beta}{r}\right)f \quad = \quad 0\ ,$$

where β is a constant you should identify. By substituting a generalized power series expansion for f, of the form $f(r)=\sum_{k=0}^{\infty}a_k r^{k+c}$, argue that $c=l$ for a non-singular wave function, and hence deduce the recurrence relation

$$a_k=\frac{2\kappa(k+l)-\kappa-\beta}{(k+l)^2-l^2}a_{k-1}$$

in this case. Hence or otherwise show that the energy levels are of the form $-\nu/(2n+1)^2$, where ν is a positive constant and n is a non-negative integer. What is the degeneracy of each energy level?

6. Show that the function

$$G(x,s)=\exp\left[-m\omega\left(\tfrac{1}{2}x^2-2sx+s^2\right)/\hbar\right]$$

satisfies the partial differential equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2}+\frac{1}{2}m\omega^2x^2G=\hbar\omega s\frac{\partial G}{\partial s}+\frac{1}{2}\hbar\omega G\ .$$

Deduce that if $G(x,s)$ is expanded as a power series $G(x,s)=\sum_{n=0}^{\infty}\frac{s^n}{n!}\chi_n(x)$, then the coefficient $\chi_n(x)$ satisfies the stationary state Schrödinger equation for the one-dimensional harmonic oscillator with potential $V(x)=\frac{1}{2}m\omega^2x^2$ and energy $E_n=(n+\frac{1}{2})\hbar\omega$.

Show that $\chi_n(x)=H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\left(\frac{m\omega}{\hbar}\right)^{n/2}\exp(-m\omega x^2/2\hbar)$ where H_n is a polynomial of degree n. Determine the polynomials H_0, H_1, and H_2.

Show that

$$\int_{\mathbb{R}}\overline{G(x,s)}G(x,u)\,\mathrm{d}x=\sqrt{\frac{\pi\hbar}{m\omega}}\exp\left(\frac{2m\omega su}{\hbar}\right)\ ,$$

and hence deduce that

$$\int_{\mathbb{R}}\overline{\chi_j(x)}\chi_k(x)\,\mathrm{d}x=\delta_{jk}k!\sqrt{\frac{\pi\hbar}{m\omega}}\left(\frac{2m\omega}{\hbar}\right)^k\ .$$

[*You may use the identity* $\int_{\mathbb{R}}\exp(-y^2/\sigma^2)\,\mathrm{d}y=\sigma\sqrt{\pi}$ *without proof.*]

Hence show that

$$\psi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp(-m\omega x^2/2\hbar)$$

are orthonormal.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.